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for the international student
Mathematics SL

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**International Baccalaureate
Diploma Programme**

Roger Dixon
Michael Haese
Robert Haese
Sandra Haese



Haese & Harris Publications

MATHEMATICS FOR THE INTERNATIONAL STUDENT

Mathematics SL – WORKED SOLUTIONS

International Baccalaureate Diploma Programme

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FOREWORD

This book gives you fully worked solutions for every question in each chapter of the Haese & Harris Publications textbook **Mathematics SL** which is one of three textbooks in our series ‘Mathematics for the International Student’. The other two textbooks are **Mathematics HL (Core)** and **Mathematical Studies SL**, and books of fully worked solutions are available for those textbooks also.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modeled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for **Mathematics SL** on our website. Please contact us if you have any additions to this list.

RLD PMH

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Background knowledge

EXERCISE A

- 1**
- a** $\sqrt{3} \times \sqrt{5}$
 $= \sqrt{3 \times 5}$
 $= \sqrt{15}$
- b** $(\sqrt{3})^2$
 $= \sqrt{3} \times \sqrt{3}$
 $= 3$
- c** $2\sqrt{2} \times \sqrt{2}$
 $= 2(\sqrt{2} \times \sqrt{2})$
 $= 2 \times 2$
 $= 4$
- d** $3\sqrt{2} \times 2\sqrt{2}$
 $= (3 \times 2)(\sqrt{2} \times \sqrt{2})$
 $= 6 \times 2$
 $= 12$
- e** $3\sqrt{7} \times 2\sqrt{7}$
 $= (3 \times 2)(\sqrt{7} \times \sqrt{7})$
 $= 6 \times 7$
 $= 42$
- f** $\frac{\sqrt{12}}{\sqrt{2}}$
 $= \sqrt{\frac{12}{2}}$
 $= \sqrt{6}$
- g** $\frac{\sqrt{12}}{\sqrt{6}}$
 $= \sqrt{\frac{12}{6}}$
 $= \sqrt{2}$
- h** $\frac{\sqrt{18}}{\sqrt{3}}$
 $= \sqrt{\frac{18}{3}}$
 $= \sqrt{6}$
- 2**
- a** $2\sqrt{2} + 3\sqrt{2}$
 $= (2 + 3)\sqrt{2}$
 $= 5\sqrt{2}$
- b** $2\sqrt{2} - 3\sqrt{2}$
 $= (2 - 3)\sqrt{2}$
 $= -\sqrt{2}$
- c** $5\sqrt{5} - 3\sqrt{5}$
 $= (5 - 3)\sqrt{5}$
 $= 2\sqrt{5}$
- d** $5\sqrt{5} + 3\sqrt{5}$
 $= (5 + 3)\sqrt{5}$
 $= 8\sqrt{5}$
- e** $3\sqrt{5} - 5\sqrt{5}$
 $= (3 - 5)\sqrt{5}$
 $= -2\sqrt{5}$
- f** $7\sqrt{3} + 2\sqrt{3}$
 $= (7 + 2)\sqrt{3}$
 $= 9\sqrt{3}$
- g** $9\sqrt{6} - 12\sqrt{6}$
 $= (9 - 12)\sqrt{6}$
 $= -3\sqrt{6}$
- h** $\sqrt{2} + \sqrt{2} + \sqrt{2}$
 $= 3 \times \sqrt{2}$
 $= 3\sqrt{2}$
- 3**
- a** $\sqrt{8}$
 $= \sqrt{4 \times 2}$
 $= \sqrt{4} \times \sqrt{2}$
 $= 2\sqrt{2}$
- b** $\sqrt{12}$
 $= \sqrt{4 \times 3}$
 $= \sqrt{4} \times \sqrt{3}$
 $= 2\sqrt{3}$
- c** $\sqrt{20}$
 $= \sqrt{4 \times 5}$
 $= \sqrt{4} \times \sqrt{5}$
 $= 2\sqrt{5}$
- d** $\sqrt{32}$
 $= \sqrt{16 \times 2}$
 $= \sqrt{16} \times \sqrt{2}$
 $= 4\sqrt{2}$
- e** $\sqrt{27}$
 $= \sqrt{9 \times 3}$
 $= \sqrt{9} \times \sqrt{3}$
 $= 3\sqrt{3}$
- f** $\sqrt{45}$
 $= \sqrt{9 \times 5}$
 $= \sqrt{9} \times \sqrt{5}$
 $= 3\sqrt{5}$
- g** $\sqrt{48}$
 $= \sqrt{16 \times 3}$
 $= \sqrt{16} \times \sqrt{3}$
 $= 4\sqrt{3}$
- h** $\sqrt{54}$
 $= \sqrt{9 \times 6}$
 $= \sqrt{9} \times \sqrt{6}$
 $= 3\sqrt{6}$
- i** $\sqrt{50}$
 $= \sqrt{25 \times 2}$
 $= \sqrt{25} \times \sqrt{2}$
 $= 5\sqrt{2}$
- j** $\sqrt{80}$
 $= \sqrt{16 \times 5}$
 $= \sqrt{16} \times \sqrt{5}$
 $= 4\sqrt{5}$
- k** $\sqrt{96}$
 $= \sqrt{16 \times 6}$
 $= \sqrt{16} \times \sqrt{6}$
 $= 4\sqrt{6}$
- l** $\sqrt{108}$
 $= \sqrt{36 \times 3}$
 $= \sqrt{36} \times \sqrt{3}$
 $= 6\sqrt{3}$
- 4**
- a** $4\sqrt{3} - \sqrt{12}$
 $= 4\sqrt{3} - \sqrt{4 \times 3}$
 $= 4\sqrt{3} - 2 \times \sqrt{3}$
 $= 4\sqrt{3} - 2\sqrt{3}$
 $= 2\sqrt{3}$
- b** $3\sqrt{2} + \sqrt{50}$
 $= 3\sqrt{2} + \sqrt{25 \times 2}$
 $= 3\sqrt{2} + 5 \times \sqrt{2}$
 $= 3\sqrt{2} + 5\sqrt{2}$
 $= 8\sqrt{2}$
- c** $3\sqrt{6} + \sqrt{24}$
 $= 3\sqrt{6} + \sqrt{4 \times 6}$
 $= 3\sqrt{6} + 2 \times \sqrt{6}$
 $= 3\sqrt{6} + 2\sqrt{6}$
 $= 5\sqrt{6}$
- d** $2\sqrt{27} + 2\sqrt{12}$
 $= 2\sqrt{9 \times 3} + 2\sqrt{4 \times 3}$
 $= 6\sqrt{3} + 4\sqrt{3}$
 $= 10\sqrt{3}$
- e** $\sqrt{75} - \sqrt{12}$
 $= \sqrt{25 \times 3} - \sqrt{4 \times 3}$
 $= 5\sqrt{3} - 2\sqrt{3}$
 $= 3\sqrt{3}$
- f** $\sqrt{2} + \sqrt{8} - \sqrt{32}$
 $= \sqrt{2} + \sqrt{4 \times 2} - \sqrt{16 \times 2}$
 $= \sqrt{2} + 2\sqrt{2} - 4\sqrt{2}$
 $= -\sqrt{2}$

5 a	$\frac{1}{\sqrt{2}}$	b	$\frac{6}{\sqrt{3}}$	c	$\frac{7}{\sqrt{2}}$	d	$\frac{10}{\sqrt{5}}$	e	$\frac{10}{\sqrt{2}}$
	$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$		$= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$		$= \frac{7}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$		$= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$		$= \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
	$= \frac{\sqrt{2}}{2}$		$= \frac{6\sqrt{3}}{3}$		$= \frac{7\sqrt{2}}{2}$		$= \frac{10\sqrt{5}}{5}$		$= \frac{10\sqrt{2}}{2}$
			$= 2\sqrt{3}$				$= 2\sqrt{5}$		$= 5\sqrt{2}$
f	$\frac{18}{\sqrt{6}}$	g	$\frac{12}{\sqrt{3}}$	h	$\frac{5}{\sqrt{7}}$	i	$\frac{14}{\sqrt{7}}$	j	$\frac{2\sqrt{3}}{\sqrt{2}}$
	$= \frac{18}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$		$= \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$		$= \frac{5}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$		$= \frac{14}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$		$= \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
	$= \frac{18\sqrt{6}}{6}$		$= \frac{12\sqrt{3}}{3}$		$= \frac{5\sqrt{7}}{7}$		$= \frac{14\sqrt{7}}{7}$		$= \frac{2\sqrt{6}}{2}$
	$= 3\sqrt{6}$		$= 4\sqrt{3}$				$= 2\sqrt{7}$		$= \sqrt{6}$

EXERCISE B

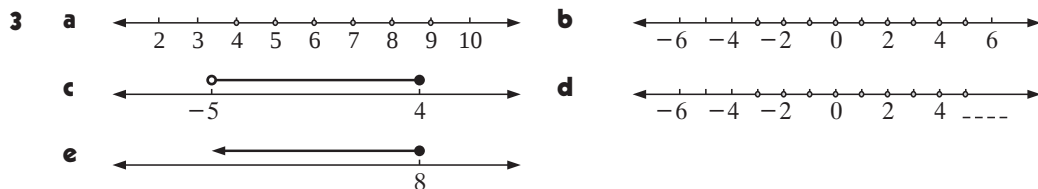
1 a	259	b	259 000	c	2.59
	$= \overbrace{2.59} \times 10^2$		$= \overbrace{2.59000} \times 10^5$		$= 2.59 \times 1$
	$= 2.59 \times 10^2$		$= 2.59 \times 10^5$		$= 2.59 \times 10^0$
d	0.259	e	0.000 259	f	40.7
	$= \overbrace{02.59} \div 10$		$= \overbrace{00002.59} \div 10^4$		$= \overbrace{4.07} \times 10$
	$= 2.59 \times 10^{-1}$		$= 2.59 \times 10^{-4}$		$= 4.07 \times 10^1$
g	4070	h	0.0407	i	407 000
	$= \overbrace{4.070} \times 10^3$		$= \overbrace{004.07} \div 10^2$		$= \overbrace{4.07000} \times 10^5$
	$= 4.07 \times 10^3$		$= 4.07 \times 10^{-2}$		$= 4.07 \times 10^5$
j	407 000 000	k	0.000 040 7		
	$= \overbrace{4.07000000} \times 10^8$		$= \overbrace{000004.07} \div 10^5$		
	$= 4.07 \times 10^8$		$= 4.07 \times 10^{-5}$		
2 a	149 500 000 000 m	b	0.0003 mm	c	0.001 mm
	$= \overbrace{1.49500000000} \times 10^{11}$		$= \overbrace{0003.} \times 10^{-4}$		$= \overbrace{001.} \times 10^{-3}$
	$= 1.495 \times 10^{11} \text{ m}$		$= 3 \times 10^{-4} \text{ mm}$		$= 1 \times 10^{-3} \text{ mm}$
d	15 million degrees	e	300 000 times		
	$= 15\,000\,000 \text{ }^\circ\text{C}$		$= 3 \times 100\,000$		
	$= 1.500\,000\,0 \times 10^7 \text{ }^\circ\text{C}$		$= 3 \times 10^5 \text{ times}$		
	$= 1.5 \times 10^7 \text{ }^\circ\text{C}$				
3 a	4×10^3	b	5×10^2	c	2.1×10^3
	$= 4 \times 1000$		$= 5 \times 100$		$= \overbrace{2.100} \times 10^3$
	$= 4000$		$= 500$		$= 2100$
d	7.8×10^4	e	3.8×10^5	f	8.6×10^1
	$= \overbrace{7.8000} \times 10^4$		$= \overbrace{3.80000} \times 10^5$		$= \overbrace{8.6} \times 10$
	$= 78\,000$		$= 380\,000$		$= 86$
g	4.33×10^7	h	6×10^7		
	$= \overbrace{4.3300000} \times 10^7$		$= 6 \times 10\,000\,000$		
	$= 43\,300\,000$		$= 60\,000\,000$		

- 4**
- a** 4×10^{-3}
 $= \overline{004.} \div 10^3$
 $= 0.004$
- b** 5×10^{-2}
 $= \overline{05.} \div 10^2$
 $= 0.05$
- c** 2.1×10^{-3}
 $= \overline{002.1} \div 10^3$
 $= 0.0021$
- d** 7.8×10^{-4}
 $= \overline{0007.8} \div 10^4$
 $= 0.00078$
- e** 3.8×10^{-5}
 $= \overline{00003.8} \div 10^5$
 $= 0.000038$
- f** 8.6×10^{-1}
 $= \overline{8.6} \div 10^1$
 $= 0.86$
- g** 4.33×10^{-7}
 $= \overline{0000004.33} \div 10^7$
 $= 0.000000433$
- h** 6×10^{-7}
 $= \overline{0000006.} \div 10^7$
 $= 0.0000006$
- 5**
- a** 9×10^{-7} m
 $= \overline{0000009.} \div 10^7$
 $= 0.0000009$ m
- b** 6.130×10^9 people
 $= \overline{6.13000000} \times 10^9$
 $= 6\,130\,000\,000$ people
- c** 1×10^5 light years
 $= 1 \times 100\,000$
 $= 100\,000$ light years
- d** 1×10^{-5} mm
 $= \overline{00001.} \div 10^5$
 $= 0.00001$ mm
- 6**
- a** $(3.42 \times 10^5) \times (4.8 \times 10^4)$
 $= (3.42 \times 4.8) \times (10^5 \times 10^4)$
 $= 16.416 \times 10^9$
 $= 1.6416 \times 10^{10}$
 $= 1.64 \times 10^{10}$ (2 d.p.)
- b** $(6.42 \times 10^{-2})^2$
 $= (6.42)^2 \times (10^{-2})^2$
 $= 41.2164 \times 10^{-4}$
 $= 4.12164 \times 10^{-3}$
 $= 4.12 \times 10^{-3}$ (2 d.p.)
- c** $\frac{3.16 \times 10^{-10}}{6 \times 10^7}$
 $= \frac{3.16}{6} \times \frac{10^{-10}}{10^7}$
 $= 0.52\overline{6} \times 10^{-17}$
 $= 5.2\overline{6} \times 10^{-18}$
 $= 5.27 \times 10^{-18}$ (2 d.p.)
- d** $(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})$
 $= \frac{9.8 \times 10^{-4}}{7.2 \times 10^{-6}}$
 $= \frac{9.8}{7.2} \times \frac{10^{-4}}{10^{-6}}$
 $= 1.36\overline{1} \times 10^2$
 $= 1.36 \times 10^2$ (2 d.p.)
- e** $\frac{1}{3.8 \times 10^5}$
 $= \frac{1}{3.8} \times 10^{-5}$
 $= 2.63 \times 10^{-6}$ (2 d.p.)
- f** $(1.2 \times 10^3)^3$
 $= (1.2)^3 \times (10^3)^3$
 $= 1.728 \times 10^9$
 $= 1.73 \times 10^9$ (2 d.p.)
- 7**
- a** 1 day = 24 hours
i.e., missile travels 5400×24
 $= 129\,600$
 $= 1.296 \times 10^5$
 $\div 1.30 \times 10^5$ km
- b** 1 week = 7 days
 $= 7 \times 24$ hours
 $= 168$ hours
i.e., missile travels 5400×168
 $= 907\,200$
 $= 9.072 \times 10^5$
 $\div 9.07 \times 10^5$ km
- c** 2 years = 2×365.25 days
 $= 730.5$ days
 $= 730.5 \times 24$ hours
 $= 17\,532$ hours
i.e., missile travels $5400 \times 17\,532$
 $= 94\,672\,800$
 $= 9.46728 \times 10^7$
 $\div 9.47 \times 10^7$ km

- 8 a** distance = speed $\times$ time
time = 1 minute = 60 seconds
so, light travels $(3 \times 10^8) \times 60$
 $= 180 \times 10^8$
 $= 1.80 \times 10^{10}$ m
- b** distance = speed $\times$ time
time = 1 day = 24 hours
 $= 24 \times 60 \times 60$ seconds
 $= 86\,400$ seconds
 $= 8.64 \times 10^4$ seconds
i.e., light travels $(3 \times 10^8) \times (8.64 \times 10^4)$
 $= 3 \times 8.64 \times 10^{12}$
 $= 25.92 \times 10^{12}$
 $\div 2.59 \times 10^{13}$ m
- c** distance = speed $\times$ time
time = 1 year = 365.25 days
 $= 365.25 \times 8.64 \times 10^4$ sec {from **b**}
 $= 3155.76 \times 10^4$
 $\div 3.16 \times 10^7$ sec
i.e., light travels $(3 \times 10^8) \times (3.156 \times 10^7)$
 $= 3 \times 3.156 \times 10^{15}$
 $= 9.468 \times 10^{15}$
 $\div 9.47 \times 10^{15}$ m

EXERCISE C

- 1 a** $\{x : x > 5\}$ reads ‘the set of all x such that x is greater than 5’
b $\{x : x \leq 3\}$ reads ‘the set of all x such that x is less than or equal to 3’
c $\{y : 0 < y < 6\}$ reads ‘the set of all y such that y lies between 0 and 6’
d $\{x : 2 \leq x \leq 4\}$ reads ‘the set of all x such that x is greater than or equal to 2, but less than or equal to 4’
e $\{t : 1 < t < 5\}$ reads ‘the set of all t such that t lies between 1 and 5’
f $\{n : n < 2 \text{ or } n \geq 6\}$ reads ‘the set of all n such that n is less than 2 or greater than or equal to 6’
- 2 a** $\{x : x > 2\}$ **b** $\{x : 1 < x \leq 5\}$ **c** $\{x : x \leq -2 \text{ or } x \geq 3\}$
d $\{x : x \in \mathbb{Z}, -1 \leq x \leq 3\}$ **e** $\{x : x \in \mathbb{Z}, 0 \leq x \leq 5\}$ **f** $\{x : x < 0\}$

**EXERCISE D**

- 1 a** $3x + 7x - 10$
 $= 10x - 10$
- b** $3x + 7x - x$
 $= 9x$
- c** $2x + 3x + 5y$
 $= 5x + 5y$
- d** $8 - 6x - 2x$
 $= 8 - 8x$
- e** $7ab + 5ba$
 $= 7ab + 5ab$
 $= 12ab$
- f** $3x^2 + 7x^3$
 $= 3x^2 + 7x^3$
i.e., cannot be simplified
- 2 a** $3(2x + 5) + 4(5 + 4x)$
 $= 6x + 15 + 20 + 16x$
 $= 22x + 35$
- b** $6 - 2(3x - 5)$
 $= 6 - 6x + 10$
 $= 16 - 6x$

$$\begin{aligned} \mathbf{c} \quad & 5(2a - 3b) - 6(a - 2b) \\ &= 10a - 15b - 6a + 12b \\ &= 4a - 3b \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 3x(x^2 - 7x + 3) - (1 - 2x - 5x^2) \\ &= 3x^3 - 21x^2 + 9x - 1 + 2x + 5x^2 \\ &= 3x^3 - 16x^2 + 11x - 1 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & 2x(3x)^2 = 2x \times 9x^2 = 18x^3 \\ \mathbf{b} \quad & \frac{3a^2b^3}{9ab^4} = \frac{\cancel{3} \times \cancel{a} \times a \times \cancel{b} \times \cancel{b} \times \cancel{b}}{\cancel{3} \times 3 \times \cancel{a} \times \cancel{b} \times \cancel{b} \times \cancel{b} \times b} \\ &= \frac{a}{3b} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \sqrt{16x^4} = \sqrt{16} \times \sqrt{x^4} = 4 \times \sqrt{(x^2)^2} = 4x^2 \\ \mathbf{d} \quad & (2a^2)^3 \times 3a^4 = 2^3 \times (a^2)^3 \times 3a^4 = 8 \times a^6 \times 3a^4 = 24a^{10} \end{aligned}$$

EXERCISE E

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & 2x + 5 = 25 \\ \therefore & 2x = 20 \\ \therefore & x = 10 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 3x - 7 > 11 \\ \therefore & 3x > 18 \\ \therefore & x > 6 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 5x + 16 = 20 \\ \therefore & 5x = 4 \\ \therefore & x = \frac{4}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \frac{x}{3} - 7 = 10 \\ \therefore & \frac{x}{3} = 17 \\ \therefore & x = 51 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 6x + 11 < 4x - 9 \\ \therefore & 2x < -20 \\ \therefore & x < -10 \\ \mathbf{f} \quad & \frac{3x - 2}{5} = 8 \\ \therefore & 3x - 2 = 40 \\ \therefore & 3x = 42 \\ \therefore & x = 14 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & 1 - 2x \geq 19 \\ \therefore & -2x \geq 18 \\ \therefore & 2x \leq -18 \\ \therefore & x \leq -9 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \frac{1}{2}x + 1 = \frac{2}{3}x - 2 \\ \therefore & \frac{3}{6}x - \frac{4}{6}x = -3 \\ \therefore & -\frac{1}{6}x = -3 \\ \therefore & x = 18 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \frac{2}{3} - \frac{3x}{4} = \frac{1}{2}(2x - 1) \quad \text{Multiplying each term by the LCD of 12 gives} \\ \therefore & 8 - 9x = 6(2x - 1) \\ \therefore & 8 - 9x = 12x - 6 \\ \therefore & 14 = 21x \quad \text{i.e., } x = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & x + 2y = 9 \quad \dots (1) \\ & x - y = 3 \quad \dots (2) \end{aligned}$$

Multiplying (2) by 2 gives

$$\begin{aligned} & x + 2y = 9 \\ & 2x - 2y = 6 \\ \hline \therefore & 3x = 15 \quad \{\text{adding}\} \\ \therefore & x = 5 \end{aligned}$$

Substituting $x = 5$ into (2) gives

$$\begin{aligned} 5 - y &= 3 \\ \therefore y &= 2 \end{aligned}$$

$$\therefore x = 5 \text{ and } y = 2$$

$$\begin{aligned} \mathbf{c} \quad & 7x + 2y = -4 \quad \dots (1) \\ & 3x + 4y = 14 \quad \dots (2) \end{aligned}$$

Multiplying (1) by -2 gives

$$\begin{aligned} & -14x - 4y = 8 \\ & 3x + 4y = 14 \\ \hline \therefore & -11x = 22 \quad \{\text{adding}\} \\ \therefore & x = -2 \end{aligned}$$

Substituting $x = -2$ into (2) gives

$$\begin{aligned} 3(-2) + 4y &= 14 \\ \therefore -6 + 4y &= 14 \\ \therefore 4y &= 20 \quad \text{and } \therefore y = 5 \end{aligned}$$

$$\therefore x = -2 \text{ and } y = 5$$

$$\begin{aligned} \mathbf{b} \quad & 2x + 5y = 28 \quad \dots (1) \\ & x - 2y = 2 \quad \dots (2) \end{aligned}$$

Multiplying (2) by -2 gives

$$\begin{aligned} & 2x + 5y = 28 \\ & -2x + 4y = -4 \\ \hline \therefore & 9y = 24 \quad \{\text{adding}\} \\ \therefore & y = \frac{24}{9} = \frac{8}{3} \end{aligned}$$

Substituting $y = \frac{8}{3}$ into (2) gives

$$x - 2\left(\frac{8}{3}\right) = 2 \quad \therefore x - \frac{16}{3} = 2 \quad \text{and so } x = \frac{22}{3}$$

$$\therefore x = \frac{22}{3} \text{ and } y = \frac{8}{3}$$

$$\begin{aligned} \mathbf{d} \quad & 5x - 4y = 27 \quad \dots (1) \\ & 3x + 2y = 9 \quad \dots (2) \end{aligned}$$

Multiplying (2) by 2 gives

$$\begin{aligned} & 5x - 4y = 27 \\ & 6x + 4y = 18 \\ \hline \therefore & 11x = 45 \quad \{\text{adding}\} \\ \therefore & x = \frac{45}{11} \end{aligned}$$

Substituting $x = \frac{45}{11}$ into (1) gives

$$\begin{aligned} 5\left(\frac{45}{11}\right) - 4y &= 27 \quad \therefore \frac{225}{11} - 27 = 4y \\ \therefore 4y &= -\frac{72}{11} \quad \text{and } \therefore y = -\frac{18}{11} \end{aligned}$$

$$\therefore x = \frac{45}{11} \text{ and } y = -\frac{18}{11}$$

$$\mathbf{e} \quad x + 2y = 5 \quad \dots (1)$$

$$2x + 4y = 1 \quad \dots (2)$$

Multiplying (1) by -2 gives

$$-2x - 4y = -10$$

$$2x + 4y = 1$$

$$\hline \therefore 0 = -9 \quad \{\text{adding}\}$$

which is absurd

$\therefore$ there are no solutions

$$\mathbf{f} \quad \frac{x}{2} + \frac{y}{3} = 5 \quad \dots (1)$$

$$\frac{x}{3} + \frac{y}{4} = 1 \quad \dots (2)$$

Multiplying (1) by 18 and (2) by -24 gives

$$9x + 6y = 90 \quad \dots (3)$$

$$-8x - 6y = -24$$

$$\hline \therefore x = 66 \quad \{\text{adding}\}$$

Substituting $x = 66$ into (3) gives

$$9 \times 66 + 6y = 90$$

$$\therefore 6y = 90 - 594 = -504$$

$$\therefore y = -84$$

$$\therefore x = 66 \text{ and } y = -84$$

EXERCISE F

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & 5 - (-11) \\ &= 5 + 11 \\ &= 16 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & |5| - |-11| \\ &= 5 - 11 \\ &= -6 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & |5 - (-11)| \\ &= |5 + 11| \\ &= |16| \\ &= 16 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & |(-2)^2 + 11(-2)| \\ &= |4 - 22| \\ &= |-18| \\ &= 18 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & |-6| - |-8| \\ &= 6 - 8 \\ &= -2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & |-6 - (-8)| \\ &= |-6 + 8| \\ &= |2| \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & |a| = |-2| \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & |b| = |3| \\ &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & |a| |b| = |-2| |3| \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & |ab| = |-2 \times 3| \\ &= |-6| \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & |a - b| = |-2 - 3| \\ &= |-5| \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & |a| - |b| = |-2| - |3| \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & |a + b| = |-2 + 3| \\ &= |1| \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & |a| + |b| = |-2| + |3| \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & |a|^2 = |-2|^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\mathbf{j} \quad a^2 = (-2)^2 = 4$$

$$\mathbf{k} \quad \left| \frac{c}{a} \right| = \left| \frac{-4}{-2} \right| = |2| = 2$$

$$\mathbf{l} \quad \frac{|c|}{|a|} = \frac{|-4|}{|-2|} = \frac{4}{2} = 2$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & |x| = 3 \\ \therefore x &= \pm 3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & |x| = -5 \\ \text{but } |x| &\geq 0 \text{ for all } x \\ &(\text{property of modulus}) \\ \therefore &\text{no solution} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & |x| = 0 \\ \therefore x &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & |x - 1| = 3 \\ \therefore x - 1 &= \pm 3 \\ \therefore x &= 1 \pm 3 \\ \therefore x &= -2 \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & |3 - x| = 4 \\ \therefore 3 - x &= \pm 4 \\ \therefore -x &= -3 \pm 4 \\ \therefore x &= 3 \mp 4 \\ \therefore x &= -1 \text{ or } 7 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & |x + 5| = -1 \\ \text{but } |x + 5| &\geq 0 \text{ for all } x \\ &(\text{property of modulus}) \\ \therefore &\text{no solution} \end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad |3x - 2| &= 1 \\ \therefore 3x - 2 &= \pm 1 \\ \therefore 3x &= 2 \pm 1 \\ \therefore 3x &= 3 \text{ or } 1 \\ \therefore x &= \frac{1}{3} \text{ or } 1\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad |3 - 2x| &= 3 \\ \therefore 3 - 2x &= \pm 3 \\ \therefore 2x &= 3 \mp 3 \\ \therefore 2x &= 0 \text{ or } 6 \\ \therefore x &= 0 \text{ or } 3\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad |2 - 5x| &= 12 \\ \therefore 2 - 5x &= \pm 12 \\ \therefore 5x &= 2 \mp 12 \\ \therefore 5x &= -10 \text{ or } 14 \\ \therefore x &= -2 \text{ or } \frac{14}{5}\end{aligned}$$

EXERCISE G

$$\begin{aligned}\mathbf{1} \quad \mathbf{a} \quad (2x + 3)(x + 1) &= 2x^2 + 2x + 3x + 3 \\ &= 2x^2 + 5x + 3 \\ \mathbf{d} \quad (x + 2)(3x - 5) &= 3x^2 - 5x + 6x - 10 \\ &= 3x^2 + x - 10 \\ \mathbf{g} \quad (3x + 4)(5x - 3) &= 15x^2 - 9x + 20x - 12 \\ &= 15x^2 + 11x - 12 \\ \mathbf{j} \quad (5 - 2x)(3 - 2x) &= 15 - 10x - 6x + 4x^2 \\ &= 4x^2 - 16x + 15\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (3x + 4)(x + 2) &= 3x^2 + 6x + 4x + 8 \\ &= 3x^2 + 10x + 8 \\ \mathbf{e} \quad (7 - 2x)(2 + 3x) &= 14 + 21x - 4x - 6x^2 \\ &= -6x^2 + 17x + 14 \\ \mathbf{h} \quad (1 - 3x)(2 - 5x) &= 2 - 5x - 6x + 15x^2 \\ &= 15x^2 - 11x + 2 \\ \mathbf{k} \quad -(x + 1)(x + 2) &= -(x^2 + 2x + x + 2) \\ &= -(x^2 + 3x + 2) \\ &= -x^2 - 3x - 2\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (5x - 2)(2x + 1) &= 10x^2 + 5x - 4x - 2 \\ &= 10x^2 + x - 2 \\ \mathbf{f} \quad (1 - 3x)(5 + 2x) &= 5 + 2x - 15x - 6x^2 \\ &= -6x^2 - 13x + 5 \\ \mathbf{i} \quad (7 - x)(3 - 2x) &= 21 - 14x - 3x + 2x^2 \\ &= 2x^2 - 17x + 21 \\ \mathbf{l} \quad -2(x - 1)(2x + 3) &= -2(2x^2 + 3x - 2x - 3) \\ &= -2(2x^2 + x - 3) \\ &= -4x^2 - 2x + 6\end{aligned}$$

$$\begin{aligned}\mathbf{2} \quad \mathbf{a} \quad (x + 6)(x - 6) &= x^2 - 6^2 \\ &= x^2 - 36 \\ \mathbf{d} \quad (3x - 2)(3x + 2) &= (3x)^2 - 2^2 \\ &= 9x^2 - 4 \\ \mathbf{g} \quad (3 - x)(3 + x) &= 3^2 - x^2 \\ &= 9 - x^2 \\ \mathbf{j} \quad (x + \sqrt{2})(x - \sqrt{2}) &= x^2 - (\sqrt{2})^2 \\ &= x^2 - 2\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (x + 8)(x - 8) &= x^2 - 8^2 \\ &= x^2 - 64 \\ \mathbf{e} \quad (4x + 5)(4x - 5) &= (4x)^2 - 5^2 \\ &= 16x^2 - 25 \\ \mathbf{h} \quad (7 - x)(7 + x) &= 7^2 - x^2 \\ &= 49 - x^2 \\ \mathbf{k} \quad (x + \sqrt{5})(x - \sqrt{5}) &= x^2 - (\sqrt{5})^2 \\ &= x^2 - 5\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (2x - 1)(2x + 1) &= (2x)^2 - 1^2 \\ &= 4x^2 - 1 \\ \mathbf{f} \quad (5x - 3)(5x + 3) &= (5x)^2 - 3^2 \\ &= 25x^2 - 9 \\ \mathbf{i} \quad (7 + 2x)(7 - 2x) &= 7^2 - (2x)^2 \\ &= 49 - 4x^2 \\ \mathbf{l} \quad (2x - \sqrt{3})(2x + \sqrt{3}) &= (2x)^2 - (\sqrt{3})^2 \\ &= 4x^2 - 3\end{aligned}$$

$$\begin{aligned}\mathbf{3} \quad \mathbf{a} \quad (x + 5)^2 &= x^2 + 2(x)(5) + 5^2 \\ &= x^2 + 10x + 25 \\ \mathbf{d} \quad (x - 6)^2 &= x^2 - 2(x)(6) + 6^2 \\ &= x^2 - 12x + 36 \\ \mathbf{g} \quad (11 - x)^2 &= 11^2 - 2(11)(x) + x^2 \\ &= x^2 - 22x + 121 \\ \mathbf{j} \quad (3x + 2)^2 &= (3x)^2 + 2(3x)(2) + 2^2 \\ &= 9x^2 + 12x + 4\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (x + 7)^2 &= x^2 + 2(x)(7) + 7^2 \\ &= x^2 + 14x + 49 \\ \mathbf{e} \quad (3 + x)^2 &= 3^2 + 2(3)(x) + x^2 \\ &= x^2 + 6x + 9 \\ \mathbf{h} \quad (10 - x)^2 &= 10^2 - 2(10)(x) + x^2 \\ &= x^2 - 20x + 100 \\ \mathbf{k} \quad (5 - 2x)^2 &= 5^2 - 2(5)(2x) + (2x)^2 \\ &= 4x^2 - 20x + 25\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (x - 2)^2 &= x^2 - 2(x)(2) + 2^2 \\ &= x^2 - 4x + 4 \\ \mathbf{f} \quad (5 + x)^2 &= 5^2 + 2(5)(x) + x^2 \\ &= x^2 + 10x + 25 \\ \mathbf{i} \quad (2x + 7)^2 &= (2x)^2 + 2(2x)(7) + 7^2 \\ &= 4x^2 + 28x + 49 \\ \mathbf{l} \quad (7 - 3x)^2 &= 7^2 - 2(7)(3x) + (3x)^2 \\ &= 9x^2 - 42x + 49\end{aligned}$$

$$\begin{aligned}4 \quad \mathbf{a} \quad y &= 2(x+2)(x+3) \\&= 2(x^2 + 3x + 2x + 6) \\&= 2(x^2 + 5x + 6) \\&= 2x^2 + 10x + 12\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad y &= -(x+1)(x-7) \\&= -(x^2 - 7x + x - 7) \\&= -(x^2 - 6x - 7) \\&= -x^2 + 6x + 7\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad y &= 4(x-1)(x-5) \\&= 4(x^2 - 5x - x + 5) \\&= 4(x^2 - 6x + 5) \\&= 4x^2 - 24x + 20\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad y &= -5(x-1)(x-6) \\&= -5(x^2 - 6x - x + 6) \\&= -5(x^2 - 7x + 6) \\&= -5x^2 + 35x - 30\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad y &= -\frac{5}{2}(x-4)^2 \\&= -\frac{5}{2}(x^2 - 2(x)(4) + 4^2) \\&= -\frac{5}{2}(x^2 - 8x + 16) \\&= -\frac{5}{2}x^2 + 20x - 40\end{aligned}$$

$$\begin{aligned}5 \quad \mathbf{a} \quad 1 + 2(x+3)^2 \\&= 1 + 2(x^2 + 2(x)(3) + 3^2) \\&= 1 + 2(x^2 + 6x + 9) \\&= 1 + 2x^2 + 12x + 18 \\&= 2x^2 + 12x + 19\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad 3 - (3-x)^2 \\&= 3 - (9 - 2(3)(x) + x^2) \\&= 3 - (x^2 - 6x + 9) \\&= 3 - x^2 + 6x - 9 \\&= -x^2 + 6x - 6\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad 1 + 2(4-x)^2 \\&= 1 + 2(4^2 - 2(4)(x) + x^2) \\&= 1 + 2(x^2 - 8x + 16) \\&= 1 + 2x^2 - 16x + 32 \\&= 2x^2 - 16x + 33\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad (x+2)^2 - (x+1)(x-4) \\&= x^2 + 2(x)(2) + 2^2 - (x^2 - 4x + x - 4) \\&= x^2 + 4x + 4 - (x^2 - 3x - 4) \\&= x^2 + 4x + 4 - x^2 + 3x + 4 \\&= 7x + 8\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad y &= 3(x-1)^2 + 4 \\&= 3(x^2 - 2(x)(1) + 1^2) + 4 \\&= 3(x^2 - 2x + 1) + 4 \\&= 3x^2 - 6x + 3 + 4 \\&= 3x^2 - 6x + 7\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad y &= -(x+2)^2 - 11 \\&= -(x^2 + 2(x)(2) + 2^2) - 11 \\&= -(x^2 + 4x + 4) - 11 \\&= -x^2 - 4x - 4 - 11 \\&= -x^2 - 4x - 15\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad y &= -\frac{1}{2}(x+4)^2 - 6 \\&= -\frac{1}{2}(x^2 + 2(x)(4) + 4^2) - 6 \\&= -\frac{1}{2}(x^2 + 8x + 16) - 6 \\&= -\frac{1}{2}x^2 - 4x - 8 - 6 \\&= -\frac{1}{2}x^2 - 4x - 14\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad y &= \frac{1}{2}(x+2)^2 - 6 \\&= \frac{1}{2}(x^2 + 2(x)(2) + 2^2) - 6 \\&= \frac{1}{2}(x^2 + 4x + 4) - 6 \\&= \frac{1}{2}x^2 + 2x + 2 - 6 \\&= \frac{1}{2}x^2 + 2x - 4\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 2 + 3(x-2)(x+3) \\&= 2 + 3(x^2 + 3x - 2x - 6) \\&= 2 + 3(x^2 + x - 6) \\&= 2 + 3x^2 + 3x - 18 \\&= 3x^2 + 3x - 16\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad 5 - (x+5)(x-4) \\&= 5 - (x^2 - 4x + 5x - 20) \\&= 5 - (x^2 + x - 20) \\&= 5 - x^2 - x + 20 \\&= -x^2 - x + 25\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad x^2 - 3x - (x+2)(x-2) \\&= x^2 - 3x - (x^2 - 2^2) \\&= x^2 - 3x - x^2 + 4 \\&= -3x + 4\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad (2x+3)^2 + 3(x+1)^2 \\&= (2x)^2 + 2(2x)(3) + 3^2 + 3(x^2 + 2(x)(1) + 1^2) \\&= 4x^2 + 12x + 9 + 3(x^2 + 2x + 1) \\&= 4x^2 + 12x + 9 + 3x^2 + 6x + 3 \\&= 7x^2 + 18x + 12\end{aligned}$$

$$\begin{array}{ll}
 \text{i} & x^2 + 3x - 2(x - 4)^2 \\
 & = x^2 + 3x - 2(x^2 - 2(x)(4) + 4^2) \\
 & = x^2 + 3x - 2(x^2 - 8x + 16) \\
 & = x^2 + 3x - 2x^2 + 16x - 32 \\
 & = -x^2 + 19x - 32 \\
 \text{j} & (3x - 2)^2 - 2(x + 1)^2 \\
 & = (3x)^2 - 2(3x)(2) + 2^2 - 2(x^2 + 2(x)(1) + 1^2) \\
 & = 9x^2 - 12x + 4 - 2(x^2 + 2x + 1) \\
 & = 9x^2 - 12x + 4 - 2x^2 - 4x - 2 \\
 & = 7x^2 - 16x + 2
 \end{array}$$

EXERCISE H

$$\begin{array}{lll}
 \text{1 a} & 3x^2 + 9x & \text{b} \quad 2x^2 + 7x \\
 & = 3x(x + 3) & = x(2x + 7) \\
 \text{d} & 6x^2 - 15x & \text{e} \quad 9x^2 - 25 \\
 & = 3x(2x - 5) & = (3x)^2 - 5^2 \\
 & & = (3x + 5)(3x - 5) \\
 \text{g} & 2x^2 - 8 & \text{h} \quad 3x^2 - 9 \\
 & = 2(x^2 - 4) & = 3(x^2 - 3) \\
 & = 2(x^2 - 2^2) & = 3(x^2 - (\sqrt{3})^2) \\
 & = 2(x + 2)(x - 2) & = 3(x + \sqrt{3})(x - \sqrt{3}) \\
 \text{j} & x^2 - 8x + 16 & \text{k} \quad x^2 - 10x + 25 \\
 & = x^2 - 2(x)(4) + 4^2 & = x^2 - 2(x)(5) + 5^2 \\
 & = (x - 4)^2 & = (x - 5)^2 \\
 \text{m} & 16x^2 + 40x + 25 & \text{n} \quad 9x^2 + 12x + 4 \\
 & = (4x)^2 + 2(4x)(5) + 5^2 & = (3x)^2 + 2(3x)(2) + 2^2 \\
 & = (4x + 5)^2 & = (3x + 2)^2 \\
 \text{o} & x^2 - 22x + 121 & \\
 & = x^2 - 2(x)(11) + 11^2 & = (x - 11)^2 \\
 \text{2 a} & x^2 + 9x + 8 & \text{b} \quad x^2 + 7x + 12 \\
 & = (x + 1)(x + 8) & = (x + 3)(x + 4) \\
 & \{\text{as sum} = 9, \text{ product} = 8\} & \{\text{as sum} = 7, \text{ product} = 12\} \\
 \text{c} & x^2 - 7x - 18 & \text{d} \quad x^2 + 4x - 21 \\
 & = (x - 9)(x + 2) & = (x + 7)(x - 3) \\
 & \{\text{as sum} = -7, \text{ product} = -18\} & \{\text{as sum} = 4, \text{ product} = -21\} \\
 \text{e} & x^2 - 9x + 18 & \text{f} \quad x^2 + x - 6 \\
 & = (x - 6)(x - 3) & = (x + 3)(x - 2) \\
 & \{\text{as sum} = -9, \text{ product} = 18\} & \{\text{as sum} = 1, \text{ product} = -6\} \\
 \text{g} & -x^2 + x + 2 & \text{h} \quad 3x^2 - 42x + 99 \\
 & = -(x^2 - x - 2) & = 3(x^2 - 14x + 33) \\
 & = -(x - 2)(x + 1) & = 3(x - 3)(x - 11) \\
 & \{\text{as sum} = -1, \text{ product} = -2\} & \{\text{as sum} = -14, \text{ product} = 33\} \\
 \text{i} & -2x^2 - 4x - 2 & \text{j} \quad 2x^2 + 6x - 20 \\
 & = -2(x^2 + 2x + 1) & = 2(x^2 + 3x - 10) \\
 & = -2(x^2 + 2(x)(1) + 1^2) & = 2(x + 5)(x - 2) \\
 & = -2(x + 1)^2 & \{\text{as sum} = 3, \text{ product} = -10\}
 \end{array}$$

k $2x^2 - 10x - 48$

$= 2(x^2 - 5x - 24)$

$= 2(x - 8)(x + 3)$

{as sum = -5, product = -24}

l $-2x^2 + 14x - 12$

$= -2(x^2 - 7x + 6)$

$= -2(x - 6)(x - 1)$

{as sum = -7, product = 6}

m $-3x^2 + 6x - 3$

$= -3(x^2 - 2x + 1)$

$= -3(x^2 - 2(x)(1) + 1^2)$

$= -3(x - 1)^2$

n $-x^2 - 2x - 1$

$= -(x^2 + 2x + 1)$

$= -(x^2 + 2(x)(1) + 1^2)$

$= -(x + 1)^2$

o $-5x^2 + 10x + 40$

$= -5(x^2 - 2x - 8)$

$= -5(x - 4)(x + 2)$

{as sum = -2, prod. = -8}

3 a $2x^2 + 5x - 12$

$= 2x^2 + 8x - 3x - 12$

$= 2x(x + 4) - 3(x + 4)$

$= (2x - 3)(x + 4)$

has $ac = 2 \times -12 = -24$

Factors of -24 which add to 5 are 8 and -3.

b $3x^2 - 5x - 2$

$= 3x^2 - 6x + x - 2$

$= 3x(x - 2) + (x - 2)$

$= (3x + 1)(x - 2)$

has $ac = 3 \times -2 = -6$

Factors of -6 which add to -5 are -6 and 1.

c $7x^2 - 9x + 2$

$= 7x^2 - 7x - 2x + 2$

$= 7x(x - 1) - 2(x - 1)$

$= (7x - 2)(x - 1)$

has $ac = 7 \times 2 = 14$

Factors of 14 which add to -9 are -7 and -2.

d $6x^2 - x - 2$

$= 6x^2 + 3x - 4x - 2$

$= 3x(2x + 1) - 2(2x + 1)$

$= (3x - 2)(2x + 1)$

has $ac = 6 \times -2 = -12$

Factors of -12 which add to -1 are 3 and -4.

e $4x^2 - 4x - 3$

$= 4x^2 + 2x - 6x - 3$

$= 2x(2x + 1) - 3(2x + 1)$

$= (2x - 3)(2x + 1)$

has $ac = 4 \times -3 = -12$

Factors of -12 which add to -4 are -6 and 2.

f $10x^2 - x - 3$

$= 10x^2 + 5x - 6x - 3$

$= 5x(2x + 1) - 3(2x + 1)$

$= (5x - 3)(2x + 1)$

has $ac = 10 \times -3 = -30$

Factors of -30 which add to -1 are -6 and 5.

g $2x^2 - 11x - 6$

$= 2x^2 - 12x + x - 6$

$= 2x(x - 6) + (x - 6)$

$= (2x + 1)(x - 6)$

has $ac = 2 \times -6 = -12$

Factors of -12 which add to -11 are -12 and 1.

h $3x^2 - 5x - 28$

$= 3x^2 - 12x + 7x - 28$

$= 3x(x - 4) + 7(x - 4)$

$= (3x + 7)(x - 4)$

has $ac = 3 \times -28 = -84$

Factors of -84 which add to -5 are -12 and 7.

$$\begin{aligned}
 \mathbf{i} \quad & 8x^2 + 2x - 3 \\
 &= 8x^2 - 4x + 6x - 3 \\
 &= 4x(2x - 1) + 3(2x - 1) \\
 &= (4x + 3)(2x - 1)
 \end{aligned}$$

has $ac = 8 \times -3 = -24$
 Factors of -24 which add to 2 are 6 and -4 .

$$\begin{aligned}
 \mathbf{j} \quad & 10x^2 - 9x - 9 \\
 &= 10x^2 - 15x + 6x - 9 \\
 &= 5x(2x - 3) + 3(2x - 3) \\
 &= (5x + 3)(2x - 3)
 \end{aligned}$$

has $ac = 10 \times -9 = -90$
 Factors of -90 which add to -9 are -15 and 6 .

$$\begin{aligned}
 \mathbf{k} \quad & 3x^2 + 23x - 8 \\
 &= 3x^2 - x + 24x - 8 \\
 &= x(3x - 1) + 8(3x - 1) \\
 &= (x + 8)(3x - 1)
 \end{aligned}$$

has $ac = 3 \times -8 = -24$
 Factors of -24 which add to 23 are 24 and -1 .

$$\begin{aligned}
 \mathbf{l} \quad & 6x^2 + 7x + 2 \\
 &= 6x^2 + 3x + 4x + 2 \\
 &= 3x(2x + 1) + 2(2x + 1) \\
 &= (3x + 2)(2x + 1)
 \end{aligned}$$

has $ac = 6 \times 2 = 12$
 Factors of 12 which add to 7 are 4 and 3 .

$$\begin{aligned}
 \mathbf{m} \quad & -4x^2 - 2x + 6 \\
 &= -2(2x^2 + x - 3) \\
 &= -2(2x^2 - 2x + 3x - 3) \\
 &= -2[2x(x - 1) + 3(x - 1)] \\
 &= -2(2x + 3)(x - 1)
 \end{aligned}$$

has $ac = 2 \times -3 = -6$
 Factors of -6 which add to 1 are 3 and -2 .

$$\begin{aligned}
 \mathbf{n} \quad & 12x^2 - 16x - 3 \\
 &= 12x^2 - 18x + 2x - 3 \\
 &= 6x(2x - 3) + (2x - 3) \\
 &= (6x + 1)(2x - 3)
 \end{aligned}$$

has $ac = 12 \times -3 = -36$
 Factors of -36 which add to -16 are -18 and 2 .

$$\begin{aligned}
 \mathbf{o} \quad & -6x^2 - 9x + 42 \\
 &= -3(2x^2 + 3x - 14) \\
 &= -3(2x^2 - 4x + 7x - 14) \\
 &= -3[2x(x - 2) + 7(x - 2)] \\
 &= -3(2x + 7)(x - 2)
 \end{aligned}$$

has $ac = 2 \times -14 = -28$
 Factors of -28 which add to 3 are 7 and -4 .

$$\begin{aligned}
 \mathbf{p} \quad & 21x - 10 - 9x^2 \\
 &= -(9x^2 - 21x + 10) \\
 &= -(9x^2 - 6x - 15x + 10) \\
 &= -[3x(3x - 2) - 5(3x - 2)] \\
 &= -(3x - 5)(3x - 2)
 \end{aligned}$$

has $ac = 9 \times 10 = 90$
 Factors of 90 which add to -21 are -6 and -15 .

$$\begin{aligned}
 \mathbf{q} \quad & 8x^2 - 6x - 27 \\
 &= 8x^2 + 12x - 18x - 27 \\
 &= 4x(2x + 3) - 9(2x + 3) \\
 &= (4x - 9)(2x + 3)
 \end{aligned}$$

has $ac = 8 \times -27 = -216$
 Factors of -216 which add to -6 are -18 and 12 .

$$\begin{aligned}
 \mathbf{r} \quad & 12x^2 + 13x + 3 \\
 &= 12x^2 + 4x + 9x + 3 \\
 &= 4x(3x + 1) + 3(3x + 1) \\
 &= (4x + 3)(3x + 1)
 \end{aligned}$$

has $ac = 12 \times 3 = 36$
 Factors of 36 which add to 13 are 9 and 4 .

$$\begin{aligned}
 \mathbf{s} \quad & 12x^2 + 20x + 3 \\
 & = 12x^2 + 2x + 18x + 3 \\
 & = 2x(6x + 1) + 3(6x + 1) \\
 & = (2x + 3)(6x + 1)
 \end{aligned}$$

has $ac = 12 \times 3 = 36$
 Factors of 36 which add to 20 are 2 and 18.

$$\begin{aligned}
 \mathbf{t} \quad & 15x^2 - 22x + 8 \\
 & = 15x^2 - 10x - 12x + 8 \\
 & = 5x(3x - 2) - 4(3x - 2) \\
 & = (5x - 4)(3x - 2)
 \end{aligned}$$

has $ac = 15 \times 8 = 120$
 Factors of 120 which add to -22 are -10 and -12 .

$$\begin{aligned}
 \mathbf{u} \quad & 14x^2 - 11x - 15 \\
 & = 14x^2 - 21x + 10x - 15 \\
 & = 7x(2x - 3) + 5(2x - 3) \\
 & = (7x + 5)(2x - 3)
 \end{aligned}$$

has $ac = 14 \times -15 = -210$
 Factors of -210 which add to -11 are -21 and 10 .

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & 3(x + 4) + 2(x + 4)(x - 1) \\
 & = (x + 4)[3 + 2(x - 1)] \\
 & = (x + 4)(3 + 2x - 2) \\
 & = (x + 4)(2x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 8(2 - x) - 3(x + 1)(2 - x) \\
 & = (2 - x)[8 - 3(x + 1)] \\
 & = (2 - x)(8 - 3x - 3) \\
 & = (2 - x)(5 - 3x)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 6(x + 2)^2 + 9(x + 2) \\
 & = (x + 2)[6(x + 2) + 9] \\
 & = (x + 2)(6x + 12 + 9) \\
 & = (x + 2)(6x + 21) \\
 & = (x + 2) \times 3(2x + 7) \\
 & = 3(x + 2)(2x + 7)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 4(x + 5) + 8(x + 5)^2 \\
 & = (x + 5)[4 + 8(x + 5)] \\
 & = (x + 5)(4 + 8x + 40) \\
 & = (x + 5)(8x + 44) \\
 & = (x + 5) \times 4(2x + 11) \\
 & = 4(x + 5)(2x + 11)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (x + 2)(x + 3) - (x + 3)(2 - x) \\
 & = (x + 3)[(x + 2) - (2 - x)] \\
 & = (x + 3)(x + 2 - 2 + x) \\
 & = (x + 3)(2x) \\
 & = 2x(x + 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & (x + 3)^2 + 2(x + 3) - x(x + 3) \\
 & = (x + 3)[(x + 3) + 2 - x] \\
 & = (x + 3)(x + 3 + 2 - x) \\
 & = (x + 3)(5) \\
 & = 5(x + 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 5(x - 2) - 3(2 - x)(x + 7) \\
 & = 5(x - 2) + 3(x - 2)(x + 7) \\
 & = (x - 2)[5 + 3(x + 7)] \\
 & = (x - 2)(5 + 3x + 21) \\
 & = (x - 2)(3x + 26)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 3(1 - x) + 2(x + 1)(x - 1) \\
 & = -3(x - 1) + 2(x + 1)(x - 1) \\
 & = (x - 1)[-3 + 2(x + 1)] \\
 & = (x - 1)(-3 + 2x + 2) \\
 & = (x - 1)(2x - 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & (x + 3)^2 - 16 \\
 & = (x + 3)^2 - 4^2 \\
 & = (x + 3 + 4)(x + 3 - 4) \\
 & = (x + 7)(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 4 - (1 - x)^2 \\
 & = 2^2 - (1 - x)^2 \\
 & = [2 + (1 - x)][2 - (1 - x)] \\
 & = (2 + 1 - x)(2 - 1 + x) \\
 & = (3 - x)(x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (x + 4)^2 - (x - 2)^2 \\
 & = [(x + 4) + (x - 2)][(x + 4) - (x - 2)] \\
 & = (x + 4 + x - 2)(x + 4 - x + 2) \\
 & = (2x + 2)(6) \\
 & = 2(x + 1)(6) \\
 & = 12(x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 16 - 4(x + 2)^2 \\
 & = 4[4 - (x + 2)^2] \\
 & = 4[2 + (x + 2)][2 - (x + 2)] \\
 & = 4(x + 4)(-x) \\
 & = -4x(x + 4)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (2x+3)^2 - (x-1)^2 \\
 &= [(2x+3) + (x-1)][(2x+3) - (x-1)] \\
 &= (2x+3+x-1)(2x+3-x+1) \\
 &= (3x+2)(x+4)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 3x^2 - 3(x+2)^2 \\
 &= 3[x^2 - (x+2)^2] \\
 &= 3[x + (x+2)][x - (x+2)] \\
 &= 3(x+x+2)(x-x-2) \\
 &= 3(2x+2)(-2) \\
 &= -6(2x+2) \\
 &= -12(x+1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & 12x^2 - 27(3+x)^2 \\
 &= 3[4x^2 - 9(3+x)^2] \\
 &= 3[(2x)^2 - 3^2(3+x)^2] \\
 &= 3[(2x)^2 - (3(3+x))^2] \\
 &= 3[2x + 3(3+x)][2x - 3(3+x)] \\
 &= 3[(2x+9+3x)(2x-9-3x)] \\
 &= 3(5x+9)(-x-9) \\
 &= -3(5x+9)(x+9)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & (x+h)^2 - x^2 \\
 &= [(x+h) + x][(x+h) - x] \\
 &= (x+h+x)(x+h-x) \\
 &= (2x+h)(h) \\
 &= h(2x+h)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 5x^2 - 20(2-x)^2 \\
 &= 5[x^2 - 4(2-x)^2] \\
 &= 5[x^2 - 2^2(2-x)^2] \\
 &= 5[x^2 - (2(2-x))^2] \\
 &= 5[x + 2(2-x)][x - 2(2-x)] \\
 &= 5(x+4-2x)(x-4+2x) \\
 &= 5(-x+4)(3x-4) \\
 &= -5(x-4)(3x-4)
 \end{aligned}$$

EXERCISE I

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & a+x=b \\
 \therefore \quad & x=b-a
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & ax=b \\
 \therefore \quad & x=\frac{b}{a}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 2x+a=d \\
 \therefore \quad & 2x=d-a \\
 \therefore \quad & x=\frac{d-a}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & c+x=t \\
 \therefore \quad & x=t-c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & 5x+2y=20 \\
 \therefore \quad & 5x=20-2y \\
 \therefore \quad & x=\frac{20-2y}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 2x+3y=12 \\
 \therefore \quad & 2x=12-3y \\
 \therefore \quad & x=\frac{12-3y}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 7x+3y=d \\
 \therefore \quad & 7x=d-3y \\
 \therefore \quad & x=\frac{d-3y}{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & ax+by=c \\
 \therefore \quad & ax=c-by \\
 \therefore \quad & x=\frac{c-by}{a}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & y=mx+c \\
 \therefore \quad & mx=y-c \\
 \therefore \quad & x=\frac{y-c}{m}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & az=\frac{b}{c} \\
 \therefore \quad & z=\frac{b}{ac}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{a}{z}=d \\
 \therefore \quad & \frac{z}{a}=\frac{1}{d} \\
 \therefore \quad & z=\frac{a}{d}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{3}{d}=\frac{2}{z} \\
 \therefore \quad & 3z=2d \\
 \therefore \quad & z=\frac{2d}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & F=ma \\
 \therefore \quad & a=\frac{F}{m}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & C=2\pi r \\
 \therefore \quad & r=\frac{C}{2\pi}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & V=ldh \\
 \therefore \quad & d=\frac{V}{lh}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & A=\frac{b}{K} \\
 \therefore \quad & KA=b \\
 \therefore \quad & K=\frac{b}{A}
 \end{aligned}$$

$$\begin{array}{llll}
 \mathbf{4} \quad \mathbf{a} & A = \pi r^2 & \mathbf{b} & N = \frac{x^5}{a} & \mathbf{c} & V = \frac{4}{3}\pi r^3 & \mathbf{d} & D = \frac{n}{x^3} \\
 \therefore \frac{A}{\pi} = r^2 & & \therefore aN = x^5 & & \therefore \frac{3}{4}V = \pi r^3 & & \therefore Dx^3 = n & \\
 \therefore r = \sqrt{\frac{A}{\pi}} & & \therefore x = \sqrt[5]{aN} & & \therefore \frac{3V}{4\pi} = r^3 & & \therefore x^3 = \frac{n}{D} & \\
 & & & & \therefore r = \sqrt[3]{\frac{3V}{4\pi}} & & \therefore x = \sqrt[3]{\frac{n}{D}} &
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{5} \quad \mathbf{a} & d = \frac{\sqrt{a}}{n} & \mathbf{b} & T = \frac{1}{5}\sqrt{l} & \mathbf{c} & c = \sqrt{a^2 - b^2} \\
 \therefore dn = \sqrt{a} & & \therefore 5T = \sqrt{l} & & \therefore c^2 = a^2 - b^2 & \\
 \therefore a = (dn)^2 & & \therefore l = (5T)^2 & & \therefore a^2 = c^2 + b^2 & \\
 \therefore a = d^2 n^2 & & \therefore l = 25T^2 & & \therefore a = \pm\sqrt{b^2 + c^2} & \\
 \mathbf{d} & T = 2\pi\sqrt{\frac{l}{g}} & \mathbf{e} & P = 2(a+b) & \mathbf{f} & A = \pi r^2 + 2\pi rh \\
 \therefore \frac{T}{2\pi} = \sqrt{\frac{l}{g}} & & \therefore \frac{P}{2} = a+b & & \therefore A - \pi r^2 = 2\pi rh & \\
 \therefore \left(\frac{T}{2\pi}\right)^2 = \frac{l}{g} & & \therefore a = \frac{P}{2} - b & & \therefore h = \frac{A - \pi r^2}{2\pi r} & \\
 \therefore \frac{T^2}{4\pi^2} = \frac{l}{g} & & \mathbf{g} & I = \frac{E}{R+r} & \mathbf{h} & A = \frac{B}{p-q} \\
 \therefore l = \frac{gT^2}{4\pi^2} & & \therefore E = I(R+r) & & \therefore A(p-q) = B & \\
 & & \therefore \frac{E}{I} = R+r & & \therefore p-q = \frac{B}{A} & \\
 & & \therefore r = \frac{E}{I} - R & & \therefore q = p - \frac{B}{A} &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{6} \quad \mathbf{a} & k = \frac{d^2}{2ab} \\
 \therefore d^2 = k(2ab) & \\
 \therefore \frac{d^2}{k} = 2ab & \\
 \therefore a = \frac{d^2}{2kb} & \\
 \mathbf{b} & \text{When } k = 112, d = 24, b = 2, \\
 & a = \frac{24^2}{2 \times 112 \times 2} \\
 & = \frac{576}{448} \\
 & \div 1.29
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{7} \quad \mathbf{a} & V = \frac{4}{3}\pi r^3 \\
 \therefore \frac{3}{4}V = \pi r^3 & \\
 \therefore r^3 = \frac{3V}{4\pi} & \\
 \therefore r = \sqrt[3]{\frac{3V}{4\pi}} & \\
 \mathbf{b} & \text{When } V = 40, \\
 & r = \sqrt[3]{\frac{3 \times 40}{4\pi}} \\
 & \div 2.122 \text{ cm}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{8} \quad \mathbf{a} & S = \frac{1}{2}at^2 \\
 \therefore t^2 = \frac{2S}{a} & \\
 \therefore t = \sqrt{\frac{2S}{a}} & \\
 \mathbf{b} & 10 \text{ m} \equiv 10 \times 100 \text{ cm} = 1000 \text{ cm} \\
 & \text{i.e., we need to find } t \text{ when } a = 8, \\
 & S = 1000 \\
 & t = \sqrt{\frac{2 \times 1000}{8}} \\
 & \therefore t \div 15.81 \text{ seconds}
 \end{array}$$

$$9 \quad \mathbf{a} \quad \frac{1}{f} = \frac{1}{u} + \frac{1}{v} \quad \therefore \quad \frac{1}{f} - \frac{1}{u} = \frac{1}{v}$$

$$\therefore \quad \frac{u-f}{uf} = \frac{1}{v} \quad \text{and so} \quad \therefore \quad v = \frac{uf}{u-f}$$

$$\mathbf{b} \quad \mathbf{i} \quad \text{When } u = 50, \quad f = 8$$

$$\text{so } v = \frac{50 \times 8}{50 - 8}$$

$$\therefore v \doteq 9.52 \text{ cm}$$

$$\mathbf{ii} \quad \text{When } u = 30, \quad f = 8$$

$$\text{so } v = \frac{30 \times 8}{30 - 8}$$

$$\therefore v \doteq 10.9 \text{ cm}$$

10 **a**

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

$$\therefore m \sqrt{1 - \left(\frac{v}{c}\right)^2} = m_0$$

$$\therefore \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{m_0}{m}$$

$$\therefore 1 - \left(\frac{v}{c}\right)^2 = \left(\frac{m_0}{m}\right)^2$$

$$\therefore \left(\frac{v}{c}\right)^2 = 1 - \frac{m_0^2}{m^2}$$

$$\therefore \frac{v}{c} = \sqrt{\frac{m^2 - m_0^2}{m^2}}$$

$$\therefore v = \frac{c}{m} \sqrt{m^2 - m_0^2}$$

$$\mathbf{b} \quad \text{When } m = 3m_0,$$

$$v = \frac{c}{3m_0} \sqrt{(3m_0)^2 - m_0^2}$$

$$= \frac{c}{3m_0} \sqrt{9m_0^2 - m_0^2}$$

$$= \frac{c}{3m_0} \sqrt{8m_0^2}$$

$$= \frac{c}{3m_0} \sqrt{8} m_0$$

$$= \frac{\sqrt{8}}{3} c$$

$$\mathbf{c} \quad \text{When } m = 30m_0, \quad c = 3 \times 10^8,$$

$$v = \frac{3 \times 10^8}{30m_0} \sqrt{(30m_0)^2 - m_0^2}$$

$$= \frac{30 \times 10^7}{30m_0} \sqrt{900m_0^2 - m_0^2}$$

$$= \frac{10^7}{m_0} \sqrt{899m_0^2}$$

$$= \frac{10^7}{m_0} \times \sqrt{899} \times m_0$$

$$= 10^7 \times \sqrt{899}$$

$$\doteq 2.998 \times 10^8 \text{ m/s}$$

EXERCISE J

$$1 \quad \mathbf{a} \quad 3 + \frac{x}{5}$$

$$= 3 \times \frac{5}{5} + \frac{x}{5}$$

$$= \frac{15}{5} + \frac{x}{5}$$

$$= \frac{x+15}{5}$$

$$\mathbf{b} \quad 1 + \frac{3}{x}$$

$$= 1 \times \frac{x}{x} + \frac{3}{x}$$

$$= \frac{x}{x} + \frac{3}{x}$$

$$= \frac{x+3}{x}$$

$$\mathbf{c} \quad 3 + \frac{x-2}{2}$$

$$= 3 \times \frac{2}{2} + \frac{x-2}{2}$$

$$= \frac{6}{2} + \frac{x-2}{2}$$

$$= \frac{6+x-2}{2}$$

$$= \frac{x+4}{2}$$

$$\mathbf{d} \quad 3 - \frac{x-2}{4}$$

$$= 3 \times \frac{4}{4} - \frac{x-2}{4}$$

$$= \frac{12}{4} - \frac{x-2}{4}$$

$$= \frac{12-x+2}{4}$$

$$= \frac{14-x}{4}$$

$$\mathbf{e} \quad \frac{2+x}{3} + \frac{x-4}{5}$$

$$= \frac{2+x}{3} \times \frac{5}{5} + \frac{x-4}{5} \times \frac{3}{3}$$

$$= \frac{5(2+x)}{15} + \frac{3(x-4)}{15}$$

$$= \frac{10+5x+3x-12}{15}$$

$$= \frac{8x-2}{15}$$

$$\mathbf{f} \quad \frac{2x+5}{4} - \frac{x-1}{6}$$

$$= \frac{2x+5}{4} \times \frac{3}{3} - \frac{x-1}{6} \times \frac{2}{2}$$

$$= \frac{3(2x+5)}{12} - \frac{2(x-1)}{12}$$

$$= \frac{6x+15-2x+2}{12}$$

$$= \frac{4x+17}{12}$$

$$\mathbf{2} \quad \mathbf{a} \quad 1 + \frac{3}{x+2}$$

$$= 1 \times \frac{x+2}{x+2} + \frac{3}{x+2}$$

$$= \frac{x+2+3}{x+2}$$

$$= \frac{x+5}{x+2}$$

$$\mathbf{b} \quad -2 + \frac{3}{x-4}$$

$$= -2 \times \frac{x-4}{x-4} + \frac{3}{x-4}$$

$$= \frac{-2(x-4)+3}{x-4}$$

$$= \frac{-2x+8+3}{x-4}$$

$$= \frac{11-2x}{x-4}$$

$$\mathbf{c} \quad -3 - \frac{2}{x-1}$$

$$= -3 \times \frac{x-1}{x-1} - \frac{2}{x-1}$$

$$= \frac{-3(x-1)-2}{x-1}$$

$$= \frac{-3x+3-2}{x-1}$$

$$= \frac{1-3x}{x-1}$$

$$\mathbf{d} \quad \frac{2x-1}{x+1} + 3$$

$$= \frac{2x-1}{x+1} + 3 \times \frac{x+1}{x+1}$$

$$= \frac{2x-1+3(x+1)}{x+1}$$

$$= \frac{2x-1+3x+3}{x+1}$$

$$= \frac{5x+2}{x+1}$$

$$\mathbf{e} \quad 3 - \frac{x}{x+1}$$

$$= 3 \times \frac{x+1}{x+1} - \frac{x}{x+1}$$

$$= \frac{3(x+1)-x}{x+1}$$

$$= \frac{3x+3-x}{x+1}$$

$$= \frac{2x+3}{x+1}$$

$$\mathbf{f} \quad -1 + \frac{4}{1-x}$$

$$= -1 \times \frac{1-x}{1-x} + \frac{4}{1-x}$$

$$= \frac{-(1-x)+4}{1-x}$$

$$= \frac{x-1+4}{1-x}$$

$$= \frac{x+3}{1-x}$$

$$\mathbf{3} \quad \mathbf{a} \quad \frac{3x}{2x-5} + \frac{2x+5}{x-2}$$

$$= \frac{3x}{2x-5} \times \frac{x-2}{x-2} + \frac{2x+5}{x-2} \times \frac{2x-5}{2x-5}$$

$$= \frac{3x(x-2) + (2x+5)(2x-5)}{(2x-5)(x-2)}$$

$$= \frac{3x^2 - 6x + (4x^2 - 5^2)}{(2x-5)(x-2)}$$

$$= \frac{3x^2 - 6x + 4x^2 - 25}{(2x-5)(x-2)}$$

$$= \frac{7x^2 - 6x - 25}{(2x-5)(x-2)}$$

$$\mathbf{b} \quad \frac{1}{x-2} - \frac{1}{x-3}$$

$$= \frac{1}{x-2} \times \frac{x-3}{x-3} - \frac{1}{x-3} \times \frac{x-2}{x-2}$$

$$= \frac{(x-3) - (x-2)}{(x-2)(x-3)}$$

$$= \frac{x-3-x+2}{(x-2)(x-3)}$$

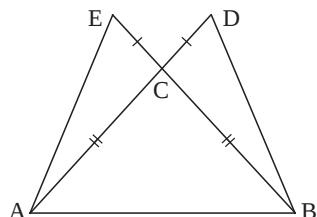
$$= -\frac{1}{(x-2)(x-3)}$$

$$\begin{aligned}
 \text{c} \quad & \frac{5x}{x-4} + \frac{3x-2}{x+4} \\
 &= \frac{5x}{x-4} \times \frac{x+4}{x+4} + \frac{3x-2}{x+4} \times \frac{x-4}{x-4} \\
 &= \frac{5x(x+4) + (3x-2)(x-4)}{(x+4)(x-4)} \\
 &= \frac{5x^2 + 20x + 3x^2 - 12x - 2x + 8}{(x+4)(x-4)} \\
 &= \frac{8x^2 + 6x + 8}{(x+4)(x-4)}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{2x+1}{x-3} - \frac{x+4}{2x+1} \\
 &= \frac{2x+1}{x-3} \times \frac{2x+1}{2x+1} - \frac{x+4}{2x+1} \times \frac{x-3}{x-3} \\
 &= \frac{(2x+1)^2 - (x+4)(x-3)}{(x-3)(2x+1)} \\
 &= \frac{(2x)^2 + 2(2x)(1) + 1^2 - (x^2 - 3x + 4x - 12)}{(x-3)(2x+1)} \\
 &= \frac{4x^2 + 4x + 1 - (x^2 + x - 12)}{(x-3)(2x+1)} \\
 &= \frac{4x^2 + 4x + 1 - x^2 - x + 12}{(x-3)(2x+1)} \\
 &= \frac{3x^2 + 3x + 13}{(x-3)(2x+1)}
 \end{aligned}$$

EXERCISE K.1

1



$\angle ECA = \angle DCB$ {opposite angles}
 Also, $EC = CD$ and $AC = BC$ {given}
 $\therefore \Delta s AEC$ and BDC are congruent (SAS)
 $\therefore AE = BD$

2 $MP = NP$ {given}

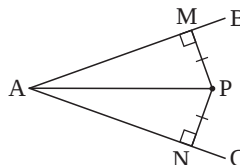
AP is common to both ΔAMP and ΔANP

$\therefore \Delta s AMP$ and ANP are congruent (RHS)

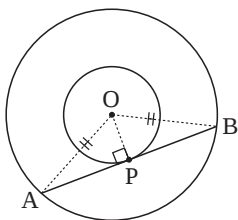
$\therefore \angle MAP = \angle NAP$ {corresponding angles}

i.e., AP bisects $\angle BAC$

$\therefore P$ lies on the bisector of $\angle BAC$



3



$OA = OB$ {both radii of circle}

$\angle OPA = \angle OPB = 90^\circ$

{since AB is a tangent to inner circle}

OP is common to both ΔAOP and ΔBOP

$\therefore \Delta s AOP$ and BOP are congruent (RHS)

$\therefore AP = BP$ {corresponding sides}

$\therefore P$ is the midpoint of AB

EXERCISE K.2

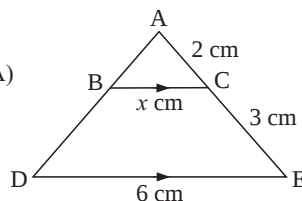
1 a $\angle ABC = \angle ADE$ {corresponding angles}

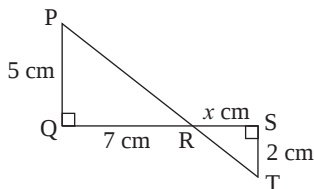
$\angle ACB = \angle AED$ {corresponding angles}

i.e., $\Delta s ABC$ and ADE are equiangular (AAA)
and hence similar.

$$\therefore \frac{x}{2} = \frac{6}{2+3} = \frac{6}{5}$$

$$\therefore x = \frac{12}{5} = 2.4$$



b $\angle PRQ = \angle TRS$ {opposite angles} $\therefore \triangle s PQR$ and TSR are equiangular (AAA) and hence similar.

$$\therefore \frac{5}{7} = \frac{2}{x}$$

$$\therefore 5x = 14$$

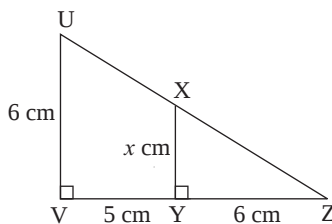
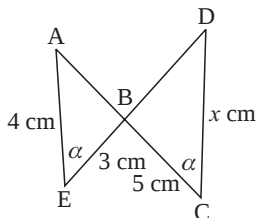
$$\therefore x = \frac{14}{5} = 2.8$$

c $\angle VUZ = \angle YXZ$ {corresponding angles} $\therefore \triangle s UVZ$ and XYZ are equiangular (AAA) and hence similar.

$$\therefore \frac{x}{6} = \frac{6}{5+6}$$

$$\therefore \frac{x}{6} = \frac{6}{11}$$

$$\therefore x = \frac{36}{11} = 3\frac{3}{11}$$

**d** $\angle ABE = \angle DBC$ {opposite angles} $\therefore \triangle s ABE$ and DBC are equiangular (AAA) and hence similar.

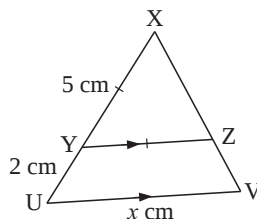
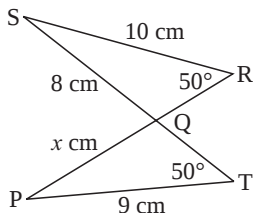
$$\therefore \frac{x}{5} = \frac{4}{3}$$

$$\therefore x = \frac{20}{3} = 6\frac{2}{3}$$

e $\angle XYZ = \angle XUV$ {corresponding angles} $\angle XZY = \angle XVU$ {corresponding angles} $\therefore \triangle s XYZ$ and XUV are equiangular (AAA) and hence similar.

$$\therefore \frac{x}{2+5} = \frac{5}{5}$$

$$\therefore \frac{x}{7} = 1 \text{ and so } x = 7$$

**f** $\angle SQR = \angle PQT$ {opposite angles} $\therefore \triangle s SQR$ and PQT are equiangular (AAA) and hence similar.

$$\therefore \frac{x}{9} = \frac{8}{10}$$

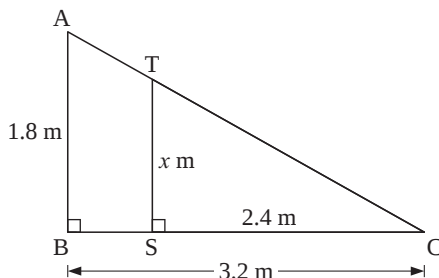
$$\therefore x = \frac{72}{10} = 7.2$$

2 $\angle BAC = \angle STC$ {corresponding angles} $\therefore \triangle s ABC$ and TSC are equiangular (AAA) and hence similar.

$$\therefore \frac{x}{2.4} = \frac{1.8}{3.2}$$

$$\therefore x = \frac{2.4 \times 1.8}{3.2} = 1.35$$

i.e., the son is 1.35 m tall

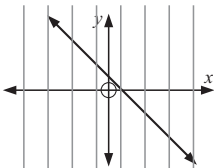
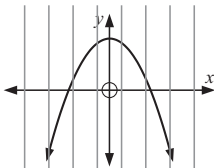
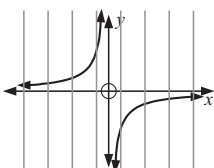
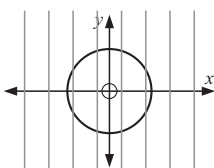
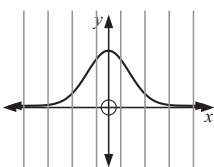
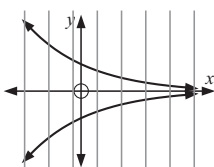
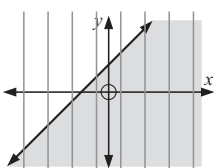
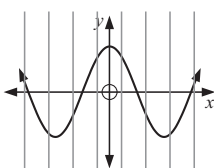
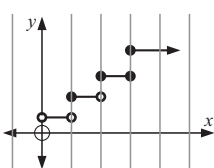


Chapter 1

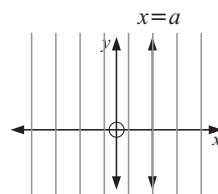
FUNCTIONS

EXERCISE 1A

- 1
 - a $(1, 3), (2, 4), (3, 5), (4, 6)$ is a function since no two ordered pairs have the same x -coordinate.
 - b $(1, 3), (3, 2), (1, 7), (-1, 4)$ is not a function since two of the ordered pairs, $(1, 3)$ and $(1, 7)$, have the same x -coordinate of 1.
 - c $(2, -1), (2, 0), (2, 3), (2, 11)$ is not a function since each ordered pair has the same x -coordinate of 2.
 - d $(7, 6), (5, 6), (3, 6), (-4, 6)$ is a function since no two ordered pairs have the same x -coordinate.
 - e $(0, 0), (1, 0), (3, 0), (5, 0)$ is a function since no two ordered pairs have the same x -coordinate.
 - f $(0, 0), (0, -2), (0, 2), (0, 4)$ is not a function since each ordered pair has the same x -coordinate of 0.

- 2
 - a  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - b  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - c  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - d  i.e., the lines cut the graph more than once $\therefore$ it is not a function
 - e  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - f  i.e., the lines cut the graph more than once $\therefore$ it is not a function
 - g  i.e., the lines cut the graph more than once $\therefore$ it is not a function
 - h  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - i  i.e., one line cuts the graph more than once $\therefore$ it is not a function

- 3 The graph of a straight line is not a function if the graph is a vertical line, i.e., $x = a$ for all a . The vertical line through $x = a$ cuts the graph at every point $\therefore$ it is not a function.



4 $x^2 + y^2 = 9$ is the equation of a circle, centre (0, 0) and radius 3.

Now $x^2 + y^2 = 9$

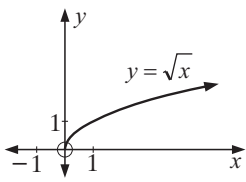
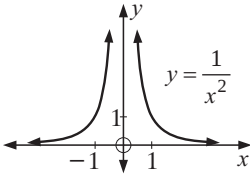
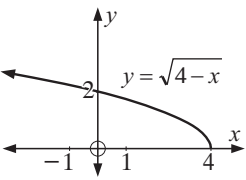
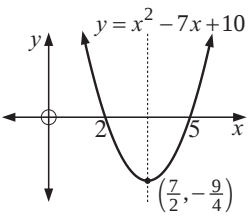
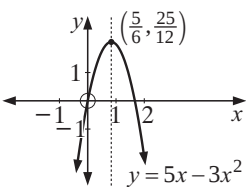
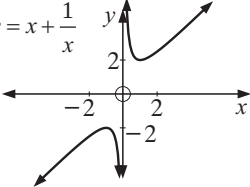
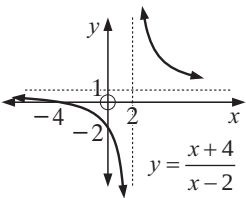
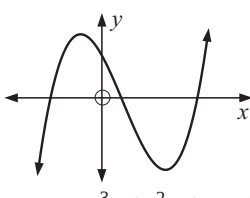
$$\therefore y^2 = 9 - x^2$$

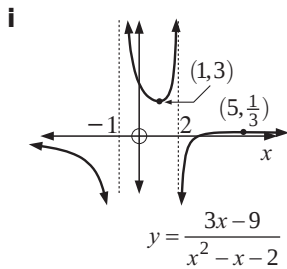
$$\therefore y = \pm\sqrt{9 - x^2}$$

Hence y has two real values for any value of x where $-3 < x < 3$.

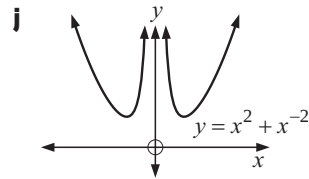
EXERCISE 1B

- 1**
- | | |
|--|---|
| a Domain is $\{x: x \geq -1\}$
Range is $\{y: y \leq 3\}$ | b Domain is $\{x: -1 < x \leq 5\}$
Range is $\{y: 1 < y \leq 3\}$ |
| c Domain is $\{x: x \neq 2\}$
Range is $\{y: y \neq -1\}$ | d Domain is $\{x: x \in \mathcal{R}\}$
Range is $\{y: 0 < y \leq 2\}$ |
| e Domain is $\{x: x \in \mathcal{R}\}$
Range is $\{y: y \geq -1\}$ | f Domain is $\{x: x \in \mathcal{R}\}$
Range is $\{y: y \leq 6\frac{1}{4}\}$ i.e., $\{y: y \leq \frac{25}{4}\}$ |
| g Domain is $\{x: x \geq -4\}$
Range is $\{y: y \geq -3\}$ | h Domain is $\{x: x \in \mathcal{R}\}$
Range is $\{y: y > -2\}$ |
| i Domain is $\{x: x \neq \pm 2\}$ Range is $\{y: y \leq -1 \text{ or } y > 0\}$ | |

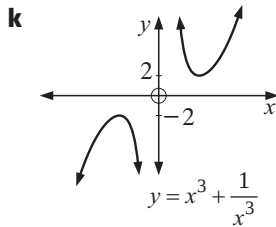
- 2**
- | | | |
|----------|---|---|
| a |  | Domain is $\{x: x \geq 0\}$
Range is $\{y: y \geq 0\}$ |
| b |  | Domain is $\{x: x \neq 0\}$
Range is $\{y: y > 0\}$ |
| c |  | Domain is $\{x: x \leq 4\}$
Range is $\{y: y \geq 0\}$ |
| d |  | Domain is $\{x: x \in \mathcal{R}\}$
Range is $\{y: y \geq -2\frac{1}{4}\}$ |
| e |  | Domain is $\{x: x \in \mathcal{R}\}$
Range is $\{y: y \leq \frac{25}{12}\}$ |
| f |  | Domain is $\{x: x \neq 0\}$
Range is $\{y: y \leq -2 \text{ or } y \geq 2\}$ |
| g |  | Domain is $\{x: x \neq 2\}$
Range is $\{y: y \neq 1\}$ |
| h |  | Domain is $\{x: x \in \mathcal{R}\}$
Range is $\{y: y \in \mathcal{R}\}$ |



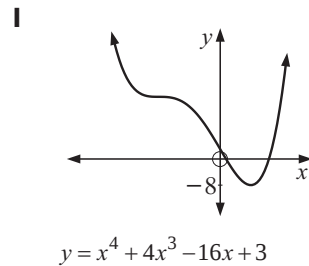
Domain is
 $\{x: x \neq -1, x \neq 2\}$
 Range is
 $\{y: y \leq \frac{1}{3} \text{ or } y \geq 3\}$



Domain is
 $\{x: x \neq 0\}$
 Range is
 $\{y: y \geq 2\}$



Domain is
 $\{x: x \neq 0\}$
 Range is
 $\{y: y \leq -2 \text{ or } y \geq 2\}$



Domain is
 $\{x: x \in \mathcal{R}\}$
 Range is
 $\{y: y \geq -8\}$

EXERCISE 1C

1 a $f(0) = 3(0) + 2$
 $= 2$

b $f(2) = 3(2) + 2$
 $= 8$

c $f(-1) = 3(-1) + 2$
 $= -1$

d $f(-5) = 3(-5) + 2$
 $= -13$

e $f(-\frac{1}{3}) = 3(-\frac{1}{3}) + 2$
 $= 1$

2 a $g(1) = 1 - \frac{4}{1}$
 $= 1 - 4$
 $= -3$

b $g(4) = 4 - \frac{4}{4}$
 $= 4 - 1$
 $= 3$

c $g(-1) = -1 - \frac{4}{-1}$
 $= -1 + 4$
 $= 3$

d $g(-4) = -4 - \frac{4}{-4}$
 $= -4 + 1$
 $= -3$

e $g(-\frac{1}{2}) = -\frac{1}{2} - \frac{4}{(-\frac{1}{2})}$
 $= -\frac{1}{2} + 8$
 $= 7\frac{1}{2}$

3 a $f(0) = 3(0) - 0^2 + 2$
 $= 2$

b $f(3) = 3(3) - 3^2 + 2$
 $= 9 - 9 + 2$
 $= 2$

c $f(-3) = 3(-3) - (-3)^2 + 2$
 $= -9 - 9 + 2$
 $= -16$

d $f(-7) = 3(-7) - (-7)^2 + 2$
 $= -21 - 49 + 2$
 $= -68$

e $f(\frac{3}{2}) = 3(\frac{3}{2}) - (\frac{3}{2})^2 + 2$
 $= \frac{9}{2} - \frac{9}{4} + 2$
 $= \frac{17}{4}$

4 a $f(a) = 7 - 3a$

b $f(-a) = 7 - 3(-a)$
 $= 7 + 3a$

c $f(a+3) = 7 - 3(a+3)$
 $= 7 - 3a - 9$
 $= -3a - 2$

d $f(b-1) = 7 - 3(b-1)$
 $= 7 - 3b + 3$
 $= 10 - 3b$

e $f(x+2) = 7 - 3(x+2)$
 $= 7 - 3x - 6$
 $= 1 - 3x$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & F(x+4) \\
 &= 2(x+4)^2 + 3(x+4) - 1 \\
 &= 2(x^2 + 8x + 16) + 3x + 12 - 1 \\
 &= 2x^2 + 16x + 32 + 3x + 11 \\
 &= 2x^2 + 19x + 43
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & F(-x) \\
 &= 2(-x)^2 + 3(-x) - 1 \\
 &= 2x^2 - 3x - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & F(x^2 - 1) \\
 &= 2(x^2 - 1)^2 + 3(x^2 - 1) - 1 \\
 &= 2(x^4 - 2x^2 + 1) + 3x^2 - 3 - 1 \\
 &= 2x^4 - 4x^2 + 2 + 3x^2 - 4 \\
 &= 2x^4 - x^2 - 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & F(2-x) \\
 &= 2(2-x)^2 + 3(2-x) - 1 \\
 &= 2(4 - 4x + x^2) + 6 - 3x - 1 \\
 &= 8 - 8x + 2x^2 + 5 - 3x \\
 &= 2x^2 - 11x + 13
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & F(x^2) \\
 &= 2(x^2)^2 + 3(x^2) - 1 \\
 &= 2x^4 + 3x^2 - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad \mathbf{i} \quad & G(2) = \frac{2(2) + 3}{2 - 4} \\
 &= \frac{7}{-2} \\
 &= -\frac{7}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad & G(0) = \frac{2(0) + 3}{0 - 4} \\
 &= \frac{3}{-4} \\
 &= -\frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iii} \quad & G\left(-\frac{1}{2}\right) = \frac{2\left(-\frac{1}{2}\right) + 3}{-\frac{1}{2} - 4} \\
 &= \frac{-1 + 3}{-\frac{9}{2}} \\
 &= \frac{2}{\left(-\frac{9}{2}\right)} \\
 &= -\frac{4}{9}
 \end{aligned}$$

$$\mathbf{b} \quad G(x) = \frac{2x+3}{x-4} \text{ is undefined when } x-4=0$$

i.e., when $x=4$

So, when $x=4$, $G(x)$ does not exist.

$$\begin{aligned}
 \mathbf{c} \quad & G(x+2) = \frac{2(x+2) + 3}{(x+2) - 4} \\
 &= \frac{2x+4+3}{x+2-4} \\
 &= \frac{2x+7}{x-2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & G(x) = -3 \\
 \text{i.e., } & \frac{2x+3}{x-4} = -3 \\
 \therefore & 2x+3 = -3(x-4) \\
 \therefore & 2x+3 = -3x+12 \\
 \therefore & 5x = 9 \text{ and so } x = \frac{9}{5}
 \end{aligned}$$

7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x .

$$\mathbf{8} \quad \text{If } f(x) = 2^x, \quad f(a)f(b) = 2^a \times 2^b \quad \text{and} \quad f(a+b) = 2^{a+b}$$

$$= 2^{a+b}$$

i.e., when $f(x) = 2^x$, $f(a)f(b) = f(a+b)$.

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad & \frac{f(x) - f(3)}{x - 3} = \frac{x^2 - 3^2}{x - 3} \\
 &= \frac{x^2 - 9}{x - 3} \\
 &= \frac{(x+3)(x-3)}{x-3} \\
 &= x+3, \quad x \neq 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{f(2+h) - f(2)}{h} = \frac{(2+h)^2 - 2^2}{h} \\
 &= \frac{4 + 4h + h^2 - 4}{h} \\
 &= \frac{h^2 + 4h}{h} \\
 &= \frac{h(h+4)}{h} \\
 &= h+4, \quad h \neq 0
 \end{aligned}$$

$$\begin{aligned} 10 \quad a \quad V(4) &= 9650 - 860(4) \\ &= 9650 - 3440 \\ &= 6210 \end{aligned}$$

i.e., the value of the photocopier 4 years after purchase is \$6210.

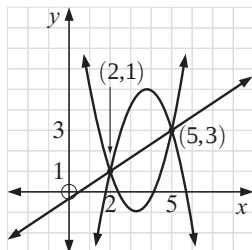
$$\begin{aligned} c \quad \text{Original purchase price is when } t &= 0, \\ \text{i.e., } V(0) &= 9650 - 860(0) \\ &= 9650 \end{aligned}$$

$$\begin{aligned} b \quad \text{If } V(t) &= 5780, \\ \text{then } 9650 - 860t &= 5780 \\ \therefore 860t &= 3870 \\ \therefore t &= 4.5 \end{aligned}$$

i.e., the value of the photocopier $4\frac{1}{2}$ years after purchase is \$5780.

i.e., the original purchase price was \$9650.

11



First sketch the linear function which passes through the two points (2, 1) and (5, 3).

Then sketch two quadratic functions which also pass through the two points.

$$12 \quad f(x) = ax + b \quad \text{where } f(2) = 1 \quad \text{and } f(-3) = 11$$

$$\begin{aligned} \text{i.e., } a(2) + b &= 1 & \text{and } a(-3) + b &= 11 \\ \therefore 2a + b &= 1 & \therefore -3a + b &= 11 \\ \therefore b &= 1 - 2a \quad \dots (1) & \therefore b &= 11 + 3a \quad \dots (2) \end{aligned}$$

$$\begin{aligned} \text{Solving (1) and (2) simultaneously, } 1 - 2a &= 11 + 3a \\ \therefore 5a &= -10 \\ \therefore a &= -2 \end{aligned}$$

$$\begin{aligned} \text{Substituting } a = -2 \text{ into (1) gives } b &= 1 - 2(-2) = 5 \quad \text{i.e., } a = -2, \quad b = 5 \\ \text{Hence } f(x) &= -2x + 5 \end{aligned}$$

$$13 \quad f(x) = ax + \frac{b}{x} \quad \text{where } f(1) = 1 \quad \text{and } f(2) = 5$$

$$\begin{aligned} \text{i.e., } a(1) + \frac{b}{1} &= 1 & \text{and } a(2) + \frac{b}{2} &= 5 \\ \therefore a + b &= 1 & \therefore 2a + \frac{b}{2} &= 5 \quad \dots (2) \\ \therefore a &= 1 - b \quad \dots (1) \end{aligned}$$

$$\text{Substituting (1) into (2) gives } 2(1 - b) + \frac{b}{2} = 5 \quad \therefore 2 - 2b + \frac{b}{2} = 5$$

$$\begin{aligned} \therefore -\frac{3b}{2} &= 3 \\ \therefore b &= -2 \end{aligned}$$

$$\text{Substituting } b = -2 \text{ into (1) gives } a = 1 - (-2) = 3 \quad \text{i.e., } a = 3, \quad b = -2$$

$$14 \quad T(x) = ax^2 + bx + c \quad \text{where } T(0) = -4, \quad T(1) = -2 \quad \text{and } T(2) = 6$$

$$\begin{aligned} \text{i.e., } a(0)^2 + b(0) + c &= -4 \\ \therefore c &= -4 \end{aligned}$$

$$\begin{aligned} \text{Also, } a(1)^2 + b(1) + c &= -2 & \text{and } a(2)^2 + b(2) + c &= 6 \\ \therefore a + b + c &= -2 & \text{and } \therefore 4a + 2b + c &= 6 \end{aligned}$$

Substituting $c = -4$ into both equations gives

$$\begin{aligned} a + b + (-4) &= -2 & \text{and} & & 4a + 2b + (-4) &= 6 \\ \therefore a + b &= 2 & & & \therefore 4a + 2b &= 10 \quad \dots (2) \\ \therefore a &= 2 - b \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{Substituting (1) into (2) gives} \quad 4(2 - b) + 2b &= 10 & \therefore 8 - 4b + 2b &= 10 \\ & & \therefore -2b &= 2 \\ & & \therefore b &= -1 \end{aligned}$$

Now, substituting $b = -1$ into (1) gives $a = 2 - (-1) = 3$ i.e., $a = 3$, $b = -1$, $c = -4$

EXERCISE 1D

- 1** **a** $(f \circ g)(x)$
 $= f(g(x))$
 $= f(1 - x)$
 $= 2(1 - x) + 3$
 $= 2 - 2x + 3$
 $= 5 - 2x$
- b** $(g \circ f)(x)$
 $= g(f(x))$
 $= g(2x + 3)$
 $= 1 - (2x + 3)$
 $= 1 - 2x - 3$
 $= -2x - 2$
- c** $(f \circ g)(-3)$
 $= f(g(-3))$
 $= f(1 - (-3))$
 $= f(4)$
 $= 2(4) + 3$
 $= 8 + 3$
 $= 11$
- 2** $(f \circ g)(x) = f(g(x))$ $(g \circ f)(x) = g(f(x))$
 $= f(2 - x)$ $= g(x^2)$
 $= (2 - x)^2$ $= 2 - x^2$
- 3** **a** $(f \circ g)(x)$
 $= f(g(x))$
 $= f(3 - x)$
 $= (3 - x)^2 + 1$
 $= 9 - 6x + x^2 + 1$
 $= x^2 - 6x + 10$
- b** $(g \circ f)(x)$
 $= g(f(x))$
 $= g(x^2 + 1)$
 $= 3 - (x^2 + 1)$
 $= 3 - x^2 - 1$
 $= -x^2 + 2$
- c** $(g \circ f)(x) = f(x)$
i.e., $-x^2 + 2 = f(x)$ {from **b**}
i.e., $-x^2 + 2 = x^2 + 1$
 $\therefore 2x^2 = 1$
 $\therefore x^2 = \frac{1}{2}$
 $\therefore x = \pm \frac{1}{\sqrt{2}}$
- 4** **a** $ax + b = cx + d$ is true for all x {given}
Let $x = 0$, Let $x = 1$,
 $\therefore a(0) + b = c(0) + d$ $\therefore a(1) + b = c(1) + d$
 $\therefore b = d \quad \dots (1)$ $\therefore a + b = c + d$
but $b = d$ (from (1))
 $\therefore a + d = c + d$
 $\therefore a = c$
- b** $(f \circ g)(x) = x$ for all x {given}
 $\therefore f(g(x)) = x$
 $\therefore f(ax + b) = x$
 $\therefore 2(ax + b) + 3 = x$
 $\therefore 2ax + 2b + 3 = x$
- When $x = 0$, When $x = 1$, $2a(1) + 2b + 3 = 1$
 $2a(0) + 2b + 3 = 0$ $\therefore 2a + 2b = -2$
 $\therefore 2b = -3$ $\therefore 2a = -2 - 2(-\frac{3}{2}) = 1$
 $\therefore b = -\frac{3}{2}$ $\therefore a = \frac{1}{2}$
- i.e., $a = \frac{1}{2}$ and $b = -\frac{3}{2}$ as required.

$$\begin{aligned}
 \text{c} \quad & \text{If } (g \circ f)(x) = x \\
 & \text{then } g(f(x)) = x \\
 & \text{i.e., } g(2x+3) = x \\
 & \text{i.e., } a(2x+3) + b = x \\
 & \therefore 2ax + 3a + b = x
 \end{aligned}$$

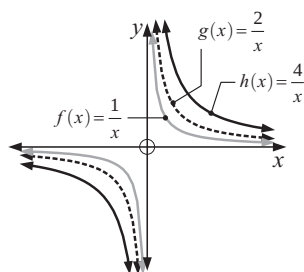
$$\begin{aligned}
 \text{When } x = 1, \\
 2a(1) + 3a + b &= 1 \\
 \therefore 2a + 3a + b &= 1 \\
 \therefore 5a + b &= 1 \\
 \therefore 5(-\frac{1}{3}b) + b &= 1 \quad (\text{from (1)}) \\
 \therefore -\frac{2}{3}b &= 1 \\
 \therefore b &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = 0, \\
 2a(0) + 3a + b &= 0 \\
 \therefore 3a &= -b \\
 \therefore a &= -\frac{1}{3}b \quad \dots (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } b = -\frac{3}{2} \text{ into (1) gives} \\
 a &= -\frac{1}{3}(-\frac{3}{2}) = \frac{1}{2} \\
 \text{i.e., } a &= \frac{1}{2} \text{ and } b = -\frac{3}{2} \text{ as required.} \\
 \therefore \text{the result in b is also true if} \\
 (g \circ f)(x) &= x \text{ for all } x.
 \end{aligned}$$

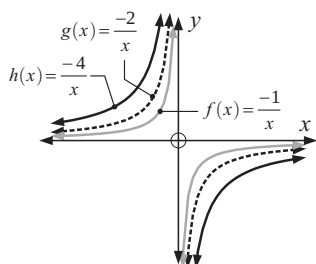
EXERCISE 1E

1



$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x - and y -axes. The graphs all lie in the 1st and 3rd quadrants. The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x) = \frac{1}{x}$ is closer to the axes than $g(x) = \frac{2}{x}$ for corresponding values of x , and $g(x) = \frac{2}{x}$ is closer to the axes than $h(x) = \frac{4}{x}$.

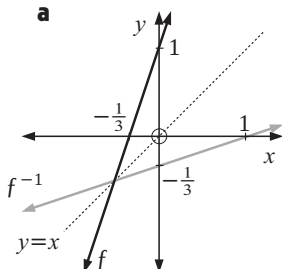
2



$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x - and y -axes. The graphs all lie in the 2nd and 4th quadrants. The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x) = -\frac{1}{x}$ is closer to the axes than $g(x) = -\frac{2}{x}$ for corresponding values of x , and $g(x) = -\frac{2}{x}$ is closer to the axes than $h(x) = -\frac{4}{x}$.

EXERCISE 1F

1 a



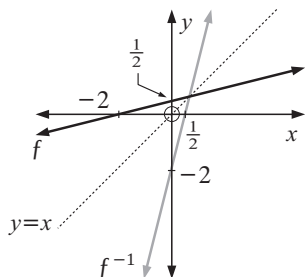
b $f(x)$ passes through $(0, 1)$ and $(-\frac{1}{3}, 0)$
 $\therefore f^{-1}(x)$ passes through $(1, 0)$ and $(0, -\frac{1}{3})$

$$f^{-1}(x) \text{ has slope } \frac{-\frac{1}{3} - 0}{0 - 1} = \frac{-\frac{1}{3}}{-1} = \frac{1}{3}$$

$$\text{So, its equation is } \frac{y - 0}{x - 1} = \frac{1}{3}$$

$$\text{i.e., } y = \frac{x - 1}{3} \quad \text{i.e., } f^{-1}(x) = \frac{x - 1}{3}$$

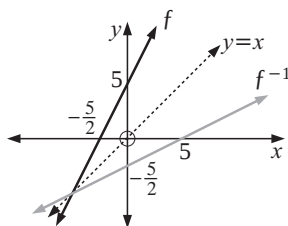
c f is $y = 3x + 1$
 so f^{-1} is $x = 3y + 1$
 $\therefore x - 1 = 3y$
 $\therefore y = \frac{x-1}{3}$ i.e., $f^{-1}(x) = \frac{x-1}{3}$

2 a

b $f(x)$ passes through $(0, \frac{1}{2})$ and $(-2, 0)$
 $\therefore f^{-1}(x)$ passes through $(\frac{1}{2}, 0)$ and $(0, -2)$
 $f^{-1}(x)$ has slope $\frac{-2-0}{0-\frac{1}{2}} = \frac{-2}{-\frac{1}{2}} = 4$
 So, its equation is $\frac{y-0}{x-\frac{1}{2}} = 4$
 i.e., $y = 4x - 2$
 i.e., $f^{-1}(x) = 4x - 2$

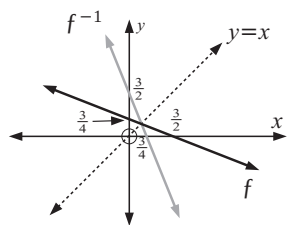
c f is $y = \frac{x+2}{4}$
 so f^{-1} is $x = \frac{y+2}{4}$
 $\therefore 4x = y + 2$
 $\therefore y = 4x - 2$
 i.e., $f^{-1}(x) = 4x - 2$

3 a i f is $y = 2x + 5$
 so f^{-1} is $x = 2y + 5$
 $\therefore x - 5 = 2y$
 $\therefore y = \frac{x-5}{2}$
 i.e., $f^{-1}(x) = \frac{x-5}{2}$

ii

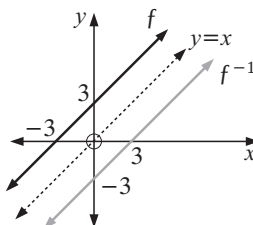
$f(x)$ passes through $(0, 5)$ and $(-\frac{5}{2}, 0)$
 $\therefore f^{-1}(x)$ passes through $(5, 0)$ and $(0, -\frac{5}{2})$

b i f is $y = \frac{3-2x}{4}$
 so f^{-1} is $x = \frac{3-2y}{4}$
 $\therefore 4x = 3 - 2y$
 $\therefore 4x - 3 = -2y$
 $\therefore y = -2x + \frac{3}{2}$
 i.e., $f^{-1}(x) = -2x + \frac{3}{2}$

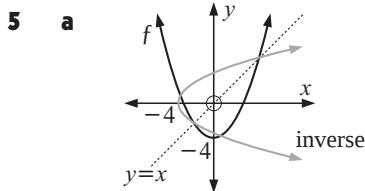
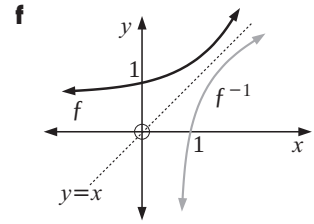
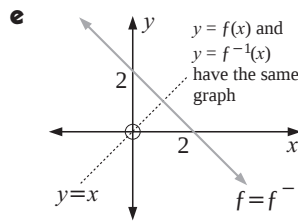
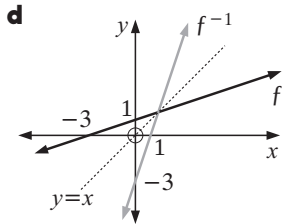
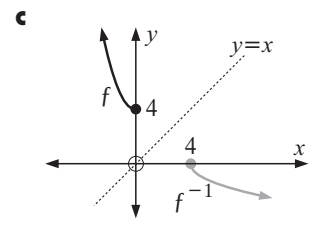
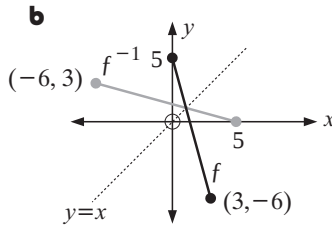
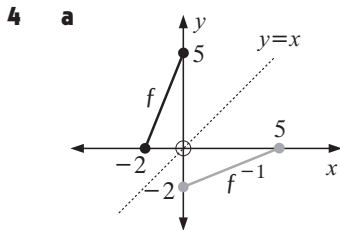
ii

$f(x)$ passes through $(0, \frac{3}{4})$ and $(\frac{3}{2}, 0)$
 $\therefore f^{-1}(x)$ passes through $(\frac{3}{4}, 0)$ and $(0, \frac{3}{2})$

c i f is $y = x + 3$
 so f^{-1} is $x = y + 3$
 $\therefore y = x - 3$
 i.e., $f^{-1}(x) = x - 3$

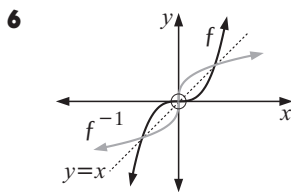
ii

$f(x)$ passes through $(0, 3)$ and $(-3, 0)$
 $\therefore f^{-1}(x)$ passes through $(3, 0)$ and $(0, -3)$



b Using the ‘horizontal line test’, f does not have an inverse function as a horizontal line through $y = x^2 - 4$ cuts it more than once.

c For $x \geq 0$, any horizontal line cuts it only once, i.e., f does have an inverse function for $x \geq 0$.

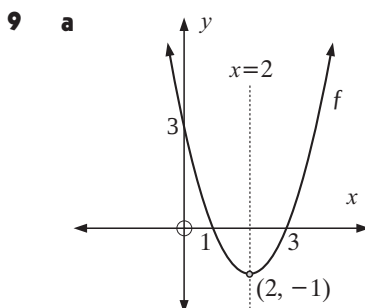
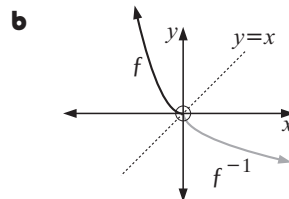


7 a If $y = f(x)$ has an inverse function, then the inverse function must also be a function. Thus, it must satisfy the ‘vertical line test’, i.e., no vertical line can cut it more than once. This condition for the inverse function cannot be satisfied if the original function does not satisfy the ‘horizontal line test’. Thus, the ‘horizontal line test’ is a valid test for the existence of an inverse function.

b i This graph satisfies the ‘horizontal line test’ and therefore has an inverse function.

ii, iii These graphs both fail the ‘horizontal line test’ so neither of these have inverse functions.

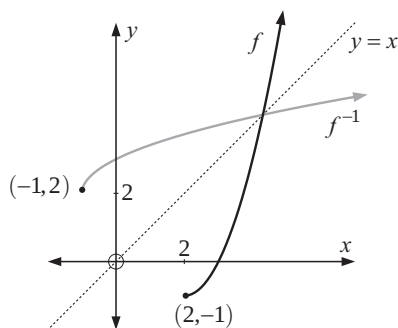
8 a f is $y = x^2, x \leq 0$
 so f^{-1} is $x = y^2, y \leq 0$
 $\therefore y = -\sqrt{x}$
 i.e., $f^{-1}(x) = -\sqrt{x}$



$f : x \rightarrow x^2 - 4x + 3$ satisfies the ‘vertical line test’ so is therefore a function. It does not however satisfy the horizontal line test as any horizontal line above the vertex cuts the graph twice. Therefore it does not have an inverse function.

b For $x \geq 2$, all horizontal lines cut the graph no more than once. Therefore f has an inverse function for $x \geq 2$.

- c** f is $y = x^2 - 4x + 3, x \geq 2$
 so f^{-1} is $x = y^2 - 4y + 3, y \geq 2$
 i.e., $x = (y-2)^2 - 4 + 3, y \geq 2$ {completing the square}
 $= (y-2)^2 - 1, y \geq 2$
 $\therefore x+1 = (y-2)^2, y \geq 2$
 $\therefore y-2 = \sqrt{x+1}, y \geq 2$
 $\therefore y = 2 + \sqrt{1+x}, y \geq 2$
 i.e., $f^{-1}(x) = 2 + \sqrt{1+x}$ as required



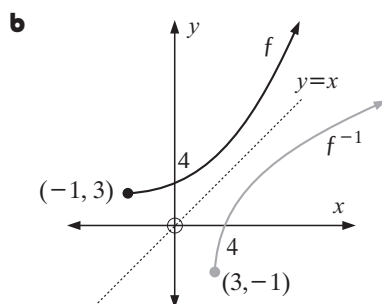
- d i** domain of f is $\{x: x \geq 2\}$,
 range is $\{y: y \geq -1\}$
ii domain of f^{-1} is $\{x: x \geq -1\}$,
 range is $\{y: y \geq 2\}$

- 10 a** f is $y = \frac{1}{2}x - 1$ so f^{-1} is $x = \frac{1}{2}y - 1$
 $\therefore y = 2x + 2$
 i.e., $f^{-1}(x) = 2x + 2$

- b i** $(f \circ f^{-1})(x) = f(f^{-1}(x))$
 $= f(2x + 2)$
 $= \frac{1}{2}(2x + 2) - 1$
 $= x + 1 - 1$
 i.e., $(f \circ f^{-1})(x) = x$
- ii** $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
 $= f^{-1}(\frac{1}{2}x - 1)$
 $= 2(\frac{1}{2}x - 1) + 2$
 $= x - 2 + 2$
 $= x$

- 11 a** f is $y = (x+1)^2 + 3, x \geq -1$
 so f^{-1} is $x = (y+1)^2 + 3, y \geq -1$
 i.e., $x-3 = (y+1)^2, y \geq -1$
 $\therefore y+1 = \sqrt{x-3}, y \geq -1, x \geq 3$
 $\therefore y = \sqrt{x-3} - 1, y \geq -1, x \geq 3$

- c i** Domain $\{x: x \geq -1\}$, Range $\{y: y \geq 3\}$
ii Domain $\{x: x \geq 3\}$, Range $\{y: y \geq -1\}$



- 12 a** g is $y = \frac{8-x}{2}$
 so g^{-1} is $x = \frac{8-y}{2}$
 $\therefore 2x = 8 - y$
 $\therefore y = 8 - 2x$
 i.e., $g^{-1}(x) = 8 - 2x$
 Now $g^{-1}(-1) = 8 - 2(-1) = 10$
- b** $(f \circ g^{-1})(x) = 9$
 $\therefore f(g^{-1}(x)) = 9$
 $\therefore f(8 - 2x) = 9$
 $\therefore 2(8 - 2x) + 5 = 9$
 $\therefore 16 - 4x + 5 = 9$
 $\therefore -4x = -12$
 $\therefore x = 3$

- 13 a i** f is $y = 5^x$ so, $f(2) = 5^2 = 25$
- ii** g is $y = \sqrt{x}$ so g^{-1} is $x = \sqrt{y}$
- $\therefore y = x^2$
 i.e., $g^{-1}(x) = x^2, x \geq 0$
 $\therefore g^{-1}(4) = 4^2$
 $\therefore g^{-1}(4) = 16$

$$\begin{array}{ll}
 \mathbf{b} & (g^{-1} \circ f)(x) = 25 \\
 & \therefore g^{-1}(f(x)) = 25 \\
 & \therefore g^{-1}(5^x) = 25 \\
 & \therefore (5^x)^2 = 25 \quad \{\text{as } g^{-1}(x) = x^2, x \geq 0\}
 \end{array}
 \quad
 \begin{array}{l}
 \text{and so } 5^{2x} = 5^2 \\
 \therefore 2x = 2 \\
 \therefore x = 1
 \end{array}$$

14 Show: $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$

$$\begin{array}{llll}
 f \text{ is } y = 2x & g \text{ is } y = 4x - 3 & (g \circ f)(x) = g(f(x)) \\
 \text{so } f^{-1} \text{ is } x = 2y & \text{so } g^{-1} \text{ is } x = 4y - 3 & = g(2x) \\
 \therefore y = \frac{x}{2} & \therefore 4y = x + 3 & = 4(2x) - 3 \\
 \text{i.e., } f^{-1}(x) = \frac{x}{2} & \therefore y = \frac{x+3}{4} & \text{i.e., } (g \circ f)(x) = 8x - 3 \\
 & \text{i.e., } g^{-1}(x) = \frac{x+3}{4} & \text{i.e., } g \circ f \text{ is } y = 8x - 3 \\
 & & \text{so } (g \circ f)^{-1} \text{ is } x = 8y - 3 \\
 & & \therefore y = \frac{x+3}{8} \\
 & & \text{i.e., } (g \circ f)^{-1}(x) = \frac{x+3}{8}
 \end{array}$$

$$\begin{aligned}
 \text{Now, } (f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x)) \\
 &= f^{-1}\left(\frac{x+3}{4}\right) \\
 &= \frac{\left(\frac{x+3}{4}\right)}{2}
 \end{aligned}$$

$$\therefore (f^{-1} \circ g^{-1})(x) = \frac{x+3}{8} = (g \circ f)^{-1}(x) \quad \text{as required}$$

$$\begin{array}{lll}
 \mathbf{15} \quad \mathbf{a} & f \text{ is } y = 2x & \mathbf{b} \quad f \text{ is } y = x \\
 \text{so } f^{-1} \text{ is } x = 2y & & \text{so } f^{-1} \text{ is } x = y \\
 \therefore y = \frac{x}{2} & & \therefore y = x \\
 \text{i.e., } f^{-1}(x) = \frac{x}{2} \neq 2x & & \text{i.e., } f^{-1}(x) = x \\
 \text{So, } f^{-1}(x) \neq f(x) & & \text{So, } f^{-1}(x) = f(x)
 \end{array}
 \quad
 \begin{array}{l}
 \mathbf{c} \quad f \text{ is } y = -x \\
 \text{so } f^{-1} \text{ is } x = -y \\
 \therefore y = -x \\
 \text{i.e., } f^{-1}(x) = -x \\
 \text{So, } f^{-1}(x) = f(x)
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{d} & f \text{ is } y = \frac{1}{x} \\
 \text{so } f^{-1} \text{ is } x = \frac{1}{y} & \mathbf{e} \quad f \text{ is } y = -\frac{6}{x} \\
 \therefore y = \frac{1}{x} & \text{so } f^{-1} \text{ is } x = -\frac{6}{y} \\
 \text{i.e., } f^{-1}(x) = \frac{1}{x} & \therefore y = -\frac{6}{x} \\
 \text{So, } f^{-1}(x) = f(x) & \text{i.e., } f^{-1}(x) = -\frac{6}{x} \\
 & \text{So, } f^{-1}(x) = f(x)
 \end{array}$$

i.e., $f^{-1}(x) = f(x)$ is true for parts **b**, **c**, **d** and **e**.

EXERCISE 1G

1 f is $y = 3x + 1$

so f^{-1} is $x = 3y + 1$

$$\therefore y = \frac{x-1}{3} \quad \text{i.e., } f^{-1}(x) = \frac{x-1}{3}$$

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) & (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f\left(\frac{x-1}{3}\right) & &= f^{-1}(3x+1) \\ &= 3\left(\frac{x-1}{3}\right) + 1 & &= \frac{3x+1-1}{3} \\ &= x-1+1 & &= \frac{3x}{3} \end{aligned}$$

i.e., $(f \circ f^{-1})(x) = x$ i.e., $(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x)$ as required

2 f is $y = \frac{x+3}{4}$ so f^{-1} is $x = \frac{y+3}{4}$

$$\therefore 4x = y + 3$$

$$\therefore y = 4x - 3$$

i.e., $f^{-1}(x) = 4x - 3$

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) & (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f(4x-3) & &= f^{-1}\left(\frac{x+3}{4}\right) \\ &= \frac{4x-3+3}{4} & &= 4\left(\frac{x+3}{4}\right) - 3 \\ &= \frac{4x}{4} & &= x+3-3 \\ &= x & &= x \end{aligned}$$

i.e., $(f \circ f^{-1})(x) = x$ i.e., $(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x)$ as required

3 f is $y = \sqrt{x}$ for $x \geq 0$ so f^{-1} is $x = \sqrt{y}$

$$\therefore y = x^2$$

i.e., $f^{-1}(x) = x^2$ for $x \geq 0$

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) & (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f(x^2) & &= f^{-1}(\sqrt{x}) \\ &= \sqrt{x^2} & &= (\sqrt{x})^2 \end{aligned}$$

i.e., $(f \circ f^{-1})(x) = x$ i.e., $(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x)$ as required

4 a $f(x)$ passes through $A(x, f(x))$, so $f^{-1}(x)$ passes through $B(f(x), x)$

b Substitute the coordinates of $B(f(x), x)$ into $y = f^{-1}(x)$:

$$\begin{aligned} \text{i.e., } x &= f^{-1}(f(x)) \\ &= (f^{-1} \circ f)(x) \end{aligned}$$

c B has coordinates $(x, f^{-1}(x))$ since it lies on $y = f^{-1}(x)$,

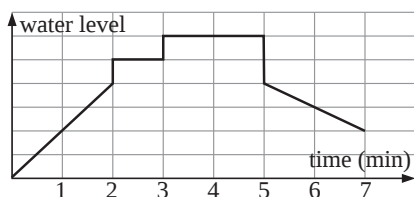
so A has coordinates $(f^{-1}(x), x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.

Substitute the coordinates of $A(f^{-1}(x), x)$ into $y = f(x)$:

$$\begin{aligned} \text{i.e., } x &= f(f^{-1}(x)) \\ \text{i.e., } f(f^{-1}(x)) &= x \text{ as required} \end{aligned}$$

REVIEW SET 1A

1



$$\begin{array}{lll}
 \mathbf{2} \quad \mathbf{a} & f(x) = 2x - x^2 & \mathbf{b} \quad f(-3) = 2(-3) - (-3)^2 \quad \mathbf{c} \quad f(-\tfrac{1}{2}) = 2(-\tfrac{1}{2}) - (-\tfrac{1}{2})^2 \\
 & f(2) = 2(2) - 2^2 & = -6 - 9 \quad = -1 - \tfrac{1}{4} \\
 & = 0 & = -15 \quad = -\tfrac{5}{4}
 \end{array}$$

- 3**
- a**
- i range is $\{y: y \geq -5\}$, domain is $\{x: x \in \mathcal{R}\}$
 - ii x -intercepts are -1 and 5 ; y -intercept is $-\frac{25}{9}$
 - iii The graph passes the 'vertical line test' so is therefore a function.
- b**
- i range is $\{y: y = 1 \text{ or } -3\}$, domain is $\{x: x \in \mathcal{R}\}$
 - ii there are no x -intercepts; y -intercept is 1
 - iii The graph passes the 'vertical line test' so is therefore a function.

- 4**
- a** domain is $\{x: x \geq -2\}$, range is $\{y: 1 \leq y < 3\}$
- b** domain is $\{x: x \in \mathcal{R}\}$, range is $\{y: y = -1, 1 \text{ or } 2\}$

$$\begin{array}{ll}
 \mathbf{5} \quad \mathbf{a} & h(x) = 7 - 3x \\
 & h(2x - 1) = 7 - 3(2x - 1) \\
 & = 7 - 6x + 3 \\
 & = 10 - 6x \\
 \mathbf{b} & h(2x - 1) = -2 \\
 & \therefore 7 - 3(2x - 1) = -2 \\
 & \therefore 7 - 6x + 3 = -2 \\
 & \therefore -6x = -12 \\
 & \therefore x = 2
 \end{array}$$

6 $f(x) = ax + b$, where $f(1) = 7$ and $f(3) = -5$

When $f(1) = 7$,

$$7 = a(1) + b$$

$$\therefore 7 = a + b$$

$$\therefore a = 7 - b \quad \dots (1)$$

When $f(3) = -5$,

$$-5 = a(3) + b$$

$$\therefore -5 = 3a + b$$

$$\therefore -5 = 3(7 - b) + b \quad \{\text{using (1)}\}$$

$$\therefore -5 = 21 - 3b + b$$

$$\therefore 2b = 26 \quad \text{and so } b = 13$$

$$\text{i.e., } a = 7 - 13 = -6$$

$$\text{i.e., } a = -6 \text{ and } b = 13$$

7 $f(x) = ax^2 + bx + c$, where $f(0) = 5$, $f(-2) = 21$ and $f(3) = -4$

When $f(0) = 5$,

$$5 = a(0)^2 + b(0) + c$$

$$\therefore 5 = c$$

$$\therefore c = 5 \quad \dots (1)$$

When $f(-2) = 21$,

$$21 = a(-2)^2 + b(-2) + c$$

$$= 4a - 2b + c$$

$$= 4a - 2b + 5 \quad \{\text{using (1)}\}$$

$$\therefore 4a - 2b = 16$$

$$\therefore 2a - b = 8 \quad \text{and so } b = 2a - 8 \quad \dots (2)$$

When $f(3) = -4$,

$$-4 = a(3)^2 + b(3) + c$$

$$\therefore -4 = 9a + 3b + c$$

$$\therefore -4 = 9a + 3b + 5 \quad \{\text{using (1)}\}$$

$$\therefore -4 = 9a + 3(2a - 8) + 5 \quad \{\text{using (2)}\}$$

$$\therefore -9 = 9a + 6a - 24$$

$$\therefore 15 = 15a$$

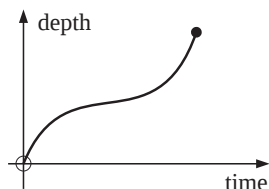
$$\therefore a = 1$$

Now, substituting $a = 1$ into (2) gives $b = 2(1) - 8 = -6$

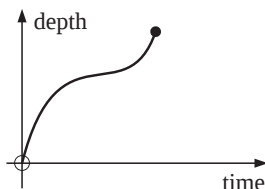
i.e., $a = 1$, $b = -6$, $c = 5$

8

a



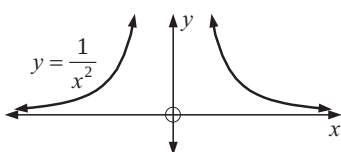
b



9

a $f(x) = \frac{1}{x^2}$ is meaningless when $x^2 = 0$
i.e., when $x = 0$

b



c domain of $f(x)$ is $\{x: x \neq 0\}$
range of $f(x)$ is $\{y: y > 0\}$

10

$$\begin{aligned} \text{a } f(g(x)) &= f(x^2 + 2) \\ &= 2(x^2 + 2) - 3 \\ &= 2x^2 + 4 - 3 \\ &= 2x^2 + 1 \end{aligned}$$

b

$$\begin{aligned} g(f(x)) &= g(2x - 3) \\ &= (2x - 3)^2 + 2 \\ &= 4x^2 - 12x + 9 + 2 \\ &= 4x^2 - 12x + 11 \end{aligned}$$

11

$$\begin{aligned} \text{a } (f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x}) \\ &= 1 - 2\sqrt{x} \end{aligned}$$

b

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(1 - 2x) \\ &= \sqrt{1 - 2x} \end{aligned}$$

12

$$\begin{aligned} \text{a } f(g(x)) &= \sqrt{1 - x^2} \\ &= f(1 - x^2) \\ \text{i.e., } f(x) &= \sqrt{x}, \\ g(x) &= 1 - x^2 \end{aligned}$$

b

$$\begin{aligned} g(f(x)) &= \left(\frac{x-2}{x+1}\right)^2 = g\left(\frac{x-2}{x+1}\right) \\ \text{i.e., } g(x) &= x^2, \\ f(x) &= \frac{x-2}{x+1} \end{aligned}$$

REVIEW SET 1B

1

$$f(x) = 5 - 2x$$

a

$$\begin{aligned} f(0) &= 5 - 2(0) \\ &= 5 \end{aligned}$$

b

$$\begin{aligned} f(5) &= 5 - 2(5) \\ &= -5 \end{aligned}$$

c

$$\begin{aligned} f(-3) &= 5 - 2(-3) \\ &= 11 \end{aligned}$$

d

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 5 - 2\left(\frac{1}{2}\right) \\ &= 4 \end{aligned}$$

2

$$g(x) = x^2 - 3x$$

a

$$\begin{aligned} g(x+1) &= (x+1)^2 - 3(x+1) \\ &= x^2 + 2x + 1 - 3x - 3 \\ &= x^2 - x - 2 \end{aligned}$$

b

$$\begin{aligned} g(x^2 - 2) &= (x^2 - 2)^2 - 3(x^2 - 2) \\ &= x^4 - 4x^2 + 4 - 3x^2 + 6 \\ &= x^4 - 7x^2 + 10 \end{aligned}$$

3 a $f(x) = 7 - 4x$
 i.e., $y = 7 - 4x$
 so $f^{-1}(x)$ is $x = 7 - 4y$
 $\therefore y = \frac{7-x}{4}$
 i.e., $f^{-1}(x) = \frac{7-x}{4}$

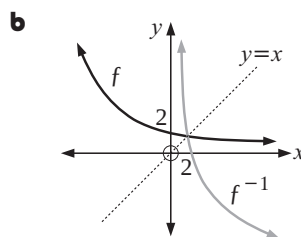
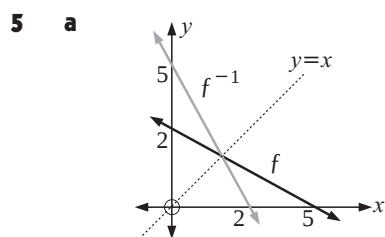
b $f(x) = \frac{3+2x}{5}$
 i.e., $y = \frac{3+2x}{5}$
 so $f^{-1}(x)$ is $x = \frac{3+2y}{5}$
 $\therefore 5x = 3 + 2y$
 $\therefore y = \frac{5x-3}{2}$
 i.e., $f^{-1}(x) = \frac{5x-3}{2}$

4 a $y = (x-1)(x-5)$
 i.e., x -intercepts are $x = 1$ and 5
 $\therefore$ vertex is at $x = 3$, $y = (3-1)(3-5)$
 $= 2 \times (-2)$
 $= -4$

i.e., vertex is at $(3, -4)$

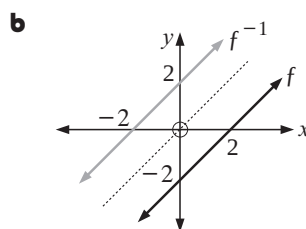
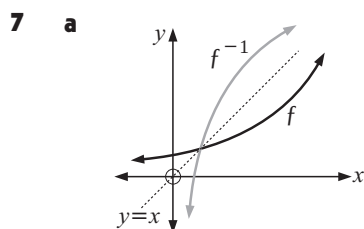
domain is $\{x: x \in \mathcal{R}\}$, range is $\{y: y \geq -4\}$

b From the graph, domain is $\{x: x \neq 0, 2\}$, range is $\{y: y \leq -1 \text{ or } y > 0\}$



6 a $f(x) = 4x + 2$
 i.e., $y = 4x + 2$
 so $f^{-1}(x)$ is $x = \frac{y-2}{4}$
 $\therefore y = \frac{x-2}{4}$
 i.e., $f^{-1}(x) = \frac{x-2}{4}$

b $f(x) = \frac{3-5x}{4}$
 i.e., $y = \frac{3-5x}{4}$
 so $f^{-1}(x)$ is $x = \frac{3-5y}{4}$
 $\therefore 4x = 3 - 5y$
 $\therefore y = \frac{3-4x}{5}$
 i.e., $f^{-1}(x) = \frac{3-4x}{5}$



8

$$f(x) = 2x + 11$$

$$\text{i.e., } y = 2x + 11$$

$$\text{so } f^{-1}(x) \text{ is } x = 2y + 11$$

$$\therefore y = \frac{x - 11}{2}$$

$$\text{i.e., } f^{-1}(x) = \frac{x - 11}{2}$$

$$g(x) = x^2$$

$$(g \circ f^{-1})(x) = g(f^{-1}(x))$$

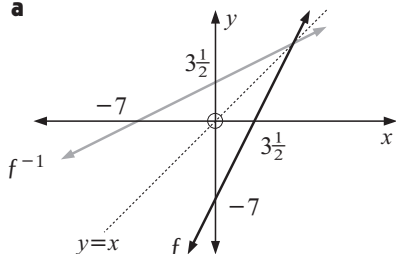
$$= g\left(\frac{x - 11}{2}\right)$$

$$= \left(\frac{x - 11}{2}\right)^2$$

$$\therefore (g \circ f^{-1})(3) = \left(\frac{3 - 11}{2}\right)^2$$

$$= (-4)^2$$

$$= 16$$

9 a**b** $f(x) = 2x - 7$ passes through $(0, -7)$ and $(\frac{7}{2}, 0)$

$$\therefore f^{-1}(x) \text{ passes through } (-7, 0) \text{ and } (0, \frac{7}{2})$$

$$f^{-1}(x) \text{ has slope } \frac{\frac{7}{2} - 0}{0 - (-7)} = \frac{\frac{7}{2}}{7} = \frac{1}{2}$$

$$\text{so its equation is } \frac{y - 0}{x - (-7)} = \frac{1}{2}$$

$$\therefore y = \frac{x + 7}{2}$$

$$\text{i.e., } f^{-1}(x) = \frac{x + 7}{2}$$

c

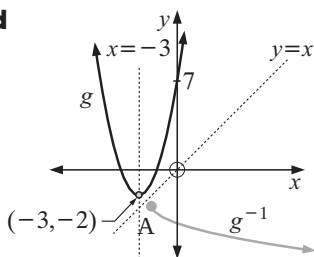
$$f(x) = 2x - 7$$

$$\text{i.e., } y = 2x - 7$$

$$\text{so } f^{-1}(x) \text{ is } x = 2y - 7$$

$$\therefore y = \frac{x + 7}{2}$$

$$\text{i.e., } f^{-1}(x) = \frac{x + 7}{2}$$

10 a, d**b** If $x \leq -3$, we have the graph to the left of $x = -3$, and any horizontal line through the graph cuts it no more than once.

Therefore it has an inverse function.

c

$$g(x) = x^2 + 6x + 7, \quad x \leq -3$$

$$\text{i.e., } y = x^2 + 6x + 7, \quad x \leq -3$$

$$\text{so } g^{-1}(x) \text{ is } x = y^2 + 6y + 7, \quad y \leq -3$$

$$= (y + 3)^2 - 9 + 7$$

$$\therefore x + 2 = (y + 3)^2$$

$$\therefore y + 3 = \pm\sqrt{x + 2}$$

$$\therefore y = -3 \pm \sqrt{x + 2}$$

$$\text{but } y \leq -3, \text{ so } y = -3 - \sqrt{x + 2}$$

$$\begin{aligned}
 \mathbf{11} \quad & h(x) = (x-4)^2 + 3, \quad x \geq 4 \\
 & \text{i.e., } y = (x-4)^2 + 3, \quad x \geq 4 \\
 \text{so } h^{-1}(x) \text{ is } & x = (y-4)^2 + 3, \quad y \geq 4 \\
 & \therefore x-3 = (y-4)^2 \\
 & \therefore y-4 = \pm\sqrt{x-3} \\
 & \therefore y = 4 \pm \sqrt{x-3} \\
 \text{but } y \geq 4, \text{ so } & y = 4 + \sqrt{x-3} \\
 \text{i.e., } h^{-1}(x) = & 4 + \sqrt{x-3}, \quad x \geq 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{12} \quad & f(x) = 3x+6 & h(x) = \frac{x}{3} \\
 & \text{i.e., } y = 3x+6 & \text{i.e., } y = \frac{x}{3} \\
 \text{so } f^{-1}(x) \text{ is } & x = 3y+6 & \text{so } h^{-1}(x) \text{ is } x = \frac{y}{3} \\
 & \therefore y = \frac{x-6}{3} & \therefore y = 3x \\
 \text{i.e., } f^{-1}(x) = & \frac{x-6}{3} & \text{i.e., } h^{-1}(x) = 3x
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } (f^{-1} \circ h^{-1})(x) &= f^{-1}(h^{-1}(x)) & (h \circ f)(x) &= h(f(x)) \\
 &= f^{-1}(3x) & &= h(3x+6) \\
 &= \frac{3x-6}{3} & &= \frac{3x+6}{3} \\
 &= x-2 & \text{i.e., } y &= x+2 \\
 & & \text{i.e., } (h \circ f)^{-1}(x) \text{ is } x &= y+2 \\
 & & \therefore y &= x-2 \\
 & & \text{i.e., } (h \circ f)^{-1}(x) &= x-2
 \end{aligned}$$

$$\text{i.e., } (f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) \quad \text{as required}$$

Chapter 2

SEQUENCES AND SERIES

EXERCISE 2A

- 1** **a** 4, 13, 22, 31, **b** 45, 39, 33, 27, **c** 2, 6, 18, 54, **d** 96, 48, 24, 12,
- 2** **a** The sequence starts at 8 and each term is 8 more than the previous term. The next two terms are 40 and 48.
b The sequence starts at 2 and each term is 3 more than the previous term. The next two terms are 14 and 17.
c The sequence starts at 36 and each term is 5 less than the previous term. The next two terms are 16 and 11.
d The sequence starts at 96 and each term is 7 less than the previous term. The next two terms are 68 and 61.
e The sequence starts at 1 and each term is 4 times the previous term. The next two terms are 256 and 1024.
f The sequence starts at 2 and each term is 3 times the previous term. The next two terms are 162 and 486.
g The sequence starts at 480 and each term is half the previous term. The next two terms are 30 and 15.
h The sequence starts at 243 and each term is one third of the previous term. The next two terms are 3 and 1.
i The sequence starts at 50 000 and each term is one fifth of the previous term. The next two terms are 80 and 16.
- 3** **a** 79, 75 **b** 1280, 5120 **c** 81, 90
- 4** **a** Each term is the square of the number of the term. The next three terms are 25, 36 and 49.
b Each term is the cube of the number of the term. The next three terms are 125, 216 and 343.
c Each term is $n \times (n + 1)$ where n is the number of the term. The next three terms are 30, 42 and 56.
- 5** **a** 625, 1296 **b** 13, 21 **c** 9, 11 **d** 13, 17 (primes) **e** 16, 22 **f** 14, 18

EXERCISE 2B

- 1** **a** $\{2n\}$ generates the sequence 2, 4, 6, 8, 10, (letting $n = 1, 2, 3, 4, 5, \dots$)
b $\{2n + 2\}$ generates the sequence 4, 6, 8, 10, 12, (letting $n = 1, 2, 3, 4, 5, \dots$)
c $\{2n - 1\}$ generates the sequence 1, 3, 5, 7, 9, (letting $n = 1, 2, 3, 4, 5, \dots$)
d $\{2n - 3\}$ generates the sequence -1, 1, 3, 5, 7, (letting $n = 1, 2, 3, 4, 5, \dots$)
e $\{2n + 3\}$ generates the sequence 5, 7, 9, 11, 13, (letting $n = 1, 2, 3, 4, 5, \dots$)
f $\{2n + 11\}$ generates the sequence 13, 15, 17, 19, 21, (letting $n = 1, 2, 3, 4, 5, \dots$)
g $\{3n + 1\}$ generates the sequence 4, 7, 10, 13, 16, (letting $n = 1, 2, 3, 4, 5, \dots$)
h $\{4n - 3\}$ generates the sequence 1, 5, 9, 13, 17, (letting $n = 1, 2, 3, 4, 5, \dots$)
i $\{5n + 4\}$ generates the sequence 9, 14, 19, 24, 29, (letting $n = 1, 2, 3, 4, 5, \dots$)
- 2** **a** $\{2^n\}$ generates the sequence 2, 4, 8, 16, 32, (letting $n = 1, 2, 3, 4, 5, \dots$)
b $\{3 \times 2^n\}$ generates the sequence 6, 12, 24, 48, 96, (letting $n = 1, 2, 3, 4, 5, \dots$)
c $\{6 \times (\frac{1}{2})^n\}$ generates the sequence 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$, (letting $n = 1, 2, 3, 4, 5, \dots$)
d $\{(-2)^n\}$ generates the sequence -2, 4, -8, 16, -32, (letting $n = 1, 2, 3, 4, 5, \dots$)

3 $\{15 - (-2)^n\}$ generates the sequence with first five terms:

$$\begin{aligned} t_1 &= 15 - (-2)^1 = 17, & t_2 &= 15 - (-2)^2 = 11, & t_3 &= 15 - (-2)^3 = 23, \\ t_4 &= 15 - (-2)^4 = -1, & t_5 &= 15 - (-2)^5 = 47 \end{aligned}$$

EXERCISE 2C

1 a $17 - 6 = 11$ So, assuming that the pattern continues, consecutive terms differ by 11.
 $28 - 17 = 11$ $\therefore$ the sequence is arithmetic with $u_1 = 6$, $d = 11$.
 $39 - 28 = 11$
 $50 - 39 = 11$

b $u_n = u_1 + (n-1)d$ **c** $u_{50} = 11(50) - 5$ **d** Let $u_n = 325 = 11n - 5$
 $= 6 + (n-1)11$ $= 545$ $\therefore 330 = 11n$
i.e., $u_n = 11n - 5$ $\therefore n = 30$

So, 325 is a member, i.e., u_{30} .

e Let $u_n = 761 = 11n - 5$
 $\therefore 766 = 11n$
 $\therefore n = 69\frac{7}{11}$, but n is an integer, so 761 is not a member of the sequence.

2 a $83 - 87 = -4$ So, assuming that the pattern continues, consecutive terms
 $79 - 83 = -4$ differ by -4 .
 $75 - 79 = -4$ $\therefore$ the sequence is arithmetic with $u_1 = 87$, $d = -4$.

b $u_n = u_1 + (n-1)d$ **c** $u_{40} = 91 - 4(40)$ **d** Let $u_n = -143 = 91 - 4n$
 $= 87 + (n-1)(-4)$ $= 91 - 160$ $\therefore 4n = 234$
 $= 87 - 4n + 4$ $= -69$ $\therefore n = 58\frac{1}{2}$
i.e., $u_n = 91 - 4n$ but n is an integer, so
 -143 is not a member
of the sequence.

3 a $u_n = 3n - 2$ $u_1 = 3(1) - 2 = 1$
 $u_{n+1} = 3(n+1) - 2$
 $= 3n + 1$
 $u_{n+1} - u_n = (3n+1) - (3n-2)$
 $= 3$ which is constant

So, assuming that the pattern continues,
consecutive terms differ by 3.

$\therefore$ the sequence is arithmetic with $u_1 = 1$
and $d = 3$.

b $u_1 = 1$, $d = 3$ **c** $u_{57} = 3(57) - 2$
 $= 169$

d Let $u_n = 450 = 3n - 2$
i.e., $3n = 452$
 $\therefore n = 150\frac{2}{3}$

So, try the two values on either side of $n = 150\frac{2}{3}$, i.e., for $n = 150$ and $n = 151$:

$$\begin{aligned} u_{150} &= 3(150) - 2 & \text{and} & & u_{151} &= 3(151) - 2 \\ &= 448 & & & &= 451 \end{aligned}$$

i.e., $u_{151} = 451$ is the least term which is greater than 450.

$$4 \quad a \quad u_n = \frac{71 - 7n}{2} = 35\frac{1}{2} - \frac{7}{2}n \quad u_1 = \frac{71 - 7(1)}{2} = 32$$

$$u_{n+1} = \frac{71 - 7(n+1)}{2}$$

$$= \frac{71 - 7n - 7}{2}$$

$$= \frac{64 - 7n}{2}$$

$$= 32 - \frac{7}{2}n$$

$$u_{n+1} - u_n = (32 - \frac{7}{2}n) - (35\frac{1}{2} - \frac{7}{2}n)$$

$$= -\frac{7}{2} \text{ which is constant}$$

So, assuming that the pattern continues,
consecutive terms differ by $-\frac{7}{2}$.

$\therefore$ the sequence is arithmetic with

$$u_1 = 32, \quad d = -\frac{7}{2}.$$

$$b \quad u_1 = 32, \quad d = -\frac{7}{2} \quad c \quad u_{75} = \frac{71 - 7(75)}{2} = -227$$

$$d \quad \text{Let } u_n = -200 = \frac{71 - 7n}{2} \quad \text{i.e., } -400 = 71 - 7n$$

$$\therefore 7n = 471$$

$$\therefore n = 67\frac{2}{7}$$

So, try the two values on either side of $n = 67\frac{2}{7}$, i.e., for $n = 67$ and $n = 68$:

$$u_{67} = \frac{71 - 7(67)}{2} = -199 \quad \text{and} \quad u_{68} = \frac{71 - 7(68)}{2} = -202\frac{1}{2}$$

i.e., terms of the sequence are less than -200 for $n \geq 68$.

5 a Since the terms are consecutive,

$$k - 32 = 3 - k$$

{equating common differences}

$$\therefore 2k = 35$$

$$\therefore k = 17\frac{1}{2}$$

b Since the terms are consecutive,

$$7 - k = 10 - 7$$

{equating common differences}

$$\therefore 7 - k = 3$$

$$\therefore k = 4$$

c Since the terms are consecutive,

$$(2k + 1) - (k + 1) = 13 - (2k + 1)$$

{equating common differences}

$$\therefore k = 12 - 2k$$

$$\therefore 3k = 12$$

$$\therefore k = 4$$

d Since the terms are consecutive,

$$(2k + 3) - (k - 1) = (7 - k) - (2k + 3)$$

{equating common differences}

$$\therefore k + 4 = 4 - 3k$$

$$\therefore 4k = 0$$

$$\therefore k = 0$$

e Since the terms are consecutive,

$$k^2 - k = k^2 + 6 - k^2$$

{equating common differences}

$$\therefore k^2 - k = 6$$

$$\therefore k^2 - k - 6 = 0$$

$$\therefore (k - 3)(k + 2) = 0$$

$$\therefore k = -2 \text{ or } 3$$

f Since the terms are consecutive,

$$k - 5 = k^2 - 8 - k$$

{equating common differences}

$$\therefore k^2 - 2k - 3 = 0$$

$$\therefore (k - 3)(k + 1) = 0$$

$$\therefore k = -1 \text{ or } 3$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad u_7 &= 41 \quad \therefore u_1 + 6d = 41 \quad \dots (1) \\ u_{13} &= 77 \quad \therefore u_1 + 12d = 77 \quad \dots (2) \end{aligned}$$

Solving simultaneously,

$$\begin{array}{rcl} -u_1 - 6d & = & -41 \\ u_1 + 12d & = & 77 \\ \hline \therefore 6d & = & 36 \quad \{\text{adding the} \\ \therefore d & = & 6 \quad \text{equations}\} \end{array}$$

$$\text{So in (1), } u_1 + 6(6) = 41$$

$$\therefore u_1 + 36 = 41$$

$$\therefore u_1 = 5$$

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore u_n = 5 + (n-1)6$$

$$\therefore u_n = 5 + 6n - 6$$

$$\therefore u_n = 6n - 1$$

$$\begin{aligned} \mathbf{b} \quad u_5 &= -2 \quad \therefore u_1 + 4d = -2 \quad \dots (1) \\ u_{12} &= -12\frac{1}{2} \quad \therefore u_1 + 11d = -12\frac{1}{2} \quad \dots (2) \end{aligned}$$

Solving simultaneously,

$$\begin{array}{rcl} -u_1 - 4d & = & 2 \\ u_1 + 11d & = & -12\frac{1}{2} \\ \hline \therefore 7d & = & -10\frac{1}{2} \quad \{\text{adding the} \\ \therefore d & = & -\frac{3}{2} \quad \text{equations}\} \end{array}$$

$$\text{So in (1), } u_1 + 4(-\frac{3}{2}) = -2$$

$$\therefore u_1 - 6 = -2$$

$$\therefore u_1 = 4$$

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore u_n = 4 + (n-1)(-\frac{3}{2})$$

$$\therefore u_n = 4 - \frac{3}{2}n + \frac{3}{2}$$

$$\therefore u_n = -\frac{3}{2}n + \frac{11}{2}$$

$$\begin{aligned} \mathbf{c} \quad u_7 &= 1 \quad \therefore u_1 + 6d = 1 \quad \dots (1) \\ u_{15} &= -39 \quad \therefore u_1 + 14d = -39 \quad \dots (2) \end{aligned}$$

Solving simultaneously,

$$\begin{array}{rcl} -u_1 - 6d & = & -1 \\ u_1 + 14d & = & -39 \\ \hline \therefore 8d & = & -40 \quad \{\text{adding the} \\ \therefore d & = & -5 \quad \text{equations}\} \end{array}$$

$$\text{So in (1), } u_1 + 6(-5) = 1$$

$$\therefore u_1 - 30 = 1$$

$$\therefore u_1 = 31$$

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore u_n = 31 + (n-1)(-5)$$

$$\therefore u_n = 31 - 5n + 5$$

$$\therefore u_n = -5n + 36$$

$$\begin{aligned} \mathbf{d} \quad u_{11} &= -16 \quad \therefore u_1 + 10d = -16 \quad \dots (1) \\ u_8 &= -11\frac{1}{2} \quad \therefore u_1 + 7d = -11\frac{1}{2} \quad \dots (2) \end{aligned}$$

Solving simultaneously,

$$\begin{array}{rcl} -u_1 - 10d & = & 16 \\ u_1 + 7d & = & -11\frac{1}{2} \\ \hline \therefore -3d & = & 4\frac{1}{2} \quad \{\text{adding the} \\ \therefore d & = & -\frac{3}{2} \quad \text{equations}\} \end{array}$$

$$\text{So in (1), } u_1 + 10(-\frac{3}{2}) = -16$$

$$\therefore u_1 - 15 = -16$$

$$\therefore u_1 = -1$$

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore u_n = -1 + (n-1)(-\frac{3}{2})$$

$$\therefore u_n = -1 - \frac{3}{2}n + \frac{3}{2}$$

$$\therefore u_n = -\frac{3}{2}n + \frac{1}{2}$$

$$\mathbf{7} \quad \mathbf{a} \quad \text{Let the numbers be } 5, 5 + d, 5 + 2d, 5 + 3d, 10.$$

$$\text{Then } 5 + 4d = 10$$

$$\therefore 4d = 5$$

$$\therefore d = \frac{5}{4} = 1\frac{1}{4}$$

$$\text{So the numbers are } 5, 6\frac{1}{4}, 7\frac{1}{2}, 8\frac{3}{4}, 10.$$

$$\mathbf{b} \quad \text{Let the numbers be } -1, -1 + d, -1 + 2d, -1 + 3d, -1 + 4d, -1 + 5d, -1 + 6d, 32.$$

$$\text{Then } -1 + 7d = 32$$

$$\therefore 7d = 33$$

$$\therefore d = \frac{33}{7} = 4\frac{5}{7}$$

$$\text{So the numbers are } -1, 3\frac{5}{7}, 8\frac{3}{7}, 13\frac{1}{7}, 17\frac{6}{7}, 22\frac{4}{7}, 27\frac{2}{7}, 32.$$

$$\begin{aligned} 8 \quad a \quad u_1 &= 36, & 35\frac{1}{3} - 36 &= -\frac{2}{3} \\ & & 34\frac{2}{3} - 35\frac{1}{3} &= -\frac{2}{3}, & \text{so } d &= -\frac{2}{3} \end{aligned}$$

$$\begin{aligned} b \quad u_n &= u_1 + (n-1)d \\ \therefore -30 &= 36 + (n-1)\left(-\frac{2}{3}\right) & \{\text{letting } u_n &= -30, \text{ the last term of the sequence}\} \\ \therefore -66 &= -\frac{2}{3}n + \frac{2}{3} \\ \therefore \frac{2}{3}n &= 66\frac{2}{3} \\ \therefore n &= 100 & \text{i.e., the sequence has 100 terms} \end{aligned}$$

$$\begin{aligned} 9 \quad u_1 &= 23, & 36 - 23 &= 13 & \therefore u_n &= u_1 + (n-1)d \\ & & 49 - 36 &= 13 & \text{i.e., } u_n &= 23 + (n-1)13 \\ & & 62 - 49 &= 13, & \text{so } d &= 13 & = 23 + 13n - 13 \\ & & & & \therefore u_n &= 13n + 10 \end{aligned}$$

$$\begin{aligned} \text{Let } u_n &= 100\,000 = 13n + 10 \\ \therefore 99\,990 &= 13n \\ \therefore n &= 7691\frac{7}{13} \end{aligned}$$

So, try the two values on either side of $n = 7691\frac{7}{13}$, i.e., for $n = 7691$ and $n = 7692$:

$$\begin{aligned} \text{i.e., } u_{7691} &= 13(7691) + 10 & \text{and } u_{7692} &= 13(7692) + 10 \\ &= 99\,993 & &= 100\,006 \end{aligned}$$

i.e., the first term to exceed 100 000 is $u_{7692} = 100\,006$.

EXERCISE 2D

$$1 \quad a \quad \frac{6}{2} = 3 \quad \therefore r = 3, \quad u_1 = 2 \quad \therefore b = 6 \times 3 = 18 \quad \text{and} \quad c = 18 \times 3 = 54$$

$$b \quad \frac{5}{10} = \frac{1}{2} \quad \therefore r = \frac{1}{2}, \quad u_1 = 10 \quad \therefore b = 5 \times \frac{1}{2} = 2\frac{1}{2} \quad \text{and} \quad c = 2\frac{1}{2} \times \frac{1}{2} = 1\frac{1}{4}$$

$$c \quad \frac{-6}{12} = -\frac{1}{2} \quad \therefore r = -\frac{1}{2}, \quad u_1 = 12 \quad \therefore b = -6 \times -\frac{1}{2} = 3 \quad \text{and} \quad c = 3 \times -\frac{1}{2} = -1\frac{1}{2}$$

$$2 \quad a \quad \frac{10}{5} = 2 \quad \text{So, assuming the pattern continues, consecutive terms have a common ratio of 2.}$$

$$\frac{20}{10} = 2 \quad \therefore \text{the sequence is geometric with } u_1 = 5 \quad \text{and} \quad r = 2.$$

$$\frac{40}{20} = 2$$

$$\begin{aligned} b \quad u_n &= u_1 r^{n-1} \\ \therefore u_n &= 5 \times 2^{n-1} \\ \text{so } u_{15} &= 5 \times 2^{14} \\ &= 81\,920 \end{aligned}$$

$$3 \quad a \quad \frac{-6}{12} = -\frac{1}{2} \quad \text{So, assuming the pattern continues, consecutive terms have a common ratio of } -\frac{1}{2}.$$

$$\frac{3}{-6} = -\frac{1}{2} \quad \therefore \text{the sequence is geometric with } u_1 = 12 \quad \text{and} \quad r = -\frac{1}{2}.$$

$$\frac{-1.5}{3} = \frac{(-\frac{3}{2})}{3} = -\frac{1}{2}$$

$$\begin{aligned}
 \text{b} \quad u_n &= u_1 r^{n-1} & \text{so } u_{13} &= 3 \times 2^{-13+3} \\
 \therefore u_n &= 12(2^{-1})^{n-1} & &= 3 \times 2^{-10} \\
 &= 12 \times 2^{-n+1} & &= 3 \times \frac{1}{1024} \\
 &= 3 \times 2^2 \times 2^{-n+1} & &= \frac{3}{1024} \\
 &= 3 \times 2^{-n+3}
 \end{aligned}$$

$$\begin{aligned}
 \text{4} \quad \frac{-6}{8} &= -\frac{3}{4} & \text{So, assuming the pattern continues, consecutive terms have a} \\
 \frac{4.5}{-6} &= -\left(\frac{9}{2}\right) = -\frac{3}{4} & \text{common ratio of } -\frac{3}{4}. \\
 \therefore \text{the sequence is geometric with } u_1 = 8 \text{ and } r = -\frac{3}{4}.
 \end{aligned}$$

$$\frac{-3.375}{4.5} = \frac{\left(-\frac{27}{8}\right)}{\left(\frac{9}{2}\right)} = -\frac{3}{4}$$

$$u_n = u_1 r^{n-1} = 8 \times \left(-\frac{3}{4}\right)^{n-1} \quad \text{i.e., } u_{10} = 8 \times \left(-\frac{3}{4}\right)^9 \div -0.600\,677\,49$$

$$\begin{aligned}
 \text{5} \quad \frac{4\sqrt{2}}{8} &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} & \text{So, assuming the pattern continues, consecutive terms have a} \\
 \frac{4}{4\sqrt{2}} &= \frac{1}{\sqrt{2}} & \text{common ratio of } \frac{1}{\sqrt{2}}. \\
 \therefore \text{the sequence is geometric with } u_1 = 8 \text{ and } r = \frac{1}{\sqrt{2}}.
 \end{aligned}$$

$$\frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$u_n = u_1 r^{n-1} = 8 \left(\frac{1}{\sqrt{2}}\right)^{n-1} = 2^3 \times \left(2^{-\frac{1}{2}}\right)^{n-1} = 2^3 \times 2^{-\frac{1}{2}n + \frac{1}{2}}$$

$$\text{i.e., } u_n = 2^{\frac{7}{2} - \frac{n}{2}}$$

6 a Since the terms are geometric,

$$\frac{k}{7} = \frac{28}{k} \quad \therefore k^2 = 196$$

$$\therefore k = \pm 14$$

c Since the terms are geometric,

$$\frac{k+8}{k} = \frac{9k}{k+8}$$

$$\therefore (k+8)^2 = 9k^2$$

$$\therefore k^2 + 16k + 64 = 9k^2$$

$$\therefore 8k^2 - 16k - 64 = 0$$

$$\therefore k^2 - 2k - 8 = 0$$

$$\therefore (k+2)(k-4) = 0 \quad \text{and so } k = -2 \text{ or } 4$$

b Since the terms are geometric,

$$\frac{3k}{k} = \frac{20-k}{3k} = 3$$

$$\therefore 20 - k = 9k$$

$$\therefore 20 = 10k$$

$$\therefore k = 2$$

$$\text{7 a } u_4 = 24 \quad \therefore u_1 \times r^3 = 24 \quad \dots (1)$$

$$u_7 = 192 \quad \therefore u_1 \times r^6 = 192 \quad \dots (2)$$

$$\text{So, } \frac{u_1 r^6}{u_1 r^3} = \frac{192}{24} \quad \{(2) \div (1)\}$$

$$\therefore r^3 = 8 \quad \therefore r = 2$$

$$\text{So in (1), } u_1 \times 2^3 = 24$$

$$\therefore u_1 \times 8 = 24$$

$$\therefore u_1 = 3$$

$$\therefore u_n = 3 \times 2^{n-1}$$

$$\text{b } u_3 = 8 \quad \therefore u_1 \times r^2 = 8 \quad \dots (1)$$

$$u_6 = -1 \quad \therefore u_1 \times r^5 = -1 \quad \dots (2)$$

$$\text{So, } \frac{u_1 r^5}{u_1 r^2} = -\frac{1}{8} \quad \{(2) \div (1)\}$$

$$\therefore r^3 = -\frac{1}{8} \quad \therefore r = -\frac{1}{2}$$

$$\text{So in (1), } u_1 \times \left(-\frac{1}{2}\right)^2 = 8$$

$$\therefore u_1 \times \frac{1}{4} = 8$$

$$\therefore u_1 = 32$$

$$\therefore u_n = 32 \times \left(-\frac{1}{2}\right)^{n-1}$$

$$\begin{aligned} \text{c} \quad u_7 &= 24 \quad \therefore u_1 \times r^6 = 24 \quad \dots (1) \\ u_{15} &= 384 \quad \therefore u_1 \times r^{14} = 384 \quad \dots (2) \end{aligned}$$

$$\text{So, } \frac{u_1 r^{14}}{u_1 r^6} = \frac{384}{24} \quad \{(2) \div (1)\}$$

$$\therefore r^8 = 16 \quad \therefore r = \sqrt{2}$$

$$\text{So in (1), } u_1 \times (\sqrt{2})^6 = 24$$

$$\therefore u_1 \times 8 = 24$$

$$\therefore u_1 = 3$$

$$\text{Now } u_n = u_1 r^{n-1}$$

$$\therefore u_n = 3 \times (\sqrt{2})^{n-1}$$

$$\begin{aligned} \text{d} \quad u_3 &= 5 \quad \therefore u_1 \times r^2 = 5 \quad \dots (1) \\ u_7 &= \frac{5}{4} \quad \therefore u_1 \times r^6 = \frac{5}{4} \quad \dots (2) \end{aligned}$$

$$\text{So, } \frac{u_1 r^6}{u_1 r^2} = \frac{(\frac{5}{4})}{5} \quad \{(2) \div (1)\}$$

$$\therefore r^4 = \frac{1}{4} \quad \therefore r = \frac{1}{\sqrt{2}}$$

$$\text{So in (1), } u_1 \times \left(\frac{1}{\sqrt{2}}\right)^2 = 5$$

$$\therefore u_1 \times \frac{1}{2} = 5$$

$$\therefore u_1 = 10$$

$$\text{Now } u_n = u_1 r^{n-1}$$

$$\therefore u_n = 10 \times \left(\frac{1}{\sqrt{2}}\right)^{n-1}$$

$$\therefore u_n = 10 \times (\sqrt{2})^{1-n}$$

8 a 2, 6, 18, 54, i.e., $u_1 = 2$ and $r = 3$

$$u_n = u_1 r^{n-1} \quad \therefore u_n = 2 \times 3^{n-1}$$

$$\text{Let } u_n = 10\,000 = 2 \times 3^{n-1} \quad \text{i.e., } 5000 = 3^{n-1}$$

$$\therefore n \div 8.7527 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n = 8.7527$, i.e., for $n = 8$ and $n = 9$:

$$\begin{aligned} u_8 &= 2 \times 3^7 & \text{and} & & u_9 &= 2 \times 3^8 \\ &= 4374 & & & &= 13\,122 \end{aligned}$$

i.e., the first term to exceed 10 000 is $u_9 = 13\,122$.

b 4, $4\sqrt{3}$, 12, $12\sqrt{3}$, i.e., $u_1 = 4$ and $r = \sqrt{3}$

$$u_n = u_1 r^{n-1} \quad \therefore u_n = 4 \times (\sqrt{3})^{n-1}$$

$$\text{Let } u_n = 4800 \quad \text{i.e., } 1200 = (\sqrt{3})^{n-1}$$

$$\therefore n \div 13.91 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 13.91$, i.e., for $n = 13$ and $n = 14$:

$$\begin{aligned} u_{13} &= 4 \times (\sqrt{3})^{12} & \text{and} & & u_{14} &= 4 \times (\sqrt{3})^{13} \\ &= 2916 & & & &\div 5050.66 \end{aligned}$$

i.e., the first term to exceed 4800 is $u_{14} \div 5050.66$.

c 12, 6, 3, 1.5, i.e., $u_1 = 12$ and $r = \frac{1}{2}$

$$u_n = u_1 r^{n-1} \quad \therefore u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$$

$$\text{Let } 0.0001 = u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$$

$$\text{i.e., } 0.000\,008\bar{3} = \left(\frac{1}{2}\right)^{n-1}$$

$$\therefore n \div 17.87 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 17.87$, i.e., for $n = 17$ and $n = 18$:

$$\begin{aligned} u_{17} &= 12 \times \left(\frac{1}{2}\right)^{16} & \text{and} & & u_{18} &= 12 \times \left(\frac{1}{2}\right)^{17} \\ &\div 0.000\,1831 & & & &\div 0.000\,091\,55 \end{aligned}$$

i.e., the first term of the sequence which is less than 0.0001 is $u_{18} \div 0.000\,091\,55$.

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_1 = 3000, \quad r = 1.1, \quad n = 3 \\ \text{i.e., } u_4 &= 3000 \times (1.1)^3 \\ &= 3993 \end{aligned}$$

So it amounts to \$3993.

$$\begin{aligned} \mathbf{b} \quad \text{Interest} &= \text{amount after 3 years} - \text{initial amount} \\ &= \$3993 - \$3000 \\ &= \$993 \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_1 = 20\,000, \quad r = 1.12, \quad n = 4 \\ \text{i.e., } u_5 &= 20\,000 \times (1.12)^4 \\ &= 31\,470.39 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= 31\,470.39 - 20\,000 \text{ Euro} \\ &= 11\,470.39 \text{ Euro} \end{aligned}$$

$$\begin{aligned} \mathbf{11} \quad \mathbf{a} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_1 = 30\,000, \quad r = 1.1, \quad n = 4 \\ \text{i.e., } u_5 &= 30\,000 \times (1.1)^4 \\ &= 43\,923, \text{ i.e., the investment amounts to } 43\,923 \text{ Yen.} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{Interest} &= \text{amount after 4 years} - \text{initial amount} \\ &= 43\,923 - 30\,000 \text{ Yen} \\ &= 13\,923 \text{ Yen} \end{aligned}$$

$$\begin{aligned} \mathbf{12} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_1 = 80\,000, \quad r = 1.09, \quad n = 3 \\ \text{i.e., } u_4 &= 80\,000 \times (1.09)^3 \\ &= 103\,602.32 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= \text{amount after 3 years} - \text{initial amount} \\ &= \$103\,602.32 - \$80\,000 \\ &= \$23\,602.32 \end{aligned}$$

$$\begin{aligned} \mathbf{13} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_1 = 100\,000, \quad r = 1 + \frac{0.08}{2} = 1.04, \quad n = 10 \\ \text{i.e., } u_{11} &= 100\,000 \times (1.04)^{10} \\ &= 148\,024.43, \text{ i.e., it amounts to } 148\,024.43 \text{ Yen.} \end{aligned}$$

$$\begin{aligned} \mathbf{14} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_1 = 45\,000, \quad r = 1 + \frac{0.075}{4} = 1.01875, \\ \text{i.e., } u_{10} &= 45\,000 \times (1.01875)^7 \quad n = 7 \\ &= 51\,249.06 \\ \text{i.e., it amounts to } &\text{£}51\,249.06 \end{aligned}$$

$$\begin{aligned} \mathbf{15} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_{n+1} = 20\,000, \quad r = 1.075, \quad n = 4 \\ \therefore 20\,000 &= u_1 \times (1.075)^4 \\ \therefore u_1 &= 14\,976.01 \end{aligned}$$

So, \$14 976.01 should be invested now.

$$\begin{aligned} \mathbf{16} \quad u_{n+1} &= u_1 \times r^n, \quad \text{where } u_{n+1} = 15\,000, \quad r = 1.055, \quad n = \frac{60}{12} = 5 \\ \therefore 15\,000 &= u_1 \times (1.055)^5 \\ \therefore u_1 &= 11\,477.02 \end{aligned}$$

So, the initial investment required is £11 477.02.

- 17** $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 25\,000$, $r = 1 + \frac{0.08}{4} = 1.02$, $n = 3 \times 4 = 12$
 $\therefore 25\,000 = u_1 \times (1.02)^{12}$
 $\therefore u_1 = 19\,712.33$
 i.e., should invest 19 712.33 Euro now.

- 18** $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 40\,000$, $r = 1 + \frac{0.09}{12} = 1.0075$, $n = 8 \times 12 = 96$
 $\therefore 40\,000 = u_1 \times (1.0075)^{96}$
 $\therefore u_1 = 19\,522.47$, i.e., initial investment should be 19 522.47 Yen.

- 19** $u_{n+1} = u_1 \times r^n$, where $u_1 = 500$, $r = 1.12$
- a i** $u_{11} = 500 \times (1.12)^{10}$
 $\div 1552.92$
 $\div 1550$ ants
- ii** $u_{21} = 500 \times (1.12)^{20}$
 $\div 4823.15$
 $\div 4820$ ants
- b** For the population to reach 2000,
 $u_{n+1} = 500 \times (1.12)^n = 2000$
 $\therefore (1.12)^n = 4$
 $\therefore n \div 12.23$ {using technology}
 i.e., it will take approximately 12.2 weeks.

- 20** $u_{n+1} = u_1 \times r^n$, where $u_1 = 555$, $r = 0.955$
- a** $u_{16} = 555 \times (0.955)^{15}$
 $\div 278.19$ i.e., the population is approximately 278 animals in the year 2000.
- b** For the population to have declined to 50,
 $u_{n+1} = 555 \times (0.955)^n = 50$
 $\therefore (0.955)^n = 0.0900$
 $\therefore n \div 52.3$ {using technology}
 So, after approximately 52 years the population is 50, i.e., in the year 2037.

EXERCISE 2E.1

- 1 a i** 3, 11, 19, 27, so $u_1 = 3$, $d = 8$, i.e., $u_n = 3 + (n-1)8$
 $= 8n - 5$
 i.e., $S_n = 3 + 11 + 19 + 27 + \dots + (8n - 5)$
- ii** $S_5 = 3 + 11 + 19 + 27 + 35 = 95$
- b i** 42, 37, 32, 27, so $u_1 = 42$, $d = -5$, i.e., $u_n = 42 + (n-1)(-5)$
 $= 47 - 5n$
 i.e., $S_n = 42 + 37 + 32 + 27 + \dots + (47 - 5n)$
- ii** $S_5 = 42 + 37 + 32 + 27 + 22 = 160$
- c i** 12, 6, 3, $1\frac{1}{2}$, so $u_1 = 12$, $r = \frac{1}{2}$, i.e., $u_n = 12 \times (\frac{1}{2})^{n-1}$
 i.e., $S_n = 12 + 6 + 3 + 1\frac{1}{2} + \dots + 12(\frac{1}{2})^{n-1}$
- ii** $S_5 = 12 + 6 + 3 + 1\frac{1}{2} + \frac{3}{4} = 23\frac{1}{4}$

- d i** $2, 3, 4\frac{1}{2}, 6\frac{3}{4}, \dots$ so $u_1 = 2, r = \frac{3}{2}$, i.e., $u_n = 2 \times (\frac{3}{2})^{n-1}$
 i.e., $S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + \dots + 2(\frac{3}{2})^{n-1}$
- ii** $S_5 = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + 10\frac{1}{8} = 26\frac{3}{8}$
- e i** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ so $u_1 = 1, r = \frac{1}{2}$, i.e., $u_n = 1 \times (\frac{1}{2})^{n-1}$
 $= 2^{1-n}$
 i.e., $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + 2^{1-n}$
- ii** $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$
- f i** $1, 8, 27, 64, \dots$
 i.e., $S_n = 1 + 8 + 27 + 64 + \dots + n^3$ {since $1 = 1^3, 8 = 2^3, 27 = 3^3, 64 = 4^3$ }
- ii** $S_5 = 1 + 8 + 27 + 64 + 125 = 225$

EXERCISE 2E.2

- 1 a** The series is arithmetic with
 $u_1 = 3, d = 4, n = 20$
 $S_n = \frac{n}{2}(2u_1 + (n-1)d)$
 So, $S_{20} = \frac{20}{2}(2 \times 3 + 19 \times 4)$
 $= 10(6 + 76)$
 $= 820$
- b** The series is arithmetic with
 $u_1 = \frac{1}{2}, d = 2\frac{1}{2}, n = 50$
 $S_n = \frac{n}{2}(2u_1 + (n-1)d)$
 So, $S_{50} = \frac{50}{2}(2 \times \frac{1}{2} + 49 \times \frac{5}{2})$
 $= 25(1 + 122\frac{1}{2})$
 $= 3087\frac{1}{2}$
- c** The series is arithmetic with
 $u_1 = 100, d = -7, n = 40$
 $S_n = \frac{n}{2}(2u_1 + (n-1)d)$
 So, $S_{40} = \frac{40}{2}(2 \times 100 + 39 \times (-7))$
 $= 20(200 - 273)$
 $= -1460$
- d** The series is arithmetic with
 $u_1 = 50, d = -1\frac{1}{2}, n = 80$
 $S_n = \frac{n}{2}(2u_1 + (n-1)d)$
 So, $S_{80} = \frac{80}{2}(2 \times 50 + 79 \times (-\frac{3}{2}))$
 $= 40(100 - \frac{237}{2})$
 $= -740$
- 2 a** The series is arithmetic with
 $u_1 = 5, d = 3, u_n = 101$
 Since $u_n = 101$,
 then $u_1 + (n-1)d = 101$
 $\therefore 5 + 3(n-1) = 101$
 $\therefore 5 + 3n - 3 = 101$
 $\therefore 3n - 3 = 96$
 $\therefore 3n = 99$
 $\therefore n = 33$
 So, $S_n = \frac{n}{2}(u_1 + u_n)$
 $= \frac{33}{2}(5 + 101)$
 $= 1749$
- b** The series is arithmetic with
 $u_1 = 50, d = -\frac{1}{2}, u_n = -20$
 Since $u_n = -20$,
 then $u_1 + (n-1)d = -20$
 $\therefore 50 + (-\frac{1}{2})(n-1) = -20$
 $\therefore -\frac{1}{2}n + \frac{1}{2} = -70$
 $\therefore -\frac{1}{2}n = -\frac{141}{2}$
 $\therefore n = 141$
 So, $S_n = \frac{n}{2}(u_1 + u_n)$
 $= \frac{141}{2}(50 + (-20))$
 $= 2115$

- c** The series is arithmetic with

$$u_1 = 8, \quad d = 2\frac{1}{2}, \quad u_n = 83$$

$$\text{Since } u_n = 83,$$

$$\text{then } u_1 + (n-1)d = 83$$

$$\therefore 8 + \frac{5}{2}(n-1) = 83$$

$$\therefore \frac{5}{2}n - \frac{5}{2} = 75$$

$$\therefore \frac{5}{2}n = \frac{155}{2}$$

$$\therefore n = 31$$

$$\text{So, } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{31}{2}(8 + 83)$$

$$= 1410\frac{1}{2}$$

$$\mathbf{3} \quad u_1 = 5, \quad n = 7, \\ u_n = 53$$

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{7}{2}(5 + 53)$$

$$= 203$$

$$\mathbf{4} \quad u_1 = 6, \quad n = 11, \\ u_n = -27$$

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{11}{2}(6 + (-27))$$

$$= -115\frac{1}{2}$$

- 5** The total number of bricks can be expressed as an arithmetic series:

$$1 + 2 + 3 + 4 + \dots + n$$

We know that the total number of bricks is 171, i.e., $S_n = 171$.

Also, $u_1 = 1$, $d = 1$ and we need to find n , the number of members (layers) of the series.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = 171$$

$$\therefore \frac{n}{2}(2 \times 1 + (n-1) \times 1) = 171$$

$$\therefore n(2 + n - 1) = 342$$

$$\therefore n(n+1) = 342$$

$$\therefore n^2 + n - 342 = 0$$

$$\therefore (n-18)(n+19) = 0$$

$$\therefore n = -19 \text{ or } 18$$

$$\text{but } n > 0, \therefore n = 18$$

So, there are 18 layers placed.

- 6** The total number of seats can be expressed as an arithmetic series:

$$22 + 23 + 24 + \dots + u_n$$

Row 1 has 22 seats, so $u_1 = 22$. Row 2 has 23 seats, so $d = 1$.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\therefore S_n = \frac{n}{2}(2 \times 22 + 1(n-1))$$

$$= \frac{n}{2}(44 + n - 1) \quad \therefore S_n = \frac{n}{2}(n + 43) \quad \text{which is the total number of seats in } n \text{ rows.}$$

- a** Number of seats in row 44 = total no. of seats in every row – no. of seats in 43 rows

$$= S_{44} - S_{43}$$

$$= \frac{44}{2}(44 + 43) - \frac{43}{2}(43 + 43)$$

$$= 1914 - 1849$$

$$= 65$$

- b** Number of seats in a section = $S_{44} = 1914$ (from **a**)

- c** Number of seats in 25 sections = $S_{44} \times 25 = 1914 \times 25 = 47\,850$

- 7 a** The first 50 multiples of 11 can be expressed as an arithmetic series:

$$11 + 22 + 33 + \dots + u_{50} \quad \text{where } u_1 = 11, \quad d = 11, \quad n = 50$$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \quad \therefore S_{50} = \frac{50}{2}(2 \times 11 + 11(50-1))$$

$$= 25(22 + 539)$$

$$= 14\,025$$

- b** The multiples of 7 between 0 and 1000 can be expressed as an arithmetic series:

$$7 + 14 + 21 + 28 + \dots + u_n \quad \text{where } u_1 = 7, \quad d = 7$$

To find u_n , we need to find the largest multiple of 7 less than 1000, i.e., $\frac{1000}{7} \div 142.9$,
so $u_n = 142 \times 7 = 994$

$$\text{Now } u_n = u_1 + (n - 1)d$$

$$\therefore 994 = 7 + 7(n - 1)$$

$$\therefore 987 = 7n - 7$$

$$\therefore 7n = 994$$

$$\therefore n = 142$$

$$\text{So, } S_{142} = \frac{142}{2}(7 + 994) = 71\,071$$

- c** The integers between 1 and 100 which are not divisible by 3 can be expressed as:

$$1, 2, 4, 5, 7, 8, \dots, 100 \quad \text{where } u_1 = 1, \quad u_n = 100.$$

Alternatively, these integers can be expressed as two separate arithmetic series:

$$\text{i.e., } S_1 = 1 + 4 + 7 + \dots + 97 + 100 \quad \text{where } u_1 = 1, \quad d = 3, \quad u_n = 100$$

$$\text{and } S_2 = 2 + 5 + 8 + \dots + 95 + 98 \quad \text{where } u_1 = 2, \quad d = 3, \quad u_n = 98$$

$$\text{Now for } S_1, \quad u_n = u_1 + (n - 1)d \quad \text{and for } S_2, \quad u_n = u_1 + (n - 1)d$$

$$\therefore 100 = 1 + 3(n - 1)$$

$$\therefore 98 = 2 + 3(n - 1)$$

$$\therefore 99 = 3n - 3$$

$$\therefore 96 = 3n - 3$$

$$\therefore 3n = 102$$

$$\therefore 3n = 99$$

$$\therefore n = 34$$

$$\therefore n = 33$$

$$\text{i.e., } S_1 = \frac{34}{2}(1 + 100) = 1717 \quad \text{and } S_2 = \frac{33}{2}(2 + 98) = 1650$$

$$\text{i.e., total sum} = S_1 + S_2 = 1717 + 1650 = 3367$$

- 8** We need to show that $1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$

The sum of the first n positive integers can be expressed as an arithmetic series:

$$1 + 2 + 3 + 4 + \dots + n, \quad \text{where } u_1 = 1, \quad d = 1, \quad u_n = n.$$

$$\text{So the sum of the series is } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{n}{2}(1 + n) \quad \text{i.e., } S_n = \frac{n(n+1)}{2} \quad \text{as required.}$$

- 9** The series of odd numbers can be expressed as an arithmetic series:

$$1 + 3 + 5 + 7 + \dots \quad \text{where } u_1 = 1, \quad d = 2$$

a Now $u_n = u_1 + (n - 1)d = 1 + 2(n - 1)$

$$\text{i.e., } u_n = 2n - 1$$

- b** We need to show that S_n is n^2 .

The sum of the first n odd numbers can be expressed as an arithmetic series:

$$1 + 3 + 5 + 7 + \dots (2n - 1) \quad \{\text{using } u_n = 2n - 1 \text{ from a}\}$$

$$\text{So, } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{n}{2}(1 + (2n - 1))$$

$$= \frac{n}{2}(2n)$$

$$\text{i.e., } S_n = n^2 \quad \text{as required}$$

c $S_1 = \frac{1}{2}(1 + 1) = 1 = 1^2 = n^2 \quad \text{for } n = 1 \quad \checkmark$

$$S_2 = \frac{2}{2}(1 + 3) = 4 = 2^2 = n^2 \quad \text{for } n = 2 \quad \checkmark$$

$$S_3 = \frac{3}{2}(1 + 5) = 9 = 3^2 = n^2 \quad \text{for } n = 3 \quad \checkmark$$

$$S_4 = \frac{4}{2}(1 + 7) = 16 = 4^2 = n^2 \quad \text{for } n = 4 \quad \checkmark$$

10 $u_6 = 21$, $S_{17} = 0$; so we need to find u_1 and u_2

$$\begin{aligned}
 S_n &= \frac{n}{2}(2u_1 + (n-1)d) & \text{Also, } u_n &= u_1 + (n-1)d \\
 \therefore S_{17} &= \frac{17}{2}(2u_1 + 16d) = 0 & \therefore u_6 &= u_1 + 5d \\
 & \therefore u_1 + 8d = 0 & \therefore 21 &= -8d + 5d \quad \{\text{using (1)}\} \\
 & \therefore u_1 = -8d \quad \dots\dots (1) & \therefore 21 &= -3d \\
 & & \therefore d &= -7 \\
 & & \text{and } u_1 &= -8(-7) = 56 \\
 & & \text{so } u_2 &= 56 - 7 = 49
 \end{aligned}$$

i.e., the first two terms are 56 and 49.

11 Let the three consecutive terms be $x - d$, x and $x + d$.

$$\begin{aligned}
 \text{Now, sum of terms} &= 12 & \text{Also, product of terms} &= -80 \\
 \text{i.e., } (x - d) + x + (x + d) &= 12 & \text{i.e., } (4 - d)4(4 + d) &= -80 \\
 \therefore 3x &= 12 & \therefore 4(4^2 - d^2) &= -80 \\
 \therefore x &= 4 & \therefore 16 - d^2 &= -20 \\
 \text{So the terms are } 4 - d, 4, 4 + d & & \therefore d^2 &= 36 \\
 & & \therefore d &= \pm 6
 \end{aligned}$$

So, the three terms could be $4 - 6$, 4 , $4 + 6$, i.e., -2 , 4 , 10
 or $4 - (-6)$, 4 , $4 + (-6)$, i.e., 10 , 4 , -2

12 Let the five consecutive terms be $n - 2d$, $n - d$, n , $n + d$, $n + 2d$.

$$\begin{aligned}
 \text{Now, sum of terms} &= 40 \\
 \text{i.e., } (n - 2d) + (n - d) + n + (n + d) + (n + 2d) &= 40 \\
 \therefore 5n &= 40 \\
 \therefore n &= 8 \\
 \text{So the terms are } 8 - 2d, 8 - d, 8, 8 + d, 8 + 2d \\
 \text{Also, product of middle and end terms} &= 8 \times (8 - 2d) \times (8 + 2d) = 224 \\
 \therefore 8(8^2 - 4d^2) &= 224 \\
 \therefore 64 - 4d^2 &= 28 \\
 \therefore 4d^2 &= 36 \\
 \therefore d^2 &= 9 \\
 \therefore d &= \pm 3
 \end{aligned}$$

So, the five terms could be $8 - 2(3)$, $8 - 3$, 8 , $8 + 3$, $8 + 2(3)$, i.e., 2 , 5 , 8 , 11 , 14
 or $8 - 2(-3)$, $8 - (-3)$, 8 , $8 + (-3)$, $8 + 2(-3)$, i.e., 14 , 11 , 8 , 5 , 2

EXERCISE 2E.3

1 a The series is geometric with

$$u_1 = 12, \quad r = \frac{1}{2}, \quad n = 10$$

$$\text{Now } S_n = \frac{u_1(1 - r^n)}{1 - r}$$

$$\text{i.e., } S_{10} = \frac{12 \left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}}$$

$$\div 23.9766$$

b The series is geometric with

$$u_1 = \sqrt{7}, \quad r = \sqrt{7}, \quad n = 12$$

$$\text{Now } S_n = \frac{u_1(r^n - 1)}{r - 1}$$

$$\text{i.e., } S_{12} = \frac{\sqrt{7}((\sqrt{7})^{12} - 1)}{\sqrt{7} - 1}$$

$$\div 189\,134$$

- c** The series is geometric with

$$u_1 = 6, \quad r = -\frac{1}{2}, \quad n = 15$$

$$\text{Now } S_n = \frac{u_1(1 - r^n)}{1 - r}$$

$$\begin{aligned} \text{i.e., } S_{15} &= \frac{6 \left(1 - \left(-\frac{1}{2}\right)^{15}\right)}{1 - \left(-\frac{1}{2}\right)} \\ &\doteq 4.000 \end{aligned}$$

- d** The series is geometric with

$$u_1 = 1, \quad r = -\frac{1}{\sqrt{2}}, \quad n = 20$$

$$\text{Now } S_n = \frac{u_1(1 - r^n)}{1 - r}$$

$$\begin{aligned} \text{i.e., } S_{20} &= \frac{1 \left(1 - \left(-\frac{1}{\sqrt{2}}\right)^{20}\right)}{1 - \left(-\frac{1}{\sqrt{2}}\right)} \\ &\doteq 0.5852 \end{aligned}$$

- 2 a** The series is geometric with

$$u_1 = \sqrt{3}, \quad r = \sqrt{3}$$

$$\begin{aligned} S_n &= \frac{u_1(r^n - 1)}{r - 1} \\ &= \frac{\sqrt{3}((\sqrt{3})^n - 1)}{\sqrt{3} - 1} \times \left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right) \\ &= \frac{(3 + \sqrt{3})((\sqrt{3})^n - 1)}{3 - 1} \\ &= \frac{3 + \sqrt{3}}{2} ((\sqrt{3})^n - 1) \end{aligned}$$

- b** The series is geometric with

$$u_1 = 12, \quad r = \frac{1}{2}$$

$$\begin{aligned} S_n &= \frac{u_1(1 - r^n)}{1 - r} \\ &= \frac{12 \left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \\ &= 24 \left(1 - \left(\frac{1}{2}\right)^n\right) \end{aligned}$$

- c** The series is geometric with

$$u_1 = 0.9, \quad r = 0.1$$

$$\begin{aligned} S_n &= \frac{u_1(1 - r^n)}{1 - r} \\ &= \frac{0.9(1 - (0.1)^n)}{1 - 0.1} \\ &= 1 - (0.1)^n \end{aligned}$$

- d** The series is geometric with

$$u_1 = 20, \quad r = -\frac{1}{2}$$

$$\begin{aligned} S_n &= \frac{u_1(1 - r^n)}{1 - r} \\ &= \frac{20 \left(1 - \left(-\frac{1}{2}\right)^n\right)}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{20 \left(1 - \left(-\frac{1}{2}\right)^n\right)}{\left(\frac{3}{2}\right)} \\ &= \frac{40}{3} \left(1 - \left(-\frac{1}{2}\right)^n\right) \end{aligned}$$

- 3 a** $A_2 = A_1 \times 1.06 + 2000$

$$\begin{aligned} &= (A_0 \times 1.06 + 2000) \times 1.06 + 2000 \\ &= (2000 \times 1.06 + 2000) \times 1.06 + 2000 \end{aligned}$$

$$\text{i.e., } A_2 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2 \quad \text{as required}$$

- b** $A_3 = A_2 \times 1.06 + 2000$

$$= [2000 + 2000 \times 1.06 + 2000 \times (1.06)^2] \times 1.06 + 2000 \quad \{\text{from a}\}$$

$$\text{i.e., } A_3 = 2000 [1 + 1.06 + (1.06)^2 + (1.06)^3] \quad \text{as required}$$

- c** $A_9 = 2000 [1 + 1.06 + (1.06)^2 + (1.06)^3 + (1.06)^4 + (1.06)^5 + (1.06)^6 + (1.06)^7 + (1.06)^8 + (1.06)^9]$

$$\text{i.e., } A_9 = 26\,361.59$$

$$\text{i.e., total bank balance after 10 years is } \$26\,361.59$$

4 a $S_1 = \frac{1}{2}, S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16},$
 $S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

b $S_n = \frac{2^n - 1}{2^n}$ **c** $S_n = \frac{u_1(1 - r^n)}{1 - r}, \text{ where } u_1 = \frac{1}{2}, r = \frac{1}{2}$
 $= \frac{\frac{1}{2} \left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}}$

i.e., $S_n = 1 - \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

d As n gets very large, i.e., as $n \rightarrow \infty$, $\left(\frac{1}{2}\right)^n \rightarrow 0$, so $S_n \rightarrow 1$ (from below)

e The diagram represents one whole unit divided into smaller and smaller fractions.
 As $n \rightarrow \infty$, the area which the fraction represents becomes smaller and smaller, and the total area approaches one whole unit.

5 a Total time of motion = $1 + (90\% \times 1) + (90\% \times 1) + (90\% \times 90\% \times 1)$
 $+ (90\% \times 90\% \times 1) + (90\% \times 90\% \times 90\% \times 1) + \dots$
 $= 1 + 0.9 + 0.9 + (0.9)^2 + (0.9)^2 + (0.9)^3 + \dots$
 $= 1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + \dots$ as required

b $S_n = \frac{u_1(1 - r^n)}{1 - r}, \text{ where } u_1 = 2(0.9), r = 0.9, 'n' = n - 1$
 (since the term u_1 , used in calculating S_n , is the second term of the series, not the first)

i.e., $S_n = \frac{2(0.9)(1 - (0.9)^{n-1})}{1 - 0.9} + 1$

$= \frac{1.8(1 - (0.9)^{n-1})}{0.1} + 1$

i.e., $S_n = 1 + 18(1 - (0.9)^{n-1})$

c For the ball to come to rest, n must approach infinity,
 thus $(0.9)^{n-1} \rightarrow 0$ so $(1 - (0.9)^{n-1}) \rightarrow 1$ (from below)

So, $S_n \rightarrow 1 + 18(1)$, i.e., $S_n \rightarrow 19$

i.e., it takes 19 seconds for the ball to come to rest.

6 a i $u_1 = \frac{3}{10}$ **ii** $r = \frac{\left(\frac{3}{100}\right)}{\left(\frac{3}{10}\right)} = \frac{1}{10}$

b We need to show that $0.\bar{3} = \frac{1}{3}$.

Now $0.\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$

So, let $S_n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$

Since $n \rightarrow \infty$, then $S_\infty = \frac{u_1}{1 - r} = \frac{\frac{3}{10}}{1 - \left(\frac{1}{10}\right)} = \frac{1}{3}$

i.e., $0.\bar{3} = \frac{1}{3}$ as required

7 Checking **4d**: Checking **5c**:

$$\begin{aligned}
 S_{\infty} &= \frac{u_1}{1-r} & S_{\infty} &= \frac{u_1}{1-r} \\
 &= \frac{\frac{1}{2}}{1-\frac{1}{2}} & &= \frac{2(0.9)}{1-0.9} + 1 \quad \{\text{since 1st term '1' is not part of the series}\} \\
 \text{i.e., } S_{\infty} &= 1 \quad \checkmark & \text{i.e., } S_{\infty} &= 19 \quad \checkmark
 \end{aligned}$$

EXERCISE 2F

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \sum_{r=1}^4 (3r-5) &= -2 + 1 + 4 + 7 \\
 &= 10 & \mathbf{b} \quad \sum_{r=1}^5 (11-2r) &= 9 + 7 + 5 + 3 + 1 \\
 & & &= 25
 \end{aligned}$$

$$\mathbf{c} \quad \sum_{r=1}^7 r(r+1) = 2 + 6 + 12 + 20 + 30 + 42 + 56 \\
 = 168$$

$$\mathbf{d} \quad \sum_{i=1}^5 10 \times 2^{i-1} = 10 + 20 + 40 + 80 + 160 \\
 = 310$$

$$\mathbf{2} \quad u_n = 3n - 1$$

$$\begin{aligned}
 u_1 + u_2 + u_3 + \dots + u_{20} &= 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 \\
 &\quad + 38 + 41 + 44 + 47 + 50 + 53 + 56 + 59
 \end{aligned}$$

This series is arithmetic with $u_1 = 2$, $d = 3$ and $n = 20$

$$\begin{aligned}
 \text{so its sum is } \frac{n}{2} [2u_1 + (n-1)d] \\
 = 10[4 + 19 \times 3] \\
 = 610
 \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad \sum_{r=1}^{10} (2r+5) = 7 + 9 + 11 + \dots + 21 + 23 + 25$$

This series is arithmetic with $u_1 = 7$, $d = 2$ and $n = 10$.

$$\begin{aligned}
 \therefore \text{sum} &= \frac{n}{2} [2u_1 + (n-1)d] \\
 &= \frac{10}{2} [14 + 9 \times 2] \\
 &= 160
 \end{aligned}$$

$$\mathbf{b} \quad \sum_{r=1}^{15} (r-50) = (-49) + (-48) + (-47) + \dots + (-37) + (-36) + (-35)$$

This series is arithmetic with $u_1 = -49$, $d = 1$ and $n = 15$.

$$\begin{aligned}
 \therefore \text{sum} &= \frac{n}{2} [2u_1 + (n-1)d] \\
 &= \frac{15}{2} [-98 + 14 \times 1] \\
 &= -630
 \end{aligned}$$

$$\mathbf{c} \quad \sum_{r=1}^{20} \left(\frac{r+3}{2} \right) = 2 + \frac{5}{2} + 3 + \dots + \frac{21}{2} + 11 + \frac{23}{2}$$

This series is arithmetic with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 20$.

$$\begin{aligned}
 \therefore \text{sum} &= \frac{n}{2} [2u_1 + (n-1)d] \\
 &= \frac{20}{2} [4 + 19 \times \frac{1}{2}] \\
 &= 135
 \end{aligned}$$

$$4 \quad a \quad \sum_{r=1}^{10} (3 \times 2^{r-1}) = 3 + 6 + 12 + \dots + 384 + 768 + 1536$$

This series is geometric with $u_1 = 3$, $r = 2$ and $n = 10$.

$$\therefore \text{sum} = \frac{u_1(r^n - 1)}{r - 1} = \frac{3(2^{10} - 1)}{1} = 3069$$

$$b \quad \sum_{r=1}^{12} \left(\frac{1}{2}\right)^{r-2} = 2 + 1 + \frac{1}{2} + \dots + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$$

This series is geometric with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 12$.

$$\therefore \text{sum} = \frac{u_1(1 - r^n)}{1 - r} = \frac{2\left(1 - \left(\frac{1}{2}\right)^{12}\right)}{\frac{1}{2}} = 4\left(1 - \left(\frac{1}{2}\right)^{12}\right)$$

$$\therefore \text{sum} \div 3.999$$

$$c \quad \sum_{r=1}^{25} (6 \times (-2)^r) = -12 + 24 + (-48) + \dots + 100\,663\,296 + (-201\,326\,592)$$

This series is geometric with $u_1 = -12$, $r = -2$ and $n = 25$.

$$\therefore \text{sum} = \frac{u_1(1 - r^n)}{1 - r} = \frac{-12(1 - (-2)^{25})}{1 + 2} = -4(1 - (-2)^{25})$$

$$\therefore \text{sum} = -134\,217\,732$$

$$5 \quad a \quad \sum_{k=1}^5 k(k+1)(k+2) = 6 + 24 + 60 + 120 + 210 = 420$$

Note: This series is neither arithmetic nor geometric so the sum is found by adding the 5 terms.

$$b \quad \sum_{k=6}^{12} (100(1.2)^{k-3}) = 172.8 + 207.36 + 248.832 + 298.5984 + 358.31808 \\ + 429.981696 + 515.9780352$$

This series is geometric with $u_1 = 172.8$, $r = 1.2$ and $n = 7$

$$\therefore \text{sum} = \frac{u_1(r^n - 1)}{r - 1} = \frac{172.8((1.2)^7 - 1)}{0.2} \div 2231.87$$

REVIEW SET 2A

$$1 \quad a \quad u_n = 3^{n-2} \quad \therefore \quad u_1 = 3^{-1} = \frac{1}{3}, \quad u_2 = 3^0 = 1, \quad u_3 = 3^1 = 3, \quad u_4 = 3^2 = 9$$

$$b \quad u_n = \frac{3n+2}{n+3} \quad \therefore \quad u_1 = \frac{5}{4}, \quad u_2 = \frac{8}{5}, \quad u_3 = \frac{11}{6}, \quad u_4 = \frac{14}{7} = 2$$

$$c \quad u_n = 2^n - (-3)^n \\ u_1 = 2 - (-3) = 5, \quad u_2 = 4 - 9 = -5, \quad u_3 = 8 - (-27) = 35, \quad u_4 = 16 - 81 = -65$$

$$2 \quad u_n = 68 - 5n$$

$$a \quad u_{n+1} - u_n = 68 - 5(n+1) - [68 - 5n] \\ = 68 - 5n - 5 - 68 + 5n \\ = -5 \quad \text{for all } n$$

$$b \quad u_1 = 68 - 5(1) = 63, \quad d = -5$$

$$c \quad u_{37} = 68 - 5(37) = -117$$

d Let $u_n = -200$, and we need to find n .

$$u_n = 68 - 5n = -200$$

$$\therefore 5n = 268$$

$$\therefore n = 53\frac{3}{5}$$

So, try the two values on either side of $n = 53\frac{3}{5}$, i.e., for $n = 53$ and $n = 54$:

$$u_{53} = 68 - 5(53) \quad \text{and} \quad u_{54} = 68 - 5(54)$$

$$= -197 \quad \quad \quad = -202$$

i.e., the first term of the sequence less than -200 is $u_{54} = -202$.

3 a 3, 12, 48, 192,

$\frac{12}{3} = 4$ So, assuming the pattern continues, consecutive terms have a common ratio of 4.

$\frac{48}{12} = 4$ $\therefore$ the sequence is geometric with $u_1 = 3$ and $r = 4$.

$$\frac{192}{48} = 4$$

$$\begin{aligned} \mathbf{b} \quad u_n &= u_1 r^{n-1} \\ \therefore u_n &= 3 \times 4^{n-1} \\ \text{i.e., } u_9 &= 3 \times 4^8 \\ &= 196\,608 \end{aligned}$$

4 Since the terms are consecutive,

then $(k-2) - 3k = k+7 - (k-2)$ {equating common differences}

$$\therefore k-2-3k = k+7-k+2$$

$$\therefore -2-2k = 9$$

$$\therefore 2k = -11$$

$$\therefore k = -\frac{11}{2}$$

$$\mathbf{5} \quad u_7 = 31 \quad \therefore u_1 + 6d = 31 \quad \dots (1)$$

$$u_{15} = -17 \quad \therefore u_1 + 14d = -17 \quad \dots (2)$$

Solving simultaneously,

$$-u_1 - 6d = -31$$

$$u_1 + 14d = -17$$

$$\hline \therefore 8d = -48 \quad \text{{adding the equations}}$$

$$\therefore d = -6$$

$$\text{So in (1), } u_1 + 6(-6) = 31$$

$$\therefore u_1 - 36 = 31$$

$$\therefore u_1 = 67$$

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore u_n = 67 + (n-1)(-6)$$

$$\therefore u_n = 67 - 6n + 6$$

$$\therefore u_n = -6n + 73$$

$$\text{So } u_{34} = -6(34) + 73 = -131$$

$$\mathbf{6} \quad u_n = 6 \left(\frac{1}{2}\right)^{n-1}$$

$$\mathbf{a} \quad \frac{u_{n+1}}{u_n} = \frac{6 \left(\frac{1}{2}\right)^{n+1-1}}{6 \left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2} \quad \text{for all } n$$

$$\mathbf{b} \quad u_1 = 6, \quad r = \frac{1}{2}$$

$$\mathbf{c} \quad u_{16} = 6 \left(\frac{1}{2}\right)^{15} = 0.000\,183$$

$\therefore \{u_n\}$ is a geometric sequence.

7 28, 23, 18, 13,

$$23 - 28 = -5$$

So, assuming that the pattern continues, consecutive terms differ by -5 .

$$18 - 23 = -5$$

 $\therefore$ the sequence is arithmetic with $u_1 = 28$, $d = -5$.

$$13 - 18 = -5$$

$$u_n = u_1 + (n - 1)d$$

$$= 28 + (n - 1)(-5)$$

$$= 28 - 5n + 5$$

$$= -5n + 33$$

$$S_n = \frac{n}{2}(2u_1 + (n - 1)d)$$

$$= \frac{n}{2}(2 \times 28 + (n - 1)(-5))$$

$$= \frac{n}{2}(56 - 5n + 5)$$

$$\text{i.e., } S_n = \frac{n}{2}(61 - 5n)$$

8 Since the terms are geometric, then $\frac{k}{4} = \frac{k^2 - 1}{k}$

$$\therefore k^2 = 4(k^2 - 1)$$

$$\therefore k^2 = 4k^2 - 4 \quad \therefore k = \pm \frac{2}{\sqrt{3}}$$

$$\therefore 3k^2 = 4$$

$$\therefore k^2 = \frac{4}{3} \quad \therefore k = \pm \frac{2\sqrt{3}}{3}$$

9 $u_6 = \frac{16}{3} \quad \therefore u_1 \times r^5 = \frac{16}{3} \quad \dots (1)$

$u_{10} = \frac{256}{3} \quad \therefore u_1 \times r^9 = \frac{256}{3} \quad \dots (2)$

So $\frac{u_1 r^9}{u_1 r^5} = \frac{(\frac{256}{3})}{(\frac{16}{3})} \quad \{(2) \div (1)\}$

$$\therefore r^4 = 16$$

$$\therefore r = \pm 2$$

Substituting $r = 2$ into (1) gives

$$u_1 \times 2^5 = \frac{16}{3}$$

$$\therefore u_1 \times 32 = \frac{16}{3}$$

$$\therefore u_1 = \frac{1}{6}$$

Substituting $r = -2$ into (1) gives

$$u_1 \times (-2)^5 = \frac{16}{3}$$

$$\therefore u_1 \times (-32) = \frac{16}{3}$$

$$\therefore u_1 = -\frac{1}{6}$$

$$\text{Now } u_n = u_1 r^{n-1} \quad \therefore u_n = \frac{1}{6} \times 2^{n-1} \quad \text{or} \quad -\frac{1}{6} \times 2^{n-1}$$

REVIEW SET 2B**1 a** 24, $23\frac{1}{4}$, $22\frac{1}{2}$,, -36 i.e., $u_1 = 24$, $u_n = -36$, and we need to find n .The sequence is arithmetic with $d = -\frac{3}{4}$.

$$\text{Now } u_n = u_1 + (n - 1)d$$

$$\therefore -36 = 24 + (n - 1)(-\frac{3}{4})$$

$$\therefore -60 = -\frac{3}{4}n + \frac{3}{4}$$

$$\therefore \frac{3}{4}n = \frac{243}{4}$$

$$\therefore n = 81$$

i.e., there are 81 terms in the sequence.

b
$$\begin{aligned} u_{35} &= 24 + (35 - 1)(-\frac{3}{4}) \\ &= 24 - \frac{102}{4} \\ &= -\frac{3}{2} \end{aligned}$$

c
$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ \text{i.e., } S_{81} &= \frac{81}{2}(24 + (-36)) \\ &= -486 \end{aligned}$$

- 2** Let the numbers be $23, 23 + d, 23 + 2d, 23 + 3d, 23 + 4d, 23 + 5d, 23 + 6d, 9$

Then $23 + 7d = 9$

$$\text{i.e., } 7d = -14$$

$$\therefore d = -2 \quad \text{i.e., the numbers are } 23, 21, 19, 17, 15, 13, 11, 9.$$

- 3 a** $86, 83, 80, 77, \dots$ i.e., the sequence is arithmetic with $u_1 = 86, d = -3$

$$u_n = u_1 + (n - 1)d$$

$$\text{i.e., } u_n = 86 + (n - 1)(-3) = 86 - 3n + 3$$

$$\text{i.e., } u_n = 89 - 3n$$

- b** $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}, \dots$ which can also be written as $\frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \frac{9}{7}, \dots$

i.e., the numerator starts at 3 and increases by 2 each time,

whilst the denominator starts at 4 and increases by 1 each time.

$$\text{i.e., the } n\text{th term is } \frac{2n + 1}{n + 3}, \quad \text{i.e., } u_n = \frac{2n + 1}{n + 3}$$

- c** $100, 90, 81, 72.9, \dots$ i.e., the sequence is geometric with $u_1 = 100, r = \frac{90}{100} = 0.9$

$$u_n = u_1 r^{n-1}$$

$$\text{i.e., } u_n = 100(0.9)^{n-1}$$

- 4 a** $\sum_{r=1}^7 r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

$$= 1 + 4 + 9 + 16 + 25 + 36 + 49$$

- b** $\sum_{r=1}^8 \frac{r+3}{r+2} = \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9} + \frac{11}{10}$

- 5 a** $4 + 11 + 18 + 25 + \dots$

i.e., the series is arithmetic with $u_1 = 4, d = 7, u_r = u_1 + (r - 1)d$

$$= 4 + 7(r - 1)$$

$$\text{i.e., } \sum_{r=1}^n (7r - 3) = 7r - 3$$

- b** $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$ i.e., the series is geometric with $u_1 = \frac{1}{4}, R = \frac{1}{2},$

$$u_r = u_1 R^{r-1} = \frac{1}{4} \times \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^{r+1}$$

$$\text{i.e., } \sum_{r=1}^n \left(\frac{1}{2}\right)^{r+1}$$

- 6 a** $3 + 9 + 15 + 21 + \dots$

i.e., the series is arithmetic with

$$u_1 = 3, d = 6, n = 23$$

$$\text{Now } S_n = \frac{n}{2} (2u_1 + (n - 1)d)$$

$$\text{i.e., } S_{23} = \frac{23}{2} (2 \times 3 + 6(23 - 1))$$

$$= \frac{23}{2} (6 + 132)$$

$$= 1587$$

- b** $24 + 12 + 6 + 3 + \dots$

i.e., the series is geometric with

$$u_1 = 24, r = \frac{1}{2}, n = 12$$

$$S_n = \frac{u_1(1 - r^n)}{1 - r}$$

$$\text{i.e., } S_{12} = \frac{24 \left(1 - \left(\frac{1}{2}\right)^{12}\right)}{1 - \frac{1}{2}}$$

$$= 48 \left(1 - \left(\frac{1}{2}\right)^{12}\right)$$

$$\text{i.e., } S_{12} \div 47.99$$

$$7 \quad a \quad \sum_{r=1}^8 \left(\frac{31-3r}{2} \right) = 14 + 12\frac{1}{2} + 11 + 9\frac{1}{2} + 8 + 6\frac{1}{2} + 5 + 3\frac{1}{2}$$

This series is arithmetic with $u_1 = 14$, $d = -1\frac{1}{2}$ and $n = 8$.

$$\therefore \text{ the sum is } \frac{8}{2} [28 + 7(-1\frac{1}{2})] = 70$$

$$b \quad \sum_{r=1}^{15} 50(0.8)^{r-1} \div 50 + 40 + 32 + \dots + 3.436 + 2.749 + 2.199$$

This series is geometric with $u_1 = 50$, $r = 0.8$ and $n = 15$.

$$\therefore \text{ the sum is } \frac{50 [1 - (0.8)^{15}]}{1 - 0.8} \div 241.2$$

$$8 \quad 5, 10, 20, 40, \dots \quad \text{i.e., the sequence is geometric with } u_1 = 5, \quad r = 2$$

$$u_n = u_1 r^{n-1} = 5 \times 2^{n-1}$$

$$\text{Let } u_n = 10\,000 = 5 \times 2^{n-1}$$

$$\therefore 2000 = 2^{n-1}$$

$$\therefore n \div 11.97 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 11.97$, i.e., for $n = 11$ and $n = 12$:

$$\begin{aligned} u_{11} &= 5 \times 2^{10} & \text{and} & & u_{12} &= 5 \times 2^{11} \\ &= 5120 & & & &= 10\,240 \end{aligned}$$

i.e., the first term to exceed 10 000 is $u_{12} = 10\,240$.

$$9 \quad a \quad u_6 = u_1 \times r^5 \quad \text{is the amount after 5 years, where } r = 1.07$$

$$= 6000 \times (1.07)^5$$

i.e., $u_6 = 8415.31$, i.e., the value of the investment will be 8415.31 Euro

$$b \quad \text{If interest is compounded quarterly, then } r = 1 + \frac{0.07}{4} = 1.0175$$

$$\text{and } n = 5 \times 4 = 20$$

$$\begin{aligned} u_{21} &= u_1 \times r^{20} \\ &= 6000 \times (1.0175)^{20} \end{aligned}$$

i.e., $u_{21} = 8488.67$, i.e., the value of the investment will be 8488.67 Euro

$$c \quad \text{If interest is compounded monthly, then } r = 1 + \frac{0.07}{12} = 1.0058\bar{3}$$

$$\begin{aligned} u_{61} &= u_1 \times r^{60} & \text{and } n &= 5 \times 12 = 60 \\ &= 6000 \times (1.0058\bar{3})^{60} \end{aligned}$$

i.e., $u_{61} = 8505.75$, i.e., the value of the investment will be 8505.75 Euro

REVIEW SET 2C

$$1 \quad u_6 = 24 \quad \therefore u_1 \times r^5 = 24 \quad \dots (1)$$

$$u_{11} = 768 \quad \therefore u_1 \times r^{10} = 768 \quad \dots (2)$$

$$\text{So } \frac{u_1 r^{10}}{u_1 r^5} = \frac{768}{24} \quad \{(2) \div (1)\}$$

$$\therefore r^5 = 32$$

$$\therefore r = 2$$

$$\text{Substituting } r = 2 \text{ into (1) gives } u_1 \times 2^5 = 24 \quad \therefore u_1 = \frac{24}{32} = \frac{3}{4}$$

$$u_n = u_1 r^{n-1} = \left(\frac{3}{4}\right) 2^{n-1}$$

$$\begin{aligned} \mathbf{a} \quad u_{17} &= \left(\frac{3}{4}\right) 2^{17-1} \\ &= 49\,152 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad S_n &= \frac{u_1(r^n - 1)}{r - 1} \\ &= \frac{\frac{3}{4}(2^n - 1)}{2 - 1} \\ &= \frac{3}{4}(2^n - 1) \\ \therefore S_{15} &= \frac{3}{4}(2^{15} - 1) \\ &= 24\,575.25 \end{aligned}$$

2 $11 + 16 + 21 + 26 + \dots$ i.e., the series is arithmetic with $u_1 = 11$, $d = 5$

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(2 \times 11 + 5(n-1)) \\ &= \frac{n}{2}(22 + 5n - 5) \\ &= \frac{n}{2}(5n + 17) \end{aligned}$$

Let $S_n = 450$ and we need to find n ,

$$\text{i.e., } S_n = \frac{n}{2}(5n + 17) = 450$$

$$\therefore \frac{5}{2}n^2 + \frac{17}{2}n - 450 = 0$$

$$\therefore 5n^2 + 17n - 900 = 0$$

$$\therefore n \div -15.2, 11.8 \quad \{\text{using technology}\}$$

$$\text{but } n > 0, \therefore n \div 11.8$$

So, try the two values on either side of $n \div 11.8$, i.e., for $n = 11$ and $n = 12$:

$$\begin{aligned} S_{11} &= \frac{11}{2}(5(11) + 17) & \text{and} & \quad S_{12} = \frac{12}{2}(5(12) + 17) \\ &= 396 & & \quad = 462 \end{aligned}$$

i.e., 12 terms of the series are required to exceed a sum of 450.

3 $24, 8, \frac{8}{3}, \frac{8}{9}, \dots$ i.e., the sequence is geometric with $u_1 = 24$, $r = \frac{1}{3}$

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ &= 24 \left(\frac{1}{3}\right)^{n-1} \end{aligned}$$

Let $u_n = 0.001$ and we need to find n .

$$\text{i.e., } u_n = 24 \left(\frac{1}{3}\right)^{n-1} = 0.001$$

$$\therefore \left(\frac{1}{3}\right)^{n-1} = \frac{0.001}{24}$$

$$\therefore n \div 10.18 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 10.18$, i.e., for $n = 10$ and $n = 11$:

$$\begin{aligned} u_{10} &= 24 \left(\frac{1}{3}\right)^9 & \text{and} & \quad u_{11} = 24 \left(\frac{1}{3}\right)^{10} \\ &= 0.001\,22 & & \quad = 0.000\,406 \end{aligned}$$

i.e., $u_{11} \div 0.000\,406$ is the first term of the sequence which is less than 0.001.

4 a 128, 64, 32, 16,, $\frac{1}{512}$

i.e., the sequence is geometric with

$$u_1 = 128, \quad r = \frac{1}{2}, \quad u_n = \frac{1}{512}$$

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ &= 128 \left(\frac{1}{2}\right)^{n-1} \end{aligned}$$

$$\text{i.e., } \frac{1}{512} = 128 \left(\frac{1}{2}\right)^{n-1}$$

$$\therefore \left(\frac{1}{2}\right)^{n-1} = \frac{1}{65\,536}$$

$$\therefore n = 17 \quad \{\text{using technology}\}$$

i.e., there are 17 terms in the sequence.

b $S_n = \frac{u_1(1-r^n)}{1-r}$

$$\text{i.e., } S_{17} = \frac{128 \left(1 - \left(\frac{1}{2}\right)^{17}\right)}{1 - \frac{1}{2}}$$

$$= 255.998$$

$$\text{i.e., } S_{17} \div 256$$

5 a $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \frac{0.0825}{2} = 1.041\,25$, $n = 5 \times 2 = 10$

$$\text{i.e., } u_{n+1} = 12\,500 \times (1.041\,25)^{10}$$

$$= 18\,726.65, \text{ i.e., the value of the investment is \$18\,726.65}$$

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \frac{0.0825}{12} = 1.006\,875$, $n = 5 \times 12 = 60$

$$\text{i.e., } u_{n+1} = 12\,500 \times (1.006\,875)^{60}$$

$$= 18\,855.74, \text{ i.e., the value of the investment is \$18\,855.74}$$

6 $u_{n+1} = u_1 \times r^n$ where $u_{n+1} = 20\,000$, $r = 1 + \frac{0.09}{12} = 1.0075$, $n = 4 \times 12 = 48$

$$\text{i.e., } 20\,000 = u_1 \times (1.0075)^{48}$$

$$\text{i.e., } u_1 = 13\,972.28, \text{ i.e., \$13\,972.28 should be invested}$$

7 a $u_{n+1} = u_1 \times r^n$, where $u_1 = 3000$, $r = 1.05$, $n = 3$

$$\text{i.e., } u_{n+1} = 3000 \times (1.05)^3$$

$$= 3472.875, \text{ i.e., approximately 3470 koalas}$$

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000$, $u_{n+1} = 5000$, $r = 1.05$

$$\text{i.e., } 5000 = 3000 \times (1.05)^n$$

$$\text{i.e., } n \div 10.47$$

i.e., after 10.47 years the population will exceed 5000, i.e., in the year 2008.

Chapter 3

EXPONENTS

EXERCISE 3A

- 1 a $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16,$
 $2^5 = 32, 2^6 = 64$
 b $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$
 c $5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625$
 d $7^1 = 7, 7^2 = 49, 7^3 = 343$

EXERCISE 3B

- 1 a $(-1)^3$
 $= (-1) \times (-1) \times (-1)$
 $= 1 \times (-1)$
 $= -1$
 b $(-1)^4$
 $= (-1)^3 \times (-1)$
 $= (-1) \times (-1)$
 $= 1$
 c $(-1)^{12}$
 $= 1$
 d $(-1)^{17}$
 $= -1$
 e $(-1)^6$
 $= 1$
 f -1^6
 $= -(1^6)$
 $= -1$
 g $-(-1)^6$
 $= -(1)$
 $= -1$
 h $(-2)^3$
 $= (-2) \times (-2) \times (-2)$
 $= 4 \times (-2)$
 $= -8$
 i -2^3
 $= -(2^3)$
 $= -8$
 j $-(-2)^3$
 $= -(-8)$
 $= 8$
 k $-(-5)^2$
 $= -(25)$
 $= -25$
 l $-(-5)^3$
 $= -(-125)$
 $= 125$

- 2 a 512 b -3125 c -243 d 16 807 e 512 f 6561 g -6561
 h 5.117 264 691 i -0.764 479 956 j -20.361 584 96

- 3 a $0.\overline{142857}$ b $0.\overline{142857}$ c $0.\overline{1}$ d $0.\overline{1}$ e 0.015 625 f 0.015 625
 g 1 h 1

We notice that $7^{-1} = \frac{1}{7^1}$, $3^{-2} = \frac{1}{3^2}$, $4^{-3} = \frac{1}{4^3}$ and $a^0 = 1$ for $a > 0$

- 4 $3^{33} = \underbrace{3^4 \times 3^4 \times 3^4 \times \dots \times 3^4}_{8 \text{ of these}} \times 3^1$ But $3^4 = 81$ i.e., ends in a 1
 $\therefore \underbrace{3^4 \times 3^4 \times 3^4 \times \dots \times 3^4}_{8 \text{ of these}}$ ends in a 1
 $\therefore 3^{33}$ ends in a 3

- 5 $7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401, 7^5 = 16\,807$

Now $7^{77} = \underbrace{7^4 \times 7^4 \times 7^4 \times \dots \times 7^4}_{19 \text{ of these}} \times 7^1$
 So, ends in a 1.

$\therefore 7^{77}$ ends in $1 \times 7 = 7$.

EXERCISE 3C

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & 7^3 \times 7^2 \\ & = 7^{3+2} \\ & = 7^5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 5^4 \times 5^3 \\ & = 5^{4+3} \\ & = 5^7 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & a^7 \times a^2 \\ & = a^{7+2} \\ & = a^9 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & a^4 \times a \\ & = a^4 \times a^1 \\ & = a^{4+1} \\ & = a^5 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & b^8 \times b^5 \\ & = b^{8+5} \\ & = b^{13} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & a^3 \times a^n \\ & = a^{3+n} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & b^7 \times b^m \\ & = b^{7+m} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & m^4 \times m^2 \times m^3 \\ & = m^{4+2+3} \\ & = m^9 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & \frac{5^9}{5^2} \\ & = 5^{9-2} \\ & = 5^7 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{11^{13}}{11^9} \\ & = 11^{13-9} \\ & = 11^4 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 7^7 \div 7^4 \\ & = 7^{7-4} \\ & = 7^3 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \frac{a^6}{a^2} \\ & = a^{6-2} \\ & = a^4 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \frac{b^{10}}{b^7} \\ & = b^{10-7} \\ & = b^3 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \frac{p^5}{p^m} \\ & = p^{5-m} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \frac{y^a}{y^5} \\ & = y^{a-5} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & b^{2x} \div b \\ & = b^{2x} \div b^1 \\ & = b^{2x-1} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & (3^2)^4 \\ & = 3^{2 \times 4} \\ & = 3^8 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & (5^3)^5 \\ & = 5^{3 \times 5} \\ & = 5^{15} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & (2^4)^7 \\ & = 2^{4 \times 7} \\ & = 2^{28} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & (a^5)^2 \\ & = a^{5 \times 2} \\ & = a^{10} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & (p^4)^5 \\ & = p^{4 \times 5} \\ & = p^{20} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & (b^5)^n \\ & = b^{5 \times n} \\ & = b^{5n} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & (x^y)^3 \\ & = x^{y \times 3} \\ & = x^{3y} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & (a^{2x})^5 \\ & = a^{2x \times 5} \\ & = a^{10x} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & 8 \\ & = 2^3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 25 \\ & = 5^2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 27 \\ & = 3^3 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 4^3 \\ & = (2^2)^3 \\ & = 2^6 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 9^2 \\ & = (3^2)^2 \\ & = 3^4 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & 3^a \times 9 \\ & = 3^a \times 3^2 \\ & = 3^{a+2} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & 5^t \div 5 \\ & = 5^t \div 5^1 \\ & = 5^{t-1} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & 3^n \times 9^n \\ & = 3^n \times (3^2)^n \\ & = 3^{n+2n} \\ & = 3^{3n} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \frac{16}{2^x} \\ & = \frac{2^4}{2^x} \\ & = 2^{4-x} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \frac{3^{x+1}}{3^{x-1}} \\ & = 3^{(x+1)-(x-1)} \\ & = 3^{x+1-x+1} \\ & = 3^2 \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & (5^4)^{x-1} \\ & = 5^{4(x-1)} \\ & = 5^{4x-4} \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & 2^x \times 2^{2-x} \\ & = 2^{x+2-x} \\ & = 2^2 \end{aligned}$$

$$\begin{aligned} \mathbf{m} \quad & \frac{2^y}{4^x} \\ & = \frac{2^y}{(2^2)^x} \\ & = \frac{2^y}{2^{2x}} \\ & = 2^{y-2x} \end{aligned}$$

$$\begin{aligned} \mathbf{n} \quad & \frac{4^y}{8^x} \\ & = \frac{(2^2)^y}{(2^3)^x} \\ & = \frac{2^{2y}}{2^{3x}} \\ & = 2^{2y-3x} \end{aligned}$$

$$\begin{aligned} \mathbf{o} \quad & \frac{3^{x+1}}{3^{1-x}} \\ & = 3^{(x+1)-(1-x)} \\ & = 3^{x+1-1+x} \\ & = 3^{2x} \end{aligned}$$

$$\begin{aligned} \mathbf{p} \quad & \frac{2^t \times 4^t}{8^{t-1}} \\ & = \frac{2^t \times (2^2)^t}{(2^3)^{t-1}} \\ & = \frac{2^t \times 2^{2t}}{2^{3t-3}} \\ & = 2^{3t-(3t-3)} \\ & = 2^3 \end{aligned}$$

5	a $(ab)^3$ $= a^3b^3$	b $(ac)^4$ $= a^4c^4$	c $(bc)^5$ $= b^5c^5$	d $(abc)^3$ $= a^3b^3c^3$
	e $(2a)^4$ $= 2^4a^4$ $= 16a^4$	f $(5b)^2$ $= 5^2b^2$ $= 25b^2$	g $(3n)^4$ $= 3^4n^4$ $= 81n^4$	h $(2bc)^3$ $= 2^3b^3c^3$ $= 8b^3c^3$
	i $(4ab)^3$ $= 4^3a^3b^3$ $= 64a^3b^3$	j $\left(\frac{a}{b}\right)^3$ $= \frac{a^3}{b^3}$	k $\left(\frac{m}{n}\right)^4$ $= \frac{m^4}{n^4}$	l $\left(\frac{2c}{d}\right)^5$ $= \frac{2^5c^5}{d^5}$ or $\frac{32c^5}{d^5}$
6	a $(2b^4)^3$ $= 2^3b^{12}$ $= 8b^{12}$	b $\left(\frac{3}{x^2y}\right)^2$ $= \frac{3^2}{x^4y^2}$ $= \frac{9}{x^4y^2}$	c $(5a^4b)^2$ $= 5^2a^8b^2$ $= 25a^8b^2$	d $\left(\frac{m^3}{2n^2}\right)^4$ $= \frac{m^{12}}{2^4n^8}$ $= \frac{m^{12}}{16n^8}$
	e $\left(\frac{3a^3}{b^5}\right)^3$ $= \frac{3^3a^9}{b^{15}}$ $= \frac{27a^9}{b^{15}}$	f $(2m^3n^2)^5$ $= 2^5m^{15}n^{10}$ $= 32m^{15}n^{10}$	g $\left(\frac{4a^4}{b^2}\right)^2$ $= \frac{4^2a^8}{b^4}$ $= \frac{16a^8}{b^4}$	h $(5x^2y^3)^3$ $= 5^3x^6y^9$ $= 125x^6y^9$
7	a $(-2a)^2$ $= (-2)^2a^2$ $= 4a^2$	b $(-6b^2)^2$ $= (-6)^2b^4$ $= 36b^4$	c $(-2a)^3$ $= (-2)^3a^3$ $= -8a^3$	d $(-3m^2n^2)^3$ $= (-3)^3m^6n^6$ $= -27m^6n^6$
	e $(-2ab^4)^4$ $= (-2)^4a^4b^{16}$ $= 16a^4b^{16}$	f $\left(\frac{-2a^2}{b^2}\right)^3$ $= \frac{(-2)^3a^6}{b^6}$ $= -\frac{8a^6}{b^6}$	g $\left(\frac{-4a^3}{b}\right)^2$ $= \frac{(-4)^2a^6}{b^2}$ $= \frac{16a^6}{b^2}$	h $\left(\frac{-3p^2}{q^3}\right)^2$ $= \frac{(-3)^2p^4}{q^6}$ $= \frac{9p^4}{q^6}$
8	a $\frac{a^3}{a}$ $= \frac{a^3}{a^1}$ $= a^{3-1}$ or a^2	b $4b^2 \times 2b^3$ $= 8b^{2+3}$ $= 8b^5$	c $\frac{m^5n^4}{m^2n^3}$ $= m^{5-2}n^{4-3}$ $= m^3n^1$ $= m^3n$	d $\frac{14a^7}{2a^2}$ $= \frac{14}{2}a^{7-2}$ $= 7a^5$
	e $\frac{12a^2b^3}{3ab}$ $= \frac{12}{3}a^{2-1}b^{3-1}$ $= 4ab^2$	f $\frac{18m^7a^3}{4m^4a^3}$ $= \frac{18}{4}m^{7-4}a^{3-3}$ $= \frac{9}{2}m^3a^0$ $= \frac{9}{2}m^3$ or $\frac{9m^3}{2}$	g $10hk^3 \times 4h^4$ $= 40 \times h^{1+4}k^3$ $= 40h^5k^3$	h $\frac{m^{11}}{(m^2)^8}$ $= \frac{m^{11}}{m^{16}}$ $= m^{11-16}$ $= m^{-5}$ or $\frac{1}{m^5}$
	i $\frac{p^2 \times p^7}{(p^3)^2} = \frac{p^{2+7}}{p^6} = \frac{p^9}{p^6} = p^{9-6} = p^3$			

9 a $5^0 = 1$

e $2^2 = 4$

i $5^2 = 25$

b $3^{-1} = \frac{1}{3}$

f $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

j $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

c $6^{-1} = \frac{1}{6}$

g $2^3 = 8$

k $10^2 = 100$

d $8^0 = 1$

h $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

l $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

10 a $\left(\frac{2}{3}\right)^0 = 1$

e 2×3^0
 $= 2 \times 1$
 $= 2$

i $\left(\frac{1}{3}\right)^{-1}$
 $= \left(\frac{3}{1}\right)^1$
 $= 3$

m $\left(\frac{2}{3}\right)^{-2}$
 $= \left(\frac{3}{2}\right)^2$
 $= \frac{9}{4} \text{ or } 2\frac{1}{4}$

b $\frac{4^3}{4^3} = 1$

f $6^0 = 1$

j $\left(\frac{2}{5}\right)^{-1}$
 $= \left(\frac{5}{2}\right)^1$
 $= 2\frac{1}{2}$

n $5^0 - 5^{-1}$
 $= 1 - \frac{1}{5}$
 $= \frac{4}{5}$

c $3y^0$
 $= 3 \times 1$
 $= 3$

g $\frac{5^2}{5^4}$
 $= 5^{2-4}$
 $= 5^{-2}$
 $= \frac{1}{5^2} \text{ or } \frac{1}{25}$

k $\left(\frac{4}{3}\right)^{-1}$
 $= \left(\frac{3}{4}\right)^1$
 $= \frac{3}{4}$

o $7^{-1} + 7^0$
 $= \frac{1}{7} + 1$
 $= 1\frac{1}{7} \text{ or } \frac{8}{7}$

d $(3y)^0$
 $= 1$

h $\frac{2^{10}}{2^{15}}$
 $= 2^{10-15}$
 $= 2^{-5}$
 $= \frac{1}{2^5} \text{ or } \frac{1}{32}$

l $\left(\frac{1}{12}\right)^{-1}$
 $= \left(\frac{12}{1}\right)^1$
 $= 12$

p $2^0 + 2^1 + 2^{-1}$
 $= 1 + 2 + \frac{1}{2}$
 $= 3\frac{1}{2} \text{ or } \frac{7}{2}$

11 a $(2a)^{-1}$
 $= \frac{1}{2a}$

e $\left(\frac{2}{b}\right)^{-2}$
 $= \left(\frac{b}{2}\right)^2$
 $= \frac{b^2}{4}$

i ab^{-1}
 $= \frac{a}{1} \times \frac{1}{b}$
 $= \frac{a}{b}$

m $(2ab)^{-1}$
 $= \frac{1}{2ab}$

b $2a^{-1}$
 $= \frac{2}{1} \times \frac{1}{a}$
 $= \frac{2}{a}$

f $(2b)^{-2}$
 $= \frac{1}{(2b)^2}$
 $= \frac{1}{4b^2}$

j $(ab)^{-1}$
 $= \frac{1}{ab}$

n $2(ab)^{-1}$
 $= \frac{2}{1} \times \frac{1}{ab}$
 $= \frac{2}{ab}$

c $3b^{-1}$
 $= \frac{3}{1} \times \frac{1}{b}$
 $= \frac{3}{b}$

g $(3n)^{-2}$
 $= \frac{1}{(3n)^2}$
 $= \frac{1}{9n^2}$

k ab^{-2}
 $= a \times \frac{1}{b^2}$
 $= \frac{a}{b^2}$

o $2ab^{-1}$
 $= \frac{2a}{1} \times \frac{1}{b}$
 $= \frac{2a}{b}$

d $(3b)^{-1}$
 $= \frac{1}{3b}$

h $(3n^{-2})^{-1}$
 $= 3^{-1}n^2$
 $= \frac{1}{3}n^2$
 $= \frac{n^2}{3}$

l $(ab)^{-2}$
 $= \frac{1}{(ab)^2}$
 $= \frac{1}{a^2b^2}$

p $\frac{(ab)^2}{b^{-1}}$
 $= \frac{a^2b^2}{b^{-1}}$
 $= a^2b^{2-(-1)}$
 $= a^2b^3$

12

a $\frac{1}{3} = 3^{-1}$

b $\frac{1}{2} = 2^{-1}$

c $\frac{1}{5} = 5^{-1}$

d $\frac{1}{4} = \frac{1}{2^2} = 2^{-2}$

e $\frac{1}{27} = \frac{1}{3^3}$
 $= 3^{-3}$

f $\frac{1}{25} = \frac{1}{5^2}$
 $= 5^{-2}$

g $\frac{1}{8^x}$
 $= \frac{1}{(2^3)^x}$
 $= \frac{1}{2^{3x}}$
 $= 2^{-3x}$

h $\frac{1}{16^y}$
 $= \frac{1}{(2^4)^y}$
 $= \frac{1}{2^{4y}}$
 $= 2^{-4y}$

i $\frac{1}{81^a}$
 $= \frac{1}{(3^4)^a}$
 $= \frac{1}{3^{4a}}$
 $= 3^{-4a}$

j $\frac{9}{3^4}$
 $= \frac{3^2}{3^4}$
 $= 3^{2-4}$
 $= 3^{-2}$

k 25×5^{-4}
 $= 5^2 \times 5^{-4}$
 $= 5^{2+(-4)}$
 $= 5^{-2}$

l $\frac{5^{-1}}{5^2}$
 $= 5^{-1-2}$
 $= 5^{-3}$

m $2 \div 2^{-3}$
 $= 2^1 \div 2^{-3}$
 $= 2^{1-(-3)}$
 $= 2^4$

n 1
 $= 2^0$ or 3^0 or 5^0

o 6^{-3}
 $= (2 \times 3)^{-3}$
 $= 2^{-3} \times 3^{-3}$

p 4×10^2
 $= 2^2 \times (2 \times 5)^2$
 $= 2^2 \times 2^2 \times 5^2$
 $= 2^4 \times 5^2$

13 1 less day, i.e., 25 days

14 For 1 coin, 1 sum
 For 2 coins, 3 sums
 For 3 coins, 7 sums
 For 4 coins, 15 sums

and $1 = 2^1 - 1$
 $3 = 2^2 - 1$
 $7 = 2^3 - 1$
 $15 = 2^4 - 1$

For coins A, B, C

ABC	AB	A
	AC	B
	BC	C

For coins A, B, C, D

ABCD	ABC	AB	A
	ABD	AC	B
	ACD	AD	C
	BCD	BC	D
		BD	
		CD	

So, for 6 coins we expect $2^6 - 1 = 63$ different sums.

15

a $5^3 = 21 + 23 + 25 + 27 + 29$

b $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$

c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155$

16 We notice that 175 and 75 have a common factor of 25.

So 2^{175} and 5^{75}

$= (2^7)^{25}$ $= (5^3)^{25}$

$= (128)^{25}$ $= (125)^{25}$ $\therefore$ as $125 < 128$, 5^{75} is smaller.

EXERCISE 3D

1

a $\sqrt[5]{2}$
 $= 2^{\frac{1}{5}}$

b $\frac{1}{\sqrt[5]{2}}$
 $= \frac{1}{2^{\frac{1}{5}}}$
 $= 2^{-\frac{1}{5}}$

c $2\sqrt{2}$
 $= 2^1 \times 2^{\frac{1}{2}}$
 $= 2^{\frac{3}{2}}$

d $4\sqrt{2}$
 $= 2^2 \times 2^{\frac{1}{2}}$
 $= 2^{\frac{5}{2}}$

e $\frac{1}{\sqrt[3]{2}}$
 $= \frac{1}{2^{\frac{1}{3}}}$
 $= 2^{-\frac{1}{3}}$

$$\begin{aligned} \mathbf{f} \quad 2 \times \sqrt[3]{2} \\ = 2^1 \times 2^{\frac{1}{3}} \\ = 2^{\frac{4}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad \frac{4}{\sqrt{2}} \\ = \frac{2^2}{2^{\frac{1}{2}}} \\ = 2^{2-\frac{1}{2}} \\ = 2^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad (\sqrt{2})^3 \\ = \left(2^{\frac{1}{2}}\right)^3 \\ = 2^{\frac{1}{2} \times 3} \\ = 2^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad \frac{1}{\sqrt[3]{16}} \\ = \frac{1}{(2^4)^{\frac{1}{3}}} \\ = \frac{1}{2^{\frac{4}{3}}} \\ = 2^{-\frac{4}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad \frac{1}{\sqrt{8}} \\ = \frac{1}{(2^3)^{\frac{1}{2}}} \\ = \frac{1}{2^{\frac{3}{2}}} \\ = 2^{-\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \sqrt[3]{3} \\ = 3^{\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \frac{1}{\sqrt[3]{3}} \\ = \frac{1}{3^{\frac{1}{3}}} \\ = 3^{-\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \sqrt[4]{3} \\ = 3^{\frac{1}{4}} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 3\sqrt{3} \\ = 3^1 \times 3^{\frac{1}{2}} \\ = 3^{1\frac{1}{2}} \\ = 3^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \frac{1}{9\sqrt{3}} \\ = \frac{1}{3^2 3^{\frac{1}{2}}} \\ = \frac{1}{3^{2\frac{1}{2}}} \\ = 3^{-\frac{5}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \sqrt[3]{7} \\ = 7^{\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \sqrt[4]{27} \\ = (3^3)^{\frac{1}{4}} \\ = 3^{\frac{3}{4}} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \sqrt[5]{16} \\ = (2^4)^{\frac{1}{5}} \\ = 2^{\frac{4}{5}} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \sqrt[3]{32} \\ = (2^5)^{\frac{1}{3}} \\ = 2^{\frac{5}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \sqrt[7]{49} \\ = (7^2)^{\frac{1}{7}} \\ = 7^{\frac{2}{7}} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{1}{\sqrt[3]{7}} \\ = \frac{1}{7^{\frac{1}{3}}} \\ = 7^{-\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad \frac{1}{\sqrt[4]{27}} \\ = \frac{1}{(3^3)^{\frac{1}{4}}} \\ = \frac{1}{3^{\frac{3}{4}}} \\ = 3^{-\frac{3}{4}} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad \frac{1}{\sqrt[5]{16}} \\ = \frac{1}{(2^4)^{\frac{1}{5}}} \\ = \frac{1}{2^{\frac{4}{5}}} \\ = 2^{-\frac{4}{5}} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad \frac{1}{\sqrt[3]{32}} \\ = \frac{1}{(2^5)^{\frac{1}{3}}} \\ = \frac{1}{2^{\frac{5}{3}}} \\ = 2^{-\frac{5}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad \frac{1}{\sqrt[7]{49}} \\ = \frac{1}{(7^2)^{\frac{1}{7}}} \\ = \frac{1}{7^{\frac{2}{7}}} \\ = 7^{-\frac{2}{7}} \end{aligned}$$

$$\mathbf{4} \quad \mathbf{a} \quad 2.280$$

$$\mathbf{b} \quad 1.834$$

$$\mathbf{c} \quad 0.794$$

$$\mathbf{d} \quad 0.435$$

$$\mathbf{5} \quad \mathbf{a} \quad 3$$

$$\mathbf{b} \quad 1.682$$

$$\mathbf{c} \quad 1.933$$

$$\mathbf{d} \quad 0.523$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad 4^{\frac{3}{2}} \\ = (2^2)^{\frac{3}{2}} \\ = 2^3 \\ = 8 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 8^{\frac{5}{3}} \\ = (2^3)^{\frac{5}{3}} \\ = 2^5 \\ = 32 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 16^{\frac{3}{4}} \\ = (2^4)^{\frac{3}{4}} \\ = 2^3 \\ = 8 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 25^{\frac{3}{2}} \\ = (5^2)^{\frac{3}{2}} \\ = 5^3 \\ = 125 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 32^{\frac{2}{5}} \\ = (2^5)^{\frac{2}{5}} \\ = 2^2 \\ = 4 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad 4^{-\frac{1}{2}} \\ = (2^2)^{-\frac{1}{2}} \\ = 2^{-1} \\ = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad 9^{-\frac{3}{2}} \\ = (3^2)^{-\frac{3}{2}} \\ = 3^{-3} \\ = \frac{1}{3^3} \\ = \frac{1}{27} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad 8^{-\frac{4}{3}} \\ = (2^3)^{-\frac{4}{3}} \\ = 2^{-4} \\ = \frac{1}{2^4} \\ = \frac{1}{16} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad 27^{-\frac{4}{3}} \\ = (3^3)^{-\frac{4}{3}} \\ = 3^{-4} \\ = \frac{1}{3^4} \\ = \frac{1}{81} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad 125^{-\frac{2}{3}} \\ = (5^3)^{-\frac{2}{3}} \\ = 5^{-2} \\ = \frac{1}{5^2} \\ = \frac{1}{25} \end{aligned}$$

EXERCISE 3E

- 1 a** $x^2(x^3 + 2x^2 + 1)$
 $= x^5 + 2x^4 + x^2$
- b** $2^x(2^x + 1)$
 $= 2^{2x} + 2^x$
- c** $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$
 $= x^1 + x^0$
 $= x + 1$
- d** $e^x(e^x + 2)$
 $= e^{2x} + 2e^x$
- e** $3^x(2 - 3^{-x})$
 $= 2 \times 3^x - 3^0$
 $= 2 \times 3^x - 1$
- f** $x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})$
 $= x^2 + 2x^1 + 3x^0$
 $= x^2 + 2x + 3$
- g** $2^{-x}(2^x + 5)$
 $= 2^0 + 5 \times 2^{-x}$
 $= 1 + 5 \times 2^{-x}$
- h** $5^{-x}(5^{2x} + 5^x)$
 $= 5^{-x+2x} + 5^{-x+x}$
 $= 5^x + 5^0$
 $= 5^x + 1$
- i** $x^{-\frac{1}{2}}(x^2 + x^1 + x^{\frac{1}{2}})$
 $= x^{1\frac{1}{2}} + x^{\frac{1}{2}} + x^0$
 $= x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$
- 2 a** $(2^x + 1)(2^x + 3)$
 $= 2^{2x} + 3 \times 2^x + 2^x + 3$
 $= 2^{2x} + 4 \times 2^x + 3$
 $= 4^x + 2^2 \times 2^x + 3$
 $= 4^x + 2^{2+x} + 3$
- b** $(3^x + 2)(3^x + 5)$
 $= 3^{2x} + 5 \times 3^x + 2 \times 3^x + 10$
 $= 9^x + 7 \times 3^x + 10$
- c** $(5^x - 2)(5^x - 4)$
 $= 5^{2x} - 4 \times 5^x - 2 \times 5^x + 8$
 $= 25^x - 6 \times 5^x + 8$
- d** $(2^x + 3)^2$
 $= (2^x)^2 + 2 \times 2^x \times 3 + 3^2$
 $= 2^{2x} + 3 \times 2^{1+x} + 9$
 $= 4^x + 3 \times 2^{x+1} + 9$
- e** $(3^x - 1)^2$
 $= (3^x)^2 - 2 \times 3^x + 1$
 $= 3^{2x} - 2 \times 3^x + 1$
 $= 9^x - 2 \times 3^x + 1$
- f** $(4^x + 7)^2$
 $= (4^x)^2 + 2 \times 4^x \times 7 + 7^2$
 $= 4^{2x} + 7 \times 2 \times 2^{2x} + 49$
 $= 2^{4x} + 7 \times 2^{2x+1} + 49$
- g** $(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$
 $= (x^{\frac{1}{2}})^2 - 2^2$
 $= x - 4$
- h** $(2^x + 3)(2^x - 3)$
 $= (2^x)^2 - 3^2$
 $= 2^{2x} - 9$
- i** $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$
 $= x^1 - x^{-1}$
 $= x - \frac{1}{x}$
- j** $\left(x + \frac{2}{x}\right)^2$
 $= x^2 + 2 \times x \times \frac{2}{x} + \frac{4}{x^2}$
 $= x^2 + 4 + \frac{4}{x^2}$
- k** $(e^x - e^{-x})^2$
 $= (e^x)^2 - 2e^x e^{-x} + e^{-2x}$
 $= e^{2x} - 2 + e^{-2x}$
- l** $(5 - 2^{-x})^2$
 $= 5^2 - 2 \times 5 \times 2^{-x} + (2^{-x})^2$
 $= 25 - 5 \times 2^{1-x} + 2^{-2x}$

EXERCISE 3F

- 1 a** $2^x = 2$
 $\therefore 2^x = 2^1$
 $\therefore x = 1$
- b** $2^x = 4$
 $\therefore 2^x = 2^2$
 $\therefore x = 2$
- c** $3^x = 27$
 $\therefore 3^x = 3^3$
 $\therefore x = 3$
- d** $2^x = 1$
 $\therefore 2^x = 2^0$
 $\therefore x = 0$

e $2^x = \frac{1}{2}$

$\therefore 2^x = 2^{-1}$

$\therefore x = -1$

f $3^x = \frac{1}{3}$

$\therefore 3^x = 3^{-1}$

$\therefore x = -1$

g $2^x = \frac{1}{8}$

$\therefore 2^x = 2^{-3}$

$\therefore x = -3$

h $2^{x+1} = 8$

$\therefore 2^{x+1} = 2^3$

$\therefore x+1 = 3$

$\therefore x = 2$

i $2^{x-2} = \frac{1}{4}$

$\therefore 2^{x-2} = 2^{-2}$

$\therefore x-2 = -2$

$\therefore x = 0$

j $3^{x+1} = \frac{1}{27}$

$\therefore 3^{x+1} = 3^{-3}$

$\therefore x+1 = -3$

$\therefore x = -4$

k $2^{x+1} = 64$

$\therefore 2^{x+1} = 2^6$

$\therefore x+1 = 6$

$\therefore x = 5$

l $2^{1-2x} = \frac{1}{2}$

$\therefore 2^{1-2x} = 2^{-1}$

$\therefore 1-2x = -1$

$\therefore -2x = -2$

$\therefore x = 1$

2 a $4^x = 32$

$\therefore 2^{2x} = 2^5$

$\therefore 2x = 5$

$\therefore x = \frac{5}{2}$

b $8^x = \frac{1}{4}$

$\therefore 2^{3x} = 2^{-2}$

$\therefore 3x = -2$

$\therefore x = -\frac{2}{3}$

c $9^x = \frac{1}{3}$

$\therefore 3^{2x} = 3^{-1}$

$\therefore 2x = -1$

$\therefore x = -\frac{1}{2}$

d $49^x = \frac{1}{7}$

$\therefore 7^{2x} = 7^{-1}$

$\therefore 2x = -1$

$\therefore x = -\frac{1}{2}$

e $4^x = \frac{1}{8}$

$\therefore 2^{2x} = 2^{-3}$

$\therefore 2x = -3$

$\therefore x = -\frac{3}{2}$

f $25^x = \frac{1}{5}$

$\therefore 5^{2x} = 5^{-1}$

$\therefore 2x = -1$

$\therefore x = -\frac{1}{2}$

g $8^{x+2} = 32$

$\therefore 2^{3(x+2)} = 2^5$

$\therefore 3x+6 = 5$

$\therefore 3x = -1$

$\therefore x = -\frac{1}{3}$

h $8^{1-x} = \frac{1}{4}$

$\therefore 2^{3(1-x)} = 2^{-2}$

$\therefore 3-3x = -2$

$\therefore -3x = -5$

$\therefore x = \frac{5}{3}$

i $4^{2x-1} = \frac{1}{2}$

$\therefore 2^{2(2x-1)} = 2^{-1}$

$\therefore 4x-2 = -1$

$\therefore 4x = 1$

$\therefore x = \frac{1}{4}$

j $9^{x-3} = 3$

$\therefore 3^{2(x-3)} = 3^1$

$\therefore 2x-6 = 1$

$\therefore 2x = 7$

$\therefore x = \frac{7}{2}$

k $\left(\frac{1}{2}\right)^{x+1} = 2$

$\therefore (2^{-1})^{x+1} = 2^1$

$\therefore -x-1 = 1$

$\therefore -x = 2$

$\therefore x = -2$

l $\left(\frac{1}{3}\right)^{x+2} = 9$

$\therefore (3^{-1})^{x+2} = 3^2$

$\therefore -x-2 = 2$

$\therefore -x = 4$

$\therefore x = -4$

m $4^x = 8^{-x}$

$\therefore 2^{2x} = (2^3)^{-x}$

$\therefore 2x = -3x$

$\therefore 5x = 0$

$\therefore x = 0$

n $\left(\frac{1}{4}\right)^{1-x} = 8$

$\therefore (2^{-2})^{1-x} = 2^3$

$\therefore -2+2x = 3$

$\therefore 2x = 5$

$\therefore x = \frac{5}{2}$

o $\left(\frac{1}{7}\right)^x = 49$

$\therefore (7^{-1})^x = 7^2$

$\therefore -x = 2$

$\therefore x = -2$

p $\left(\frac{1}{2}\right)^{x+1} = 32$

$\therefore (2^{-1})^{x+1} = 2^5$

$\therefore -x-1 = 5$

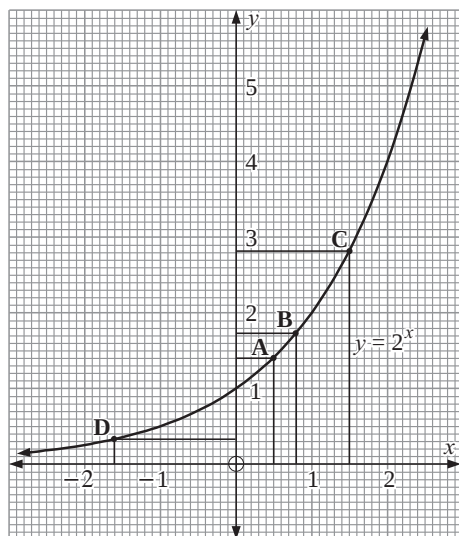
$\therefore -x = 6$

$\therefore x = -6$

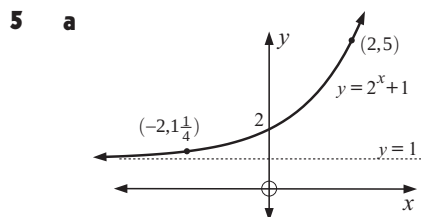
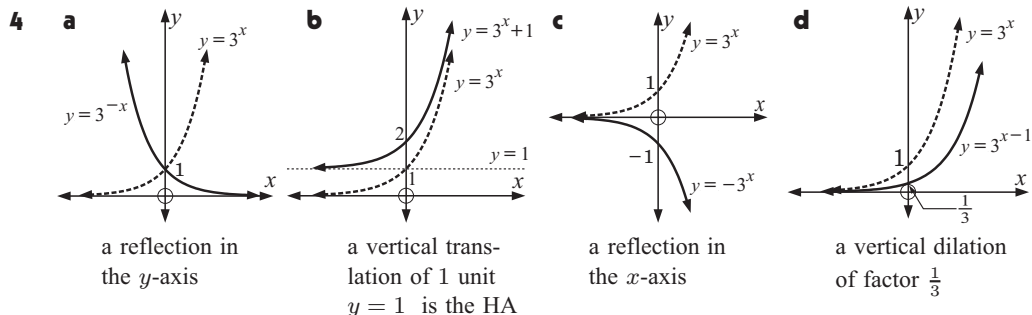
- 3 a** $4^{2x+1} = 8^{1-x}$
 $\therefore (2^2)^{2x+1} = (2^3)^{1-x}$
 $\therefore 4x + 2 = 3 - 3x$
 $\therefore 7x = 1$
 $\therefore x = \frac{1}{7}$
- b** $9^{2-x} = \left(\frac{1}{3}\right)^{2x+1}$
 $\therefore (3^2)^{2-x} = (3^{-1})^{2x+1}$
 $\therefore 4 - 2x = -2x - 1$
 $\therefore 4 = -1$
 So, no solutions exist.
- c** $2^x \times 8^{1-x} = \frac{1}{4}$
 $\therefore 2^x \times (2^3)^{1-x} = 2^{-2}$
 $\therefore x + 3 - 3x = -2$
 $\therefore -2x = -5$
 $\therefore x = \frac{5}{2}$
- 4 a** $3 \times 2^x = 24$
 $\therefore 2^x = 8$
 $\therefore 2^x = 2^3$
 $\therefore x = 3$
- b** $7 \times 2^x = 56$
 $\therefore 2^x = 8$
 $\therefore 2^x = 2^3$
 $\therefore x = 3$
- c** $3 \times 2^{x+1} = 24$
 $\therefore 2^{x+1} = 8$
 $\therefore 2^{x+1} = 2^3$
 $\therefore x + 1 = 3$
 $\therefore x = 2$
- d** $12 \times 3^{-x} = \frac{4}{3}$
 $\therefore 3^{-x} = \frac{4}{3} \div 12$
 $\therefore 3^{-x} = \frac{4}{3} \times \frac{1}{12}$
 $\therefore 3^{-x} = \frac{1}{9}$
 $\therefore 3^{-x} = 3^{-2}$
 $\therefore x = 2$
- e** $4 \times \left(\frac{1}{3}\right)^x = 36$
 $\therefore \left(\frac{1}{3}\right)^x = 9$
 $\therefore (3^{-1})^x = 3^2$
 $\therefore 3^{-x} = 3^2$
 $\therefore -x = 2$
 $\therefore x = -2$
- f** $5 \times \left(\frac{1}{2}\right)^x = 20$
 $\therefore \left(\frac{1}{2}\right)^x = 4$
 $\therefore (2^{-1})^x = 2^2$
 $\therefore -x = 2$
 $\therefore x = -2$

EXERCISE 3G

- 1 a** When $x = \frac{1}{2}$, $y = 2^{\frac{1}{2}}$
 from point A, $y \div 1.4$
 $\therefore 2^{\frac{1}{2}} \div 1.4$
- b** When $x = 0.8$, $y = 2^{0.8}$
 from point B, $y \div 1.7$
 $\therefore 2^{0.8} \div 1.7$
- c** When $x = 1.5$, $y = 2^{1.5}$
 from point C, $y \div 2.8$
 $\therefore 2^{1.5} \div 2.8$
- d** When $x = -1.6$, $y = 2^{-1.6}$
 from point D, $y \div 0.3$
 $\therefore 2^{-1.6} \div 0.3$



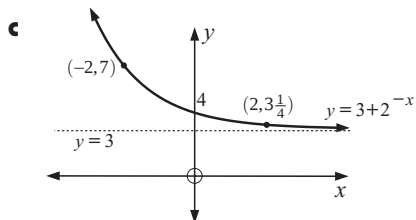
- 2 a**
 a vertical translation of 2 units downwards
 $y = -2$ is a HA
- b**
 a reflection in the y-axis
- c**
 a vertical dilation of factor $\frac{1}{4}$
- d**
 a vertical dilation of factor 2



a vertical translation of 1 unit
Horizontal asymptote is $y = 1$.

$$\text{When } x = 2, \quad y = 4 + 1 = 5$$

$$\text{When } x = -2, \quad y = \frac{1}{4} + 1 = 1\frac{1}{4}$$

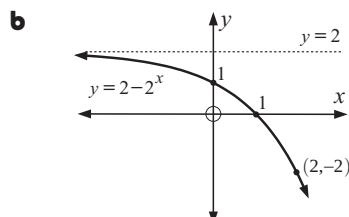


The HA is $y = 3$.

$$\text{When } x = 0, \quad y = 3 + 1 = 4$$

$$\text{When } x = 2, \quad y = 3 + \frac{1}{4} = 3\frac{1}{4}$$

$$\text{When } x = -2, \quad y = 3 + 2^2 = 7$$



The horizontal asymptote is $y = 2$.

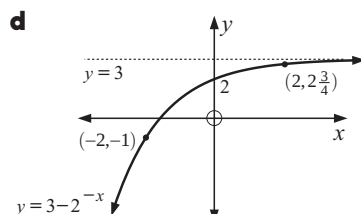
$$\text{When } x = 0, \quad y = 2 - 2^0 = 2 - 1 = 1$$

$\therefore$ y -intercept is 1

$$\text{When } x = 1, \quad y = 2 - 2 = 0$$

$$\text{When } x = 2, \quad y = 2 - 4 = -2$$

$$\text{When } x = -2, \quad y = 2 - \frac{1}{4} = 1\frac{3}{4}$$



The HA is $y = 3$.

$$\text{When } x = 0, \quad y = 3 - 1 = 2$$

$$\text{When } x = 2, \quad y = 3 - \frac{1}{4} = 2\frac{3}{4}$$

$$\text{When } x = -2, \quad y = 3 - 4 = -1$$

EXERCISE 3H

1 a When $t = 0$, $W_0 = 100$ grams

b i When $t = 4$,

$$W_4 = 100 \times 2^{0.1 \times 4}$$

$$= 100 \times 2^{0.4}$$

$$\doteq 132 \text{ grams}$$

ii When $t = 10$,

$$W_{10} = 100 \times 2^1$$

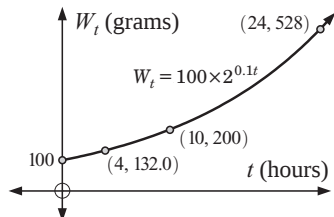
$$= 200 \text{ grams}$$

iii When $t = 24$,

$$W_{24} = 100 \times 2^{0.1 \times 24}$$

$$= 100 \times 2^{2.4}$$

$$\doteq 528 \text{ grams}$$



2 a $P_0 = 50$ (the initial population)

b i When $t = 2$,

$$P_2 = 50 \times 2^{0.3 \times 2}$$

$$= 50 \times 2^{0.6}$$

$$\doteq 75.785\dots$$

So, the expected population is 76 possums.

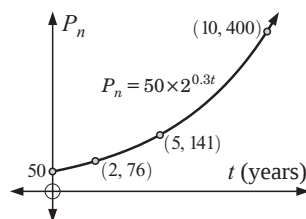
ii When $t = 5$,

$$P_5 = 50 \times 2^{0.3 \times 5}$$

$$= 50 \times 2^{1.5}$$

$$\doteq 141.421\dots$$

So, the expected population is 141 possums.



iii When $t = 10$,

$$P_{10} = 50 \times 2^{0.3 \times 10}$$

$$= 50 \times 2^3$$

$$= 400$$

So, the expected population is 400 possums.

3 a When $t = 0$

$$V_0 = V_0 \times 2^0$$

$$= V_0$$

So, the speed is V_0 .

b When $t = 20$

$$V_{20} = V_0 \times 2^{0.05 \times 20}$$

$$= V_0 \times 2^1$$

$$= 2V_0$$

So, the speed is $2V_0$.

c V_0 becomes $2V_0$
a 100% increase.

d

$$\left(\frac{V_{50} - V_{20}}{V_{20}} \right) \times 100\%$$

$$= \left(\frac{V_0 \times 2^{2.5} - V_0 \times 2^1}{V_0 \times 2^1} \right) \times 100\%$$

$$= \left(\frac{2^{2.5} - 2^1}{2^1} \right) \times 100\%$$

$$\doteq 183\%$$

This expression is the percentage increase in the speed from a speed at 20°C increased to the speed at 50°C .
 $V_{50} - V_{20}$ is the increase in speed.

4 a $B_0 = 6$ pairs = 12 bears

c At year 2008, $t = 10$

b At year 2018, $t = 20$

$$\therefore B_{20} = 12 \times 2^{0.18 \times 20}$$

$$= 12 \times 2^{3.6}$$

$$\doteq 145.508\dots$$

$$\doteq 146 \text{ bears}$$

$$\therefore \% \text{ increase} = \left(\frac{B_{20} - B_{10}}{B_{10}} \right) \times 100\%$$

$$= \left(\frac{12 \times 2^{3.6} - 12 \times 2^{1.8}}{12 \times 2^{1.8}} \right) \times 100\%$$

$$= \left(\frac{2^{3.6} - 2^{1.8}}{2^{1.8}} \right) \times 100\%$$

$$\doteq 248\%$$

EXERCISE 31

1 $W_t = 250 \times (0.998)^t$ grams

a $W_0 = 250 \times (0.998)^0$

$$= 250 \times 1$$

$$= 250 \text{ grams}$$

b i When $t = 400$

$$W_{400}$$

$$= 250 \times (0.998)^{400}$$

$$\doteq 112 \text{ grams}$$

ii When $t = 800$

$$W_{800}$$

$$= 250 \times (0.998)^{800}$$

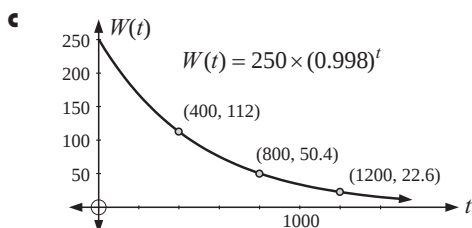
$$\doteq 50.4 \text{ grams}$$

iii When $t = 1200$

$$W_{1200}$$

$$= 250 \times (0.998)^{1200}$$

$$\doteq 22.6 \text{ grams}$$



- d** When $W(t) = 125$
 $250 \times (0.998)^t = 125$
 $\therefore (0.998)^t = 0.5$
 $\therefore t \doteq 346.2$ {technology}
 i.e., it takes approximately 346 years

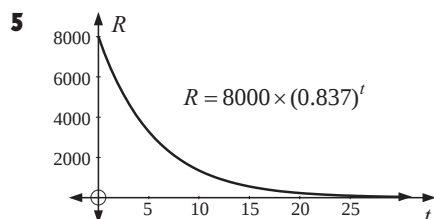
2 $R = 8000 \times (0.837)^t$

1 When $t = 0$, $R = 8000 \times (0.837)^0$
 $= 8000 \times 1$
 $= 8000$ rabbits

2 'to the power 3.5' means that
 $(0.837)^{3.5} = (0.837)^3 \times \sqrt{0.837}$

3 When $R = 80$,
 $8000 \times (0.837)^t = 80$
 $\therefore (0.837)^t = 0.01$
 and by trial-and-error methods, $t \doteq 26$
 i.e., it would take 26 weeks (approx)

- 4** Yes, after a very long time.
 Notice that when $R = 1$
 $8000(0.837)^t = 1$
 $\therefore (0.837)^t = 0.000125$
 and by trial and error $t \doteq 51$
 i.e., after 51 weeks

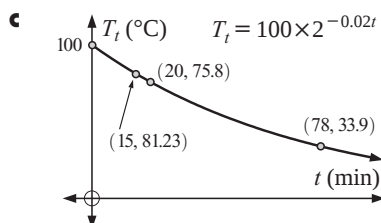


3 $T_t = 100 \times 2^{-0.02t}$

a $T_0 = 100 \times 2^0$
 $= 100 \times 1$
 $= 100^\circ\text{C}$

b i $T_{15} = 100 \times 2^{-0.02 \times 15}$
 $= 100 \times 2^{-0.3}$
 $\doteq 81.2^\circ\text{C}$

ii $T_{20} = 100 \times 2^{-0.02 \times 20}$
 $= 100 \times 2^{-0.4}$
 $\doteq 75.8^\circ\text{C}$

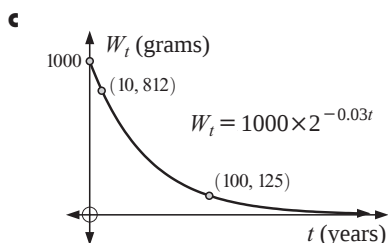


iii $T_{78} = 100 \times 2^{-0.02 \times 78}$
 $= 100 \times 2^{-1.56}$
 $\doteq 33.9^\circ\text{C}$

4 $W_t = 1000 \times 2^{-0.03t}$

a $W_0 = 1000 \times 2^0$
 $= 1000 \times 1$
 $= 1000$ grams

b i $W_{10} = 1000 \times 2^{-0.3}$
 $\doteq 812$ g



ii $W_{100} = 1000 \times 2^{-3}$
 $= 125$ g

iii $W_{1000} = 1000 \times 2^{-30}$
 $\doteq 9.31 \times 10^{-7}$ g

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & \text{When } t = 0, \quad W_0 = W_0 2^0 \\ & = W_0 \\ \therefore & \text{the original weight was } W_0 \end{aligned}$$

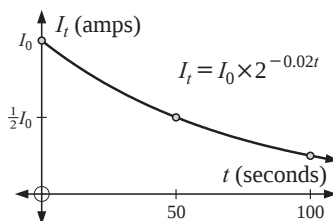
$$\begin{aligned} \mathbf{b} \quad \% \text{ loss} &= \left(\frac{W_{1000} - W_0}{W_0} \right) \times 100\% \\ &= \left(\frac{W_0 \times 2^{-0.2} - W_0}{W_0} \right) \times 100\% \\ &= (2^{-0.2} - 1) \times 100\% \\ &\div -12.9\% \\ \text{i.e.,} \quad & \text{a } 12.9\% \text{ weight loss} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad & \text{When } t = 0, \quad I_0 = I_0 \times 2^0 \\ & = I_0 \times 1 \\ & = I_0 \\ \therefore & \text{the original current is } I_0 \text{ amps} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \% \text{ change} &= \left(\frac{I_1 - I_0}{I_0} \right) \times 100\% \\ &\div \left(\frac{0.986I_0 - I_0}{I_0} \right) \times 100\% \\ &\div (0.986 - 1) \times 100\% \\ &\div -0.0138 \times 100\% \\ &\div 1.38\% \text{ loss} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \text{When } t = 1, \quad I_1 = I_0 \times 2^{-0.02} \\ & \div 0.9862 \times I_0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad I_{50} &= I_0 \times 2^{-1} = \frac{1}{2} I_0 \\ I_{100} &= I_0 \times 2^{-2} = \frac{1}{4} I_0 \end{aligned}$$



REVIEW SET 3A

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & -(-1)^{10} \\ & = -1 \quad \{(-1)^{10} = 1\} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & -(-3)^3 \\ & = -[-27] \\ & = 27 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 3^0 - 3^{-1} \\ & = 1 - \frac{1}{3} \\ & = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & a^4 b^5 \times a^2 b^2 \\ & = a^{4+2} \times b^{5+2} \\ & = a^6 b^7 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 6xy^5 \div 9x^2y^5 \\ & = \frac{6}{9} x^{1-2} y^{5-5} \\ & = \frac{2}{3} x^{-1} y^0 \\ & = \frac{2}{3x} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \frac{5(x^2y)^2}{(5x^2)^2} \\ & = \frac{5 \times x^4 y^2}{25x^4} \\ & = \frac{1}{5} x^0 y^2 \text{ or } \frac{y^2}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & 2 \times 2^{-4} \\ & = 2^1 \times 2^{-4} \\ & = 2^{1+(-4)} \\ & = 2^{-3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 16 \div 2^{-3} \\ & = 2^4 \div 2^{-3} \\ & = 2^{4-(-3)} \\ & = 2^7 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 8^4 \\ & = (2^3)^4 \\ & = 2^{12} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & b^{-3} \\ & = \frac{1}{b^3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & (ab)^{-1} \\ & = \left(\frac{1}{ab} \right) \\ & = \frac{1}{ab} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & ab^{-1} \\ & = \frac{a}{1} \times \frac{1}{b} \\ & = \frac{a}{b} \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & 2^{x-3} = \frac{1}{32} \\ \therefore & 2^{x-3} = 2^{-5} \\ \therefore & x - 3 = -5 \\ \therefore & x = -2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 9^x = 27^{2-2x} \\ \therefore & (3^2)^x = (3^3)^{2-2x} \\ \therefore & 2x = 6 - 6x \\ \therefore & 8x = 6 \\ \therefore & x = \frac{6}{8} = \frac{3}{4} \end{aligned}$$

$$6 \quad a \quad 8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$$

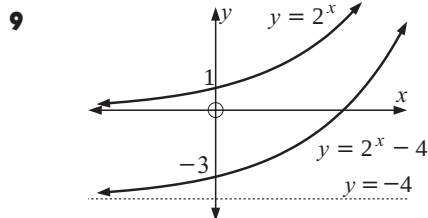
$$b \quad 27^{-\frac{2}{3}} = (3^3)^{-\frac{2}{3}} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$7 \quad a \quad 2.28 \quad b \quad 0.517 \quad c \quad 3.16$$

$$8 \quad f(x) = 3 \times 2^x \quad a \quad f(0) = 3 \times 2^0 \\ = 3 \times 1 \\ = 3$$

$$b \quad f(3) = 3 \times 2^3 \\ = 3 \times 8 \\ = 24$$

$$c \quad f(-2) = 3 \times 2^{-2} \\ = 3 \times \frac{1}{2^2} \\ = \frac{3}{4}$$



a $y = 2^x$ has y -intercept 1 and horizontal asymptote $y = 0$

b $y = 2^x - 4$ has y -intercept -3 and horizontal asymptote $y = -4$

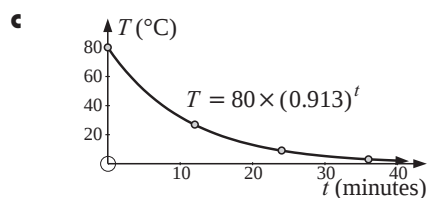
$$10 \quad T = 80 \times (0.913)^t \text{ } ^\circ\text{C}$$

a When $t = 0$, $T = 80 \times (0.913)^0$
 $= 80 \times 1$
 $= 80 \quad \therefore \text{initial temperature is } 80^\circ\text{C}$

b i When $t = 12$,
 $T = 80 \times (0.913)^{12}$
 $\div 26.8^\circ\text{C}$

ii When $t = 24$,
 $T = 80 \times (0.913)^{24}$
 $\div 9.00^\circ\text{C}$

iii When $t = 36$,
 $T = 80 \times (0.913)^{36}$
 $\div 3.02^\circ\text{C}$



d When $T = 25$
 $80 \times (0.913)^t = 25$
 $\therefore 0.913^t = 0.3125$
 $\therefore t \div 12.8 \text{ sec } \{\text{technology}\}$

REVIEW SET 3B

$$1 \quad a \quad -(-2)^3 \\ = -[-8] \\ = 8$$

$$b \quad 5^{-1} - 5^0 \\ = \frac{1}{5} - 1 \\ = -\frac{4}{5}$$

$$2 \quad a \quad (a^7)^3 \\ = a^{7 \times 3} \\ = a^{21}$$

$$b \quad pq^2 \times p^3q^4 \\ = p^{1+3}q^{2+4} \\ = p^4q^6$$

$$c \quad \frac{8ab^5}{2a^4b^4} \\ = \frac{8}{2}a^{1-4}b^{5-4} \\ = 4a^{-3}b^1 \\ = \frac{4}{1} \times \frac{1}{a^3} \times b \\ = \frac{4b}{a^3}$$

$$3 \quad a \quad \frac{1}{16} \\ = \frac{1}{2^4} \\ = 2^{-4}$$

$$b \quad 2^x \times 4 \\ = 2^x \times 2^2 \\ = 2^{x+2}$$

$$c \quad 4^x \div 8 \\ = (2^2)^x \div 2^3 \\ = 2^{2x} \div 2^3 \\ = 2^{2x-3}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & x^{-2} \times x^{-3} \\
 &= x^{-2+(-3)} \\
 &= x^{-5} \\
 &= \frac{1}{x^5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 2(ab)^{-2} \\
 &= 2 \times \frac{1}{(ab)^2} \\
 &= \frac{2}{a^2b^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 2ab^{-2} \\
 &= 2a \times \left(\frac{1}{b^2}\right) \\
 &= \frac{2a}{b^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & 2^{x+1} = 32 \\
 \therefore & 2^{x+1} = 2^5 \\
 \therefore & x+1 = 5 \\
 \therefore & x = 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 4^{x+1} = \left(\frac{1}{8}\right)^x \\
 \therefore & (2^2)^{x+1} = (2^{-3})^x \\
 \therefore & 2x+2 = -3x \\
 \therefore & 5x = -2 \\
 \therefore & x = -\frac{2}{5}
 \end{aligned}$$

$$\mathbf{6} \quad \mathbf{a} \quad 81 = 3^4$$

$$\mathbf{b} \quad 1 = 3^0$$

$$\mathbf{c} \quad \frac{1}{27} = \frac{1}{3^3} = 3^{-3}$$

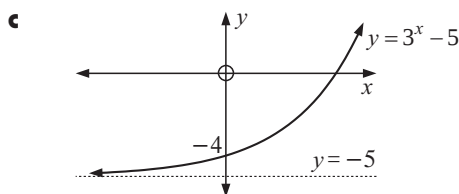
$$\mathbf{d} \quad \frac{1}{243} = \frac{1}{3^5} = 3^{-5}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad & \frac{27}{9^a} = \frac{3^3}{(3^2)^a} \\
 &= 3^{3-2a}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (\sqrt{3})^{1-x} \times 9^{1-2x} = \left(3^{\frac{1}{2}}\right)^{1-x} \times (3^2)^{1-2x} \\
 &= 3^{\frac{1}{2}-\frac{1}{2}x+2-4x} \\
 &= 3^{\frac{5}{2}-\frac{9}{2}x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & \text{When } x=0, \quad y=3^0-5=1-5=-4 \\
 & \text{When } x=1, \quad y=3^1-5=3-5=-2 \\
 & \text{When } x=2, \quad y=3^2-5=9-5=4 \\
 & \text{When } x=-1, \quad y=3^{-1}-5=\frac{1}{3}-5=-4\frac{2}{3} \\
 & \text{When } x=-2, \quad y=3^{-2}-5=\frac{1}{9}-5=-4\frac{8}{9}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{as } x \rightarrow \infty, \\
 & \quad 3^x \rightarrow \infty, \quad \therefore y \rightarrow \infty \\
 & \text{as } x \rightarrow -\infty, \\
 & \quad 3^x \rightarrow 0, \quad \therefore y \rightarrow -5 \quad (\text{from above})
 \end{aligned}$$



$$\mathbf{d} \quad y = -5 \text{ is the horizontal asymptote}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad & 27^x = 3 \\
 \therefore & (3^3)^x = 3^1 \\
 \therefore & 3x = 1 \\
 \therefore & x = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 9^{1-x} = 27^{x+2} \\
 \therefore & (3^2)^{1-x} = (3^3)^{x+2} \\
 \therefore & 2-2x = 3x+6 \\
 \therefore & -5x = 4 \quad \text{and so } x = -\frac{4}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad & 4^x \times 2^y = 16 \\
 \therefore & (2^2)^x \times 2^y = 2^4 \\
 \therefore & 2x+y = 4 \quad \dots (1)
 \end{aligned}$$

$$\begin{aligned}
 & 8^x = 2^{\frac{y}{2}} \\
 \therefore & (2^3)^x = 2^{\frac{y}{2}} \\
 \therefore & 3x = \frac{y}{2} \quad \dots (2)
 \end{aligned}$$

$$\begin{aligned}
 & \text{From (2), } y = 6x \\
 \therefore & \text{in (1), } 2x+6x = 4 \\
 \therefore & 8x = 4 \\
 \therefore & x = \frac{1}{2} \\
 \text{and so } & y = 6 \times \frac{1}{2} = 3
 \end{aligned}$$

REVIEW SET 3C

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \frac{1}{4} \\
 &= \frac{1}{2^2} \\
 &= 2^{-2}
 \end{aligned}$$

$$\mathbf{b} \quad 32 = 2^5$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{1}{\sqrt{2}} \\
 &= \frac{1}{2^{\frac{1}{2}}} \\
 &= 2^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \sqrt{8} \\
 &= (2^3)^{\frac{1}{2}} \\
 &= 2^{\frac{3}{2}}
 \end{aligned}$$

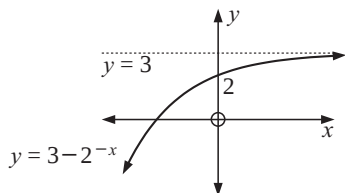
$$\begin{aligned}
 2 \quad a \quad & 4^a \times 8^b \\
 &= (2^2)^a \times (2^3)^b \\
 &= 2^{2a} \times 2^{3b} \\
 &= 2^{2a+3b}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \frac{2^{1-x}}{4^{x+2}} \\
 &= \frac{2^{1-x}}{(2^2)^{x+2}} \\
 &= \frac{2^{1-x}}{2^{2x+4}} \\
 &= 2^{1-x-(2x+4)} \\
 &= 2^{1-x-2x-4} \\
 &= 2^{-3x-3}
 \end{aligned}$$

$$\begin{aligned}
 3 \quad a \quad & \text{When } x = 0, \quad y = 3 - 2^0 = 3 - 1 = 2 \\
 & \text{When } x = 1, \quad y = 3 - 2^{-1} = 3 - \frac{1}{2} = 2\frac{1}{2} \\
 & \text{When } x = 2, \quad y = 3 - 2^{-2} = 3 - \frac{1}{4} = 2\frac{3}{4} \\
 & \text{When } x = -1, \quad y = 3 - 2^1 = 3 - 2 = 1 \\
 & \text{When } x = -2, \quad y = 3 - 2^2 = 3 - 4 = -1
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \text{as } x \rightarrow \infty, \\
 & \quad 2^{-x} \rightarrow 0, \quad \therefore y \rightarrow 3 \text{ (below)} \\
 & \text{as } x \rightarrow -\infty, \\
 & \quad 2^{-x} \rightarrow \infty, \quad \therefore y \rightarrow -\infty
 \end{aligned}$$

c

d HA is $y = 3$

$$\begin{aligned}
 4 \quad a \quad & 8^x = \frac{1}{\sqrt{2}} \\
 \therefore (2^3)^x &= \frac{1}{2^{\frac{1}{2}}} \\
 \therefore 2^{3x} &= 2^{-\frac{1}{2}} \\
 \therefore 3x &= -\frac{1}{2} \\
 \therefore x &= -\frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 4^{2x+1} = \left(\frac{1}{2}\right)^{2-x} \\
 \therefore (2^2)^{2x+1} &= (2^{-1})^{2-x} \\
 \therefore 4x + 2 &= -2 + x \\
 \therefore 3x &= -4 \\
 \therefore x &= -\frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad a \quad & mn^{-2} \\
 &= m \times \frac{1}{n^2} \\
 &= \frac{m}{n^2}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & (mn)^{-3} \\
 &= \frac{1}{(mn)^3} \\
 &= \frac{1}{m^3n^3}
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \frac{m^2n^{-1}}{p^{-2}} \\
 &= m^2 \left(\frac{1}{n}\right) p^2 \\
 &= \frac{m^2p^2}{n}
 \end{aligned}$$

$$\begin{aligned}
 d \quad & (4m^{-1}n)^2 \\
 &= 4^2 m^{-2} n^2 \\
 &= \frac{16n^2}{m^2}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad a \quad & \sqrt[4]{5} \\
 &= 5^{\frac{1}{4}}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \sqrt[3]{27} \\
 &= (3^3)^{\frac{1}{3}} \\
 &= 3^{\frac{3}{3}}
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \frac{1}{\sqrt[3]{16}} \\
 &= \frac{1}{(2^4)^{\frac{1}{3}}} \\
 &= \frac{1}{2^{\frac{4}{3}}} \\
 &= 2^{-\frac{4}{3}}
 \end{aligned}$$

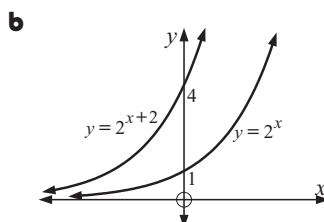
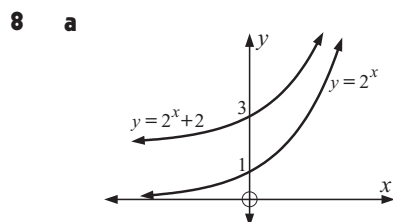
$$\begin{aligned}
 d \quad & \frac{1}{\sqrt{125}} \\
 &= \frac{1}{(5^3)^{\frac{1}{2}}} \\
 &= \frac{1}{5^{\frac{3}{2}}} \\
 &= 5^{-\frac{3}{2}}
 \end{aligned}$$

$$\begin{aligned}
 7 \quad a \quad & 8^{\frac{4}{3}} \\
 &= (2^3)^{\frac{4}{3}} \\
 &= 2^4 \\
 &= 16
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 27^{\frac{4}{3}} \\
 &= (3^3)^{\frac{4}{3}} \\
 &= 3^4 \\
 &= 81
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 32^{-\frac{2}{5}} \\
 &= (2^5)^{-\frac{2}{5}} \\
 &= 2^{-2} \\
 &= \frac{1}{2^2} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 d \quad & (25)^{-\frac{3}{2}} \\
 &= (5^2)^{-\frac{3}{2}} \\
 &= 5^{-3} \\
 &= \frac{1}{5^3} \\
 &= \frac{1}{125}
 \end{aligned}$$



9 $y = 2^{-x} - 5$

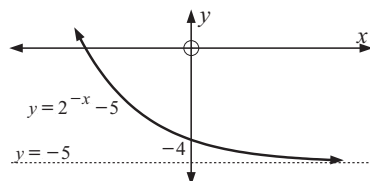
When $x = 0$, $y = 2^0 - 5 = 1 - 5 = -4$

$\therefore$ y -intercept is -4

as $x \rightarrow \infty$, $2^{-x} \rightarrow 0 \therefore y \rightarrow -5$

So, $y = -5$ is a HA

as $x \rightarrow -\infty$, $2^{-x} \rightarrow \infty \therefore y \rightarrow \infty$



10 $3^x \times 3^y = 9$ $9^x = 3^{y-2}$

$\therefore 3^{x+y} = 3^2$ $\therefore (3^2)^x = 3^{y-2}$

$\therefore x + y = 2$ (1) $\therefore 2x = y - 2$ (2)

From (1), $y = 2 - x$

$\therefore 2x = 2 - x - 2$

$\therefore 3x = 0$

$\therefore x = 0$

So, $x = 0$, $y = 2$

REVIEW SET 3D

1 a 4×2^n

$= 2^2 \times 2^n$

$= 2^{2+n}$

b $7^{-1} - 7^0$

$= \frac{1}{7} - 1$

$= -\frac{6}{7}$

c $(\frac{2}{3})^{-3}$

$= (\frac{3}{2})^3$

$= \frac{27}{8}$

$= 3\frac{3}{8}$

d $(\frac{2a^{-1}}{b^2})^2$

$= \frac{2^2 a^{-2}}{b^4}$

$= \frac{4}{a^2 b^4}$

2 a

2	288
2	144
2	72
2	36
2	18
2	9
3	

$\therefore 288 = 2^5 \times 3^2$

b $\frac{2^{x+1}}{2^{1-x}} = 2^{x+1-(1-x)}$

$= 2^{x+1-1+x}$

$= 2^{2x}$

3 a $1 = 5^0$

b $5\sqrt{5}$

$= 5^1 \times 5^{\frac{1}{2}}$

$= 5^{1\frac{1}{2}}$

$= 5^{\frac{3}{2}}$

c $\frac{1}{\sqrt[4]{5}}$

$= \frac{1}{5^{\frac{1}{4}}}$

$= 5^{-\frac{1}{4}}$

d 25^{a+3}

$= (5^2)^{a+3}$

$= 5^{2a+6}$

$$\begin{aligned}
 4 \quad a \quad & -(-2)^2 \\
 & = -[4] \\
 & = -4
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \left(-\frac{1}{2}a^{-3}\right)^2 \\
 & = \left(-\frac{1}{2}\right)^2 a^{-6} \\
 & = \frac{1}{4}a^{-6} \\
 & = \frac{1}{4} \times \frac{1}{a^6} \\
 & = \frac{1}{4a^6}
 \end{aligned}$$

$$\begin{aligned}
 c \quad & (-3b^{-1})^{-3} \\
 & = (-3)^{-3}b^3 \\
 & = \frac{1}{(-3)^3}b^3 \\
 & = \frac{1}{-27}b^3 \\
 & = -\frac{b^3}{27}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad a \quad & e^x(e^{-x} + e^x) \\
 & = e^0 + e^{2x} \\
 & = 1 + e^{2x}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & (2^x + 5)^2 \\
 & = (2^x)^2 + 2 \times 2^x \times 5 + 5^2 \\
 & = 2^{2x} + 5 \times 2^{x+1} + 25 \\
 & = 4^x + 5 \times 2^{x+1} + 25 \\
 & \text{(or } 2^{2x} + 10 \times 2^x + 25)
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \left(x^{\frac{1}{2}} - 7\right)\left(x^{\frac{1}{2}} + 7\right) \\
 & = \left(x^{\frac{1}{2}}\right)^2 - 7^2 \\
 & = x^1 - 49 \\
 & = x - 49
 \end{aligned}$$

$$\begin{aligned}
 6 \quad a \quad & (3 - 2^a)^2 \\
 & = 3^2 - 2 \times 3 \times 2^a + (2^a)^2 \\
 & = 9 - 3 \times 2^{a+1} + 2^{2a} \\
 & \text{(or } 9 - 6 \times 2^a + 2^{2a})
 \end{aligned}$$

$$\begin{aligned}
 b \quad & (\sqrt{x} + 2)(\sqrt{x} - 2) \\
 & = (\sqrt{x})^2 - 2^2 \\
 & = x - 4
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 2^{-x}(2^{2x} + 2^x) \\
 & = 2^{-x+2x} + 2^{-x+x} \\
 & = 2^x + 2^0 \\
 & = 2^x + 1
 \end{aligned}$$

$$\begin{aligned}
 7 \quad a \quad & 6 \times 2^x = 192 \\
 & \therefore 2^x = 32 \\
 & \therefore 2^x = 2^5 \\
 & \therefore x = 5
 \end{aligned}$$

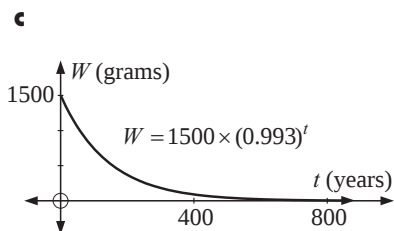
$$\begin{aligned}
 b \quad & 4 \times \left(\frac{1}{3}\right)^x = 324 \\
 & \therefore \left(\frac{1}{3}\right)^x = 81 \\
 & \therefore (3^{-1})^x = 3^4 \\
 & \therefore 3^{-x} = 3^4 \\
 & \therefore x = -4
 \end{aligned}$$

$$8 \quad W = 1500 \times (0.993)^t \text{ grams}$$

$$\begin{aligned}
 a \quad & \text{When } t = 0, \\
 & W = 1500 \times (0.993)^0 \\
 & = 1500 \times 1 \\
 & = 1500 \text{ grams}
 \end{aligned}$$

$$\begin{aligned}
 b \quad i \quad & \text{When } t = 400, \\
 & W = 1500 \times (0.993)^{400} \\
 & \div 90.3 \text{ grams}
 \end{aligned}$$

$$\begin{aligned}
 ii \quad & \text{When } t = 800, \\
 & W = 1500 \times (0.993)^{800} \\
 & \div 5.4 \text{ grams}
 \end{aligned}$$



$$\begin{aligned}
 d \quad & \text{When } W = 100, \\
 & 1500 \times (0.993)^t = 100 \\
 & \therefore (0.993)^t \div 0.0667 \\
 & \text{By trial and error} \\
 & (0.993)^{300} \div 0.1216 \\
 & (0.993)^{350} \div 0.085\,56 \\
 & (0.993)^{380} \div 0.069\,30 \\
 & (0.993)^{385} \div 0.066\,91 \\
 & (0.993)^{386} \div 0.066\,44
 \end{aligned}$$

i.e., about 386 years

Chapter 4

LOGARITHMS

EXERCISE 4A

- 1** **a** $10^4 = 10\,000$ **b** $10^{-1} = 0.1$ **c** $10^{\frac{1}{2}} = \sqrt{10}$ **d** $2^3 = 8$ **e** $2^{-2} = \frac{1}{4}$
f $3^{1.5} = \sqrt{27}$
- 2** **a** $\log_2 4 = 2$ **b** $\log_2(\frac{1}{8}) = -3$ **c** $\log_{10}(0.01) = -2$ **d** $\log_7 49 = 2$ **e** $\log_2 64 = 6$
f $\log_3(\frac{1}{27}) = -3$
- 3** **a** As $10^5 = 100\,000$
then $\log_{10}(100\,000) = 5$ **b** As $10^{-2} = 0.01$
then $\log_{10}(0.01) = -2$ **c** As $3^{\frac{1}{2}} = \sqrt{3}$
then $\log_3(\sqrt{3}) = \frac{1}{2}$
d As $2^3 = 8$ **e** As $2^6 = 64$ **f** As $2^7 = 128$
then $\log_2 8 = 3$ then $\log_2 64 = 6$ then $\log_2 128 = 7$
g As $5^2 = 25$ **h** As $5^3 = 125$ **i** As $2^{-3} = \frac{1}{8} = 0.125$
then $\log_5 25 = 2$ then $\log_5 125 = 3$ then $\log_2(0.125) = -3$
j As $9^{\frac{1}{2}} = 3$ **k** As $4^2 = 16$ **l** As $36^{\frac{1}{2}} = \sqrt{36} = 6$
then $\log_9 3 = \frac{1}{2}$ then $\log_4 16 = 2$ then $\log_{36} 6 = \frac{1}{2}$
m As $243 = 3^5$ **n** As $\sqrt[3]{2} = 2^{\frac{1}{3}}$ **o** As $a^n = a^n$
then $\log_3 243 = 5$ then $\log_2 \sqrt[3]{2} = \frac{1}{3}$ then $\log_n a^n = n$
p As $2 = 8^{\frac{1}{3}}$ **q** As $\frac{1}{t} = t^{-1}$ **r** As $6\sqrt{6} = 6^1 \times 6^{\frac{1}{2}} = 6^{1\frac{1}{2}}$
then $\log_8 2 = \frac{1}{3}$ then $\log_t \left(\frac{1}{t}\right) = -1$ then $\log_6(6\sqrt{6}) = 1\frac{1}{2}$
s As $1 = 4^0$ **t** As $9 = 9^1$
then $\log_4 1 = 0$ then $\log_9 9 = 1$
- 4** **a** $\div 2.18$ **b** $\div 1.40$ **c** $\div 1.87$ **d** $\div -0.0969$
- 5** **a** $\log_2 x = 3$ **b** $\log_4 x = \frac{1}{2}$ **c** $\log_x 81 = 4$ **d** $\log_2(x-6) = 3$
 $\therefore x = 2^3$ $\therefore x = 4^{\frac{1}{2}}$ $\therefore 81 = x^4$ $\therefore x-6 = 2^3$
 $\therefore x = 8$ $\therefore x = 2$ $\therefore x = \pm \sqrt[4]{81}$ $\therefore x-6 = 8$
 $\therefore x = \pm 3$ $\therefore x = 14$
but $x > 0$
 $\therefore x = 3$
- 6** **a** $\log_2 4$ **b** $\log_3(\frac{1}{3})$ **c** $\log_{10}(0.001)$ **d** $\log_3 \left(\frac{1}{\sqrt{3}}\right)$
 $= \log_2 2^2$ $= \log_3 3^{-1}$ $= \log_{10} 10^{-3}$ $= \log_3 \left(\frac{1}{3^{\frac{1}{2}}}\right)$
 $= 2$ $= -1$ $= -3$ $= \log_3 3^{-\frac{1}{2}}$
 $= \log_3 3^{-\frac{1}{2}}$
 $= -\frac{1}{2}$

$$\begin{aligned} \mathbf{e} \quad & \log_{10} \sqrt[3]{100} \\ &= \log_{10} (10^2)^{\frac{1}{3}} \\ &= \log_{10} 10^{\frac{2}{3}} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log_2 (2\sqrt{2}) \\ &= \log_2 \left(2^1 2^{\frac{1}{2}} \right) \\ &= \log_2 2^{\frac{3}{2}} \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \log_5 (25\sqrt{5}) \\ &= \log_5 \left(5^2 5^{\frac{1}{2}} \right) \\ &= \log_5 5^{\frac{5}{2}} \\ &= \frac{5}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log_2 \left(\frac{1}{\sqrt[3]{2}} \right) \\ &= \log_2 2^{-\frac{1}{3}} \\ &= -\frac{1}{3} \end{aligned}$$

EXERCISE 4B

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \log 10\,000 \\ &= \log_{10} 10^4 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log 0.001 \\ &= \log_{10} 10^{-3} \\ &= -3 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log 10 \\ &= \log_{10} 10^1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log 1 \\ &= \log_{10} 10^0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log \sqrt{10} \\ &= \log_{10} 10^{\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log \sqrt[3]{10} \\ &= \log_{10} 10^{\frac{1}{3}} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \log \left(\frac{1}{\sqrt[4]{10}} \right) \\ &= \log_{10} 10^{-\frac{1}{4}} \\ &= -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log 10\sqrt{10} \\ &= \log_{10} 10^1 10^{\frac{1}{2}} \\ &= \log_{10} 10^{\frac{3}{2}} \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \log \sqrt[3]{100} \\ &= \log_{10} (10^2)^{\frac{1}{3}} \\ &= \log_{10} 10^{\frac{2}{3}} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \log \left(\frac{100}{\sqrt{10}} \right) \\ &= \log_{10} \left(\frac{10^2}{10^{\frac{1}{2}}} \right) \\ &= \log_{10} 10^{\frac{3}{2}} \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \log (10 \times \sqrt[3]{10}) \\ &= \log_{10} \left(10^1 \times 10^{\frac{1}{3}} \right) \\ &= \log_{10} 10^{\frac{4}{3}} \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & \log 1000\sqrt{10} \\ &= \log_{10} (10^3 \times 10^{\frac{1}{2}}) \\ &= \log_{10} 10^{\frac{7}{2}} \\ &= \frac{7}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{m} \quad & \log 10^n \\ &= \log_{10} 10^n \\ &= n \end{aligned}$$

$$\begin{aligned} \mathbf{n} \quad & \log (10^a \times 100) \\ &= \log_{10} (10^a \times 10^2) \\ &= \log_{10} (10^{a+2}) \\ &= a + 2 \end{aligned}$$

$$\begin{aligned} \mathbf{o} \quad & \log \left(\frac{10}{10^m} \right) \\ &= \log_{10} (10^{1-m}) \\ &= 1 - m \end{aligned}$$

$$\begin{aligned} \mathbf{p} \quad & \log \left(\frac{10^a}{10^b} \right) \\ &= \log_{10} (10^{a-b}) \\ &= a - b \end{aligned}$$

$$\mathbf{2} \text{ Use: } \mathbf{a} \quad \boxed{\log} \quad 10\,000 \quad \boxed{\text{ENTER}} \quad \mathbf{b} \quad \boxed{\log} \quad 0.001 \quad \boxed{\text{ENTER}}$$

$$\mathbf{c} \quad \boxed{\log} \quad \boxed{2\text{nd}} \quad \boxed{\sqrt{}} \quad 10 \quad \boxed{)} \quad \boxed{)} \quad \boxed{\text{ENTER}}$$

$$\mathbf{d} \quad \boxed{\log} \quad 10 \quad \boxed{\wedge} \quad \boxed{(} \quad 1 \quad \boxed{\div} \quad 3 \quad \boxed{)} \quad \boxed{)} \quad \boxed{\text{ENTER}}$$

$$\mathbf{e} \quad \boxed{\log} \quad 100 \quad \boxed{\wedge} \quad \boxed{(} \quad 1 \quad \boxed{\div} \quad 3 \quad \boxed{)} \quad \boxed{)} \quad \boxed{\text{ENTER}}$$

$$\mathbf{f} \quad \boxed{\log} \quad 10 \quad \boxed{\times} \quad \boxed{2\text{nd}} \quad \boxed{\sqrt{}} \quad 10 \quad \boxed{)} \quad \boxed{)} \quad \boxed{\text{ENTER}}$$

$$\mathbf{g} \quad \boxed{\log} \quad 1 \quad \boxed{\div} \quad \boxed{2\text{nd}} \quad \boxed{\sqrt{}} \quad 10 \quad \boxed{)} \quad \boxed{)} \quad \boxed{\text{ENTER}}$$

$$\mathbf{h} \quad \boxed{\log} \quad 1 \quad \boxed{\div} \quad 10 \quad \boxed{\wedge} \quad 0.25 \quad \boxed{)} \quad \boxed{\text{ENTER}}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & 6 \\ &= 10^{\log 6} \\ &\div 10^{0.7782} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 60 \\ &= 10^{\log 60} \\ &\div 10^{1.7782} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 6000 \\ &= 10^{\log 6000} \\ &\div 10^{3.7782} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 0.6 \\ &= 10^{\log(0.6)} \\ &= 10^{-0.2218} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 0.006 \\ &= 10^{\log(0.006)} \\ &= 10^{-2.2218} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & 15 \\ &= 10^{\log 15} \\ &\div 10^{1.1761} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & 1500 \\ &= 10^{\log 1500} \\ &= 10^{3.1761} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & 1.5 \\ &= 10^{\log 1.5} \\ &= 10^{0.1761} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & 0.15 \\ &= 10^{\log(0.15)} \\ &= 10^{-0.8239} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & 0.000\,15 \\ &= 10^{\log(0.000\,15)} \\ &= 10^{-3.8239} \end{aligned}$$

$$4 \quad \text{a} \quad \log 3 \div 0.477 \quad \text{ii} \quad \log 300 \div 2.477$$

$$\begin{aligned} \text{b} \quad 300 &= 3 \times 10^2 \\ &= 10^{\log 3} \times 10^2 \\ &= 10^{\log 3 + 2} \\ \therefore \log 300 &= \log 3 + 2 \end{aligned}$$

$$5 \quad \text{a} \quad \log 5 \div 0.699 \quad \text{ii} \quad \log 0.05 = -1.301$$

$$\begin{aligned} \text{b} \quad 0.05 &= 5 \times 10^{-2} \\ &= 10^{\log 5} \times 10^{-2} \\ &= 10^{\log 5 - 2} \\ \therefore \log(0.05) &= \log 5 - 2 \end{aligned}$$

$$6 \quad \text{a} \quad \log x = 2 \\ \therefore x = 10^2 \\ \therefore x = 100$$

$$\text{b} \quad \log x = 1 \\ \therefore x = 10^1 \\ \therefore x = 10$$

$$\text{c} \quad \log x = 0 \\ \therefore x = 10^0 \\ \therefore x = 1$$

$$\text{d} \quad \log x = -1 \\ \therefore x = 10^{-1} \\ \therefore x = \frac{1}{10}$$

$$\text{e} \quad \log x = \frac{1}{2} \\ \therefore x = 10^{\frac{1}{2}} \\ \therefore x = \sqrt{10}$$

$$\begin{aligned} \text{f} \quad \log x &= -\frac{1}{2} \\ \therefore x &= 10^{-\frac{1}{2}} \\ \therefore x &= \frac{1}{10^{\frac{1}{2}}} \\ \therefore x &= \frac{1}{\sqrt{10}} \end{aligned}$$

$$\begin{aligned} \text{g} \quad \log x &= 4 \\ \therefore x &= 10^4 \\ \therefore x &= 10\,000 \end{aligned}$$

$$\begin{aligned} \text{h} \quad \log x &= -5 \\ \therefore x &= 10^{-5} \\ \therefore x &= 0.000\,01 \end{aligned}$$

$$\begin{aligned} \text{i} \quad \log x &\div 0.8351 \\ \therefore x &\div 10^{0.8351} \\ \therefore x &\div 6.84 \end{aligned}$$

$$\begin{aligned} \text{j} \quad \log x &\div 2.1457 \\ \therefore x &\div 10^{2.1457} \\ \therefore x &\div 140 \end{aligned}$$

$$\begin{aligned} \text{k} \quad \log x &\div -1.378 \\ \therefore x &\div 10^{-1.378} \\ \therefore x &\div 0.0419 \end{aligned}$$

$$\begin{aligned} \text{l} \quad \log x &\div -3.1997 \\ \therefore x &\div 10^{-3.1997} \\ \therefore x &\div 0.000\,631 \end{aligned}$$

EXERCISE 4C

$$1 \quad \text{a} \quad \log 8 + \log 2 \\ = \log(8 \times 2) \\ = \log 16$$

$$\begin{aligned} \text{b} \quad \log 8 - \log 2 \\ &= \log\left(\frac{8}{2}\right) \\ &= \log 4 \end{aligned}$$

$$\begin{aligned} \text{c} \quad \log 40 - \log 5 \\ &= \log\left(\frac{40}{5}\right) \\ &= \log 8 \end{aligned}$$

$$\begin{aligned} \text{d} \quad \log 4 + \log 5 \\ &= \log(4 \times 5) \\ &= \log 20 \end{aligned}$$

$$\begin{aligned} \text{e} \quad \log 5 + \log(0.4) \\ &= \log(5 \times 0.4) \\ &= \log 2 \end{aligned}$$

$$\begin{aligned} \text{f} \quad \log 2 + \log 3 + \log 4 \\ &= \log(2 \times 3 \times 4) \\ &= \log 24 \end{aligned}$$

$$\begin{aligned} \text{g} \quad 1 + \log 3 \\ &= \log 10^1 + \log 3 \\ &= \log(10 \times 3) \\ &= \log 30 \end{aligned}$$

$$\begin{aligned} \text{h} \quad \log 4 - 1 \\ &= \log 4 - \log 10^1 \\ &= \log\left(\frac{4}{10}\right) \\ &= \log(0.4) \end{aligned}$$

$$\begin{aligned} \text{i} \quad \log 5 + \log 4 - \log 2 \\ &= \log\left(\frac{5 \times 4}{2}\right) \\ &= \log 10 \end{aligned}$$

$$\begin{aligned} \text{j} \quad 2 + \log 2 \\ &= \log 10^2 + \log 2 \\ &= \log(100 \times 2) \\ &= \log 200 \end{aligned}$$

$$\begin{aligned} \text{k} \quad \log 40 - 2 \\ &= \log 40 - \log 10^2 \\ &= \log\left(\frac{40}{100}\right) \\ &= \log(0.4) \end{aligned}$$

$$\begin{aligned} \text{l} \quad \log 6 - \log 2 - \log 3 \\ &= \log(6 \div 2 \div 3) \\ &= \log 1 \end{aligned}$$

$$\begin{aligned} \text{m} \quad \log 50 - 4 \\ &= \log 50 - \log 10^4 \\ &= \log\left(\frac{50}{10^4}\right) \\ &= \log(0.005) \end{aligned}$$

$$\begin{aligned} \text{n} \quad 3 - \log 50 \\ &= \log 10^3 - \log 50 \\ &= \log\left(\frac{1000}{50}\right) \\ &= \log 20 \end{aligned}$$

$$\begin{aligned} \text{o} \quad \log\left(\frac{4}{3}\right) + \log 3 + \log 7 \\ &= \log\left(\frac{4}{3} \times 3 \times 7\right) \\ &= \log 28 \end{aligned}$$

$$2 \quad \text{a} \quad 5 \log 2 + \log 3 \\ = \log 2^5 + \log 3 \\ = \log(2^5 \times 3) \\ = \log 96$$

$$\begin{aligned} \text{b} \quad 2 \log 3 + 3 \log 2 \\ &= \log 3^2 + \log 2^3 \\ &= \log(9 \times 8) \\ &= \log 72 \end{aligned}$$

$$\begin{aligned} \text{c} \quad 3 \log 4 - \log 8 \\ &= \log 4^3 - \log 8 \\ &= \log\left(\frac{64}{8}\right) \\ &= \log 8 \end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & 2 \log 5 - 3 \log 2 \\ &= \log 5^2 - \log 2^3 \\ &= \log \left(\frac{25}{8} \right)\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad & \frac{1}{2} \log 4 + \log 3 \\ &= \log 4^{\frac{1}{2}} + \log 3 \\ &= \log(2 \times 3) \\ &= \log 6\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad & \frac{1}{3} \log \left(\frac{1}{8} \right) \\ &= \log \left(\frac{1}{8} \right)^{\frac{1}{3}} \\ &= \log (2^{-3})^{\frac{1}{3}} \\ &= \log 2^{-1} \\ &= \log \left(\frac{1}{2} \right) \text{ or } -\log 2\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad & 3 - \log 2 - 2 \log 5 \\ &= \log 10^3 - \log 2 - \log 5^2 \\ &= \log(1000 \div 2 \div 25) \\ &= \log 20\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad & 1 - 3 \log 2 + \log 20 \\ &= \log 10^1 - \log 2^3 + \log 20 \\ &= \log(10 \div 8 \times 20) \\ &= \log 25\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad & 2 - \frac{1}{2} \log 4 - \log 5 \\ &= \log 10^2 - \log 4^{\frac{1}{2}} - \log 5 \\ &= \log(100 \div 2 \div 5) \\ &= \log 10 \\ &= 1\end{aligned}$$

$$\begin{aligned}\mathbf{3} \quad \mathbf{a} \quad & \frac{\log 4}{\log 2} \\ &= \frac{\log 2^2}{\log 2} \\ &= \frac{2 \log 2}{\log 2} \\ &= 2\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & \frac{\log 27}{\log 9} \\ &= \frac{\log 3^3}{\log 3^2} \\ &= \frac{3 \log 3}{2 \log 3} \\ &= \frac{3}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & \frac{\log 8}{\log 2} \\ &= \frac{\log 2^3}{\log 2} \\ &= \frac{3 \log 2}{\log 2} \\ &= 3\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & \frac{\log 3}{\log 9} \\ &= \frac{\log 3}{\log 3^2} \\ &= \frac{\log 3}{2 \log 3} \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad & \frac{\log 25}{\log(0.2)} \\ &= \frac{\log 5^2}{\log 5^{-1}} \\ &= \frac{2 \log 5}{-1 \log 5} \\ &= -2\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad & \frac{\log 8}{\log(0.25)} \\ &= \frac{\log 2^3}{\log 2^{-2}} \quad \{0.25 = \frac{1}{4} = \frac{1}{2^2}\} \\ &= \frac{3 \log 2}{-2 \log 2} \\ &= -\frac{3}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{4} \quad \mathbf{a} \quad & \log 9 = \log 3^2 \\ &= 2 \log 3\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & \log \sqrt{2} = \log 2^{\frac{1}{2}} \\ &= \frac{1}{2} \log 2\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & \log \left(\frac{1}{8} \right) = \log \left(\frac{1}{2^3} \right) \\ &= \log 2^{-3} \\ &= -3 \log 2\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & \log \left(\frac{1}{5} \right) = \log 5^{-1} \\ &= -1 \log 5 \\ &= -\log 5\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad & \log 5 = \log \left(\frac{10}{2} \right) \\ &= \log 10^1 - \log 2 \\ &= 1 - \log 2\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad & \log 5000 = \log \left(\frac{10000}{2} \right) \\ &= \log 10^4 - \log 2 \\ &= 4 - \log 2\end{aligned}$$

$$\begin{aligned}\mathbf{5} \quad \mathbf{a} \quad & y = 2^x \\ \therefore \log y &= \log 2^x \\ \therefore \log y &= x \log 2\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & y = 20b^3 \\ \therefore \log y &= \log(20b^3) \\ \therefore \log y &= \log 20 + \log b^3 \\ \therefore \log y &= \log 20 + 3 \log b\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & M = ad^4 \\ \therefore \log M &= \log(ad^4) \\ \therefore \log M &= \log a + \log d^4 \\ \therefore \log M &= \log a + 4 \log d\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & T = 5\sqrt{d} = 5d^{\frac{1}{2}} \\ \therefore \log T &= \log(5d^{\frac{1}{2}}) \\ \therefore \log T &= \log 5 + \log d^{\frac{1}{2}} \\ \therefore \log T &= \log 5 + \frac{1}{2} \log d\end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad R &= b\sqrt{l} = bl^{\frac{1}{2}} \\ \therefore \log R &= \log(bl^{\frac{1}{2}}) \\ \therefore \log R &= \log b + \log l^{\frac{1}{2}} \\ \therefore \log R &= \log b + \frac{1}{2} \log l \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad y &= ab^x \\ \therefore \log y &= \log(ab^x) \\ \therefore \log y &= \log a + \log b^x \\ \therefore \log y &= \log a + x \log b \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad L &= \frac{ab}{c} \\ \therefore \log L &= \log\left(\frac{ab}{c}\right) \\ \therefore \log L &= \log ab - \log c \\ \therefore \log L &= \log a + \log b - \log c \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad S &= 200 \times 2^t \\ \therefore \log S &= \log(200 \times 2^t) \\ \therefore \log S &= \log 200 + \log 2^t \\ \therefore \log S &= \log 200 + t \log 2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad Q &= \frac{a}{b^n} \\ \therefore \log Q &= \log\left(\frac{a}{b^n}\right) \\ \therefore \log Q &= \log a - \log b^n \\ \therefore \log Q &= \log a - n \log b \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad F &= \frac{20}{\sqrt{n}} = \frac{20}{n^{\frac{1}{2}}} \\ \therefore \log F &= \log\left(\frac{20}{n^{\frac{1}{2}}}\right) \\ \therefore \log F &= \log 20 - \log n^{\frac{1}{2}} \\ \therefore \log F &= \log 20 - \frac{1}{2} \log n \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad N &= \sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} \\ \therefore \log N &= \log\left(\frac{a}{b}\right)^{\frac{1}{2}} \\ \therefore \log N &= \frac{1}{2} \log\left(\frac{a}{b}\right) \\ \therefore \log N &= \frac{1}{2} \{\log a - \log b\} \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad y &= \frac{a^m}{b^n} \\ \therefore \log y &= \log\left(\frac{a^m}{b^n}\right) \\ \therefore \log y &= \log a^m - \log b^n \\ \therefore \log y &= m \log a - n \log b \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad \log D &= \log e + \log 2 \\ &= \log(e \times 2) \\ \therefore D &= 2e \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log F &= \log 5 - \log t \\ &= \log\left(\frac{5}{t}\right) \\ \therefore F &= \frac{5}{t} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log P &= \frac{1}{2} \log x \\ &= \log x^{\frac{1}{2}} \\ \therefore P &= \sqrt{x} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \log M &= 2 \log b + \log c \\ &= \log b^2 + \log c \\ &= \log(b^2 c) \\ \therefore M &= b^2 c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \log B &= 3 \log m - 2 \log n \\ &= \log m^3 - \log n^2 \\ &= \log\left(\frac{m^3}{n^2}\right) \\ \therefore B &= \frac{m^3}{n^2} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \log N &= -\frac{1}{3} \log p \\ &= \log p^{-\frac{1}{3}} \\ &= \log\left(\frac{1}{\sqrt[3]{p}}\right) \\ \therefore N &= \frac{1}{\sqrt[3]{p}} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad \log P &= 3 \log x + 1 \\ &= \log x^3 + \log 10^1 \\ &= \log(10x^3) \\ \therefore P &= 10x^3 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad \log Q &= 2 - \log x \\ &= \log 10^2 - \log x \\ &= \log\left(\frac{100}{x}\right) \quad \therefore Q = \frac{100}{x} \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad \log_b 6 \\ &= \log_b(2 \times 3) \\ &= \log_b 2 + \log_b 3 \\ &= p + q \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log_b 108 \\ &= \log_b(2^2 3^3) \\ &= 2 \log_b 2 + 3 \log_b 3 \\ &= 2p + 3q \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log_b 45 \\ &= \log_b(3^2 5) \\ &= 2 \log_b 3 + \log_b 5 \\ &= 2q + r \end{aligned}$$

$$\mathbf{d} \quad \log_b \left(\frac{5\sqrt{3}}{2} \right)$$

$$\begin{aligned} &= \log_b (5 \times 3^{\frac{1}{2}}) - \log_b 2 \\ &= \log_b 5 + \frac{1}{2} \log_b 3 - \log_b 2 \\ &= r + \frac{1}{2}q - p \end{aligned}$$

$$\mathbf{e} \quad \log_b \left(\frac{5}{32} \right)$$

$$\begin{aligned} &= \log_b 5 - \log_b 2^5 \\ &= \log_b 5 - 5 \log_b 2 \\ &= r - 5p \end{aligned}$$

$$\mathbf{f} \quad \log_b (0.\bar{2})$$

$$\begin{aligned} &= \log_b \left(\frac{2}{9} \right) \\ &= \log_b 2 - \log_b 3^2 \\ &= p - 2q \end{aligned}$$

$$\mathbf{8} \quad \mathbf{a} \quad \log_2 (PR)$$

$$\begin{aligned} &= \log_2 P + \log_2 R \\ &= x + z \end{aligned}$$

$$\mathbf{b} \quad \log_2 (RQ^2)$$

$$\begin{aligned} &= \log_2 R + \log_2 Q^2 \\ &= \log_2 R + 2 \log_2 Q \\ &= z + 2y \end{aligned}$$

$$\mathbf{c} \quad \log_2 \left(\frac{PR}{Q} \right)$$

$$\begin{aligned} &= \log_2 (PR) - \log_2 Q \\ &= \log_2 P + \log_2 R - \log_2 Q \\ &= x + z - y \end{aligned}$$

$$\mathbf{d} \quad \log_2 (P^2 \sqrt{Q})$$

$$\begin{aligned} &= \log_2 P^2 + \log_2 Q^{\frac{1}{2}} \\ &= 2 \log_2 P + \frac{1}{2} \log_2 Q \\ &= 2x + \frac{1}{2}y \end{aligned}$$

$$\mathbf{e} \quad \log_2 \left(\frac{Q^3}{\sqrt{R}} \right)$$

$$\begin{aligned} &= \log_2 Q^3 - \log_2 R^{\frac{1}{2}} \\ &= 3 \log_2 Q - \frac{1}{2} \log_2 R \\ &= 3y - \frac{1}{2}z \end{aligned}$$

$$\mathbf{f} \quad \log_2 \left(\frac{R^2 \sqrt{Q}}{P^3} \right)$$

$$\begin{aligned} &= \log_2 R^2 + \log_2 Q^{\frac{1}{2}} - \log_2 P^3 \\ &= 2 \log_2 R + \frac{1}{2} \log_2 Q - 3 \log_2 P \\ &= 2z + \frac{1}{2}y - 3x \end{aligned}$$

$$\mathbf{9} \quad \mathbf{a} \quad \log_t N^2 = 1.72$$

$$\therefore 2 \log_t N = 1.72$$

$$\begin{aligned} \therefore \log_t N &= 1.72 \div 2 \\ &= 0.86 \end{aligned}$$

$$\mathbf{b} \quad \log_t (MN)$$

$$\begin{aligned} &= \log_t M + \log_t N \\ &= 1.29 + 0.86 \\ &= 2.15 \end{aligned}$$

$$\mathbf{c} \quad \log_t \left(\frac{N^2}{\sqrt{M}} \right)$$

$$\begin{aligned} &= \log_t N^2 - \log_t M^{\frac{1}{2}} \\ &= 2 \log_t N - \frac{1}{2} \log_t M \\ &= 2(0.86) - \frac{1}{2}(1.29) \\ &= 1.075 \end{aligned}$$

$$\mathbf{10} \quad \mathbf{a} \quad \log_3 27 + \log_3 \left(\frac{1}{3} \right) = \log_3 x$$

$$\therefore \log_3 (27 \times \frac{1}{3}) = \log_3 x$$

$$\therefore \log_3 9 = \log_3 x$$

$$\therefore x = 9$$

$$\mathbf{b} \quad \log_5 x = \log_5 8 - \log_5 (6 - x)$$

$$\therefore \log_5 x = \log_5 \left(\frac{8}{6 - x} \right)$$

$$\therefore x = \frac{8}{6 - x} \quad \text{Note: } x > 0$$

$$\therefore 6x - x^2 = 8 \quad \text{and } 6 - x > 0$$

$$\therefore x^2 - 6x + 8 = 0 \quad \text{i.e., } 0 < x < 6$$

$$\therefore (x - 2)(x - 4) = 0$$

$$\therefore x = 2 \text{ or } 4$$

$$\mathbf{c} \quad \log_5 125 - \log_5 \sqrt{5} = \log_5 x$$

$$\therefore \log_5 \left(\frac{125}{\sqrt{5}} \right) = \log_5 x$$

$$\therefore x = \frac{125}{\sqrt{5}} \text{ or } 25\sqrt{5}$$

$$\mathbf{d} \quad \log_{20} x = 1 + \log_{20} 10$$

$$\begin{aligned} \therefore \log_{20} x &= \log_{20} 20^1 + \log_{20} 10 \\ &= \log_{20} 200 \end{aligned}$$

$$\therefore x = 200$$

$$\mathbf{e} \quad \log x + \log(x + 1) = \log 30$$

$$\therefore \log[x(x + 1)] = \log 30$$

$$\therefore x^2 + x = 30$$

$$\therefore x^2 + x - 30 = 0$$

$$\therefore (x + 6)(x - 5) = 0$$

$$\therefore x = -6 \text{ or } 5$$

$$\text{but } x > 0 \text{ for } \log x \text{ to exist}$$

$$\therefore x = 5$$

$$\mathbf{f} \quad \log(x + 2) - \log(x - 2) = \log 5$$

$$\therefore \log \left(\frac{x + 2}{x - 2} \right) = \log 5$$

$$\therefore \frac{x + 2}{x - 2} = 5$$

$$\therefore x + 2 = 5x - 10$$

$$\therefore -4x = -12$$

$$\therefore x = 3$$

$$\text{Note: } x + 2 > 0 \text{ and } x - 2 > 0 \quad \therefore x > 2 \quad \checkmark$$

EXERCISE 4D

- 1 a** $2^x = 10$
 $\therefore \log 2^x = \log 10$
 $\therefore x \log 2 = \log 10$
 $\therefore x = \frac{\log 10}{\log 2} \div 3.32$
- b** $3^x = 20$
 $\therefore \log 3^x = \log 20$
 $\therefore x \log 3 = \log 20$
 $\therefore x = \frac{\log 20}{\log 3} \div 2.73$
- c** $4^x = 100$
 $\therefore \log 4^x = \log 100$
 $\therefore x \log 4 = \log 100$
 $\therefore x = \frac{\log 100}{\log 4} \div 3.32$
- d** $(1.2)^x = 1000$
 $\therefore x \log(1.2) = \log 1000$
 $\therefore x = \frac{\log 1000}{\log(1.2)} \div 37.9$
- e** $2^x = 0.08$
 $\therefore \log 2^x = \log(0.08)$
 $\therefore x \log 2 = \log(0.08)$
 $\therefore x = \frac{\log(0.08)}{\log 2} \div -3.64$
- f** $3^x = 0.000\ 25$
 $\therefore \log 3^x = \log(0.000\ 25)$
 $\therefore x \log 3 = \log(0.000\ 25)$
 $\therefore x = \frac{\log(0.000\ 25)}{\log 3} \div -7.55$
- g** $\left(\frac{1}{2}\right)^x = 0.005$
 $\therefore \log(0.5)^x = \log(0.005)$
 $\therefore x \log(0.5) = \log(0.005)$
 $\therefore x = \frac{\log(0.005)}{\log(0.5)} \div 7.64$
- h** $\left(\frac{3}{4}\right)^x = 10^{-4}$
 $\therefore \log(0.75)^x = -4$
 $\therefore x \log(0.75) = -4$
 $\therefore x = \frac{-4}{\log(0.75)} \div 32.0$
- i** $(0.99)^x = 0.000\ 01$
 $\therefore \log(0.99)^x = \log(0.000\ 01)$
 $\therefore x \log(0.99) = \log(0.000\ 01)$
 $\therefore x = \frac{\log(0.000\ 01)}{\log(0.99)}$
 $\therefore x \div 1146 \text{ or } 1150 \text{ (3 s.f.)}$
- 2 a** $200 \times 2^{0.25t} = 600$
 $\therefore 2^{0.25t} = 3$
 $\therefore \log(2^{0.25t}) = \log 3$
 $\therefore 0.25t \log 2 = \log 3$
 $\therefore t = \frac{\log 3}{0.25 \times \log 2}$
 $\therefore t \div 6.34$
- b** $20 \times 2^{0.06t} = 450$
 $\therefore 2^{0.06t} = 22.5$
 $\therefore \log(2^{0.06t}) = \log(22.5)$
 $\therefore 0.06t \log 2 = \log(22.5)$
 $\therefore t = \frac{\log(22.5)}{0.06 \times \log 2}$
 $\therefore t \div 74.9$
- c** $30 \times 3^{-0.25t} = 3$
 $\therefore 3^{-0.25t} = \frac{1}{10}$
 $\therefore \log 3^{-0.25t} = \log(0.1)$
 $\therefore -0.25t \log 3 = \log(0.1)$
 $\therefore t = \frac{\log(0.1)}{-0.25 \times \log 3} \div 8.38$
- d** $12 \times 2^{-0.05t} = 0.12$
 $\therefore 2^{-0.05t} = \frac{1}{100}$
 $\therefore \log 2^{-0.05t} = \log(0.01)$
 $\therefore -0.05t \log 2 = \log(0.01)$
 $\therefore t = \frac{\log(0.01)}{-0.05 \times \log 2} \div 133$
- e** $50 \times 5^{-0.02t} = 1$
 $\therefore 5^{-0.02t} = \frac{1}{50} = 0.02$
 $\therefore \log 5^{-0.02t} = \log(0.02)$
 $\therefore -0.02t \log 5 = \log(0.02)$
 $\therefore t = \frac{\log(0.02)}{-0.02 \times \log 5}$
 $\therefore t \div 122$
- f** $300 \times 2^{0.005t} = 1000$
 $\therefore 2^{0.005t} = \frac{10}{3}$
 $\therefore \log 2^{0.005t} = \log\left(\frac{10}{3}\right)$
 $\therefore 0.005t \log 2 = \log\left(\frac{10}{3}\right)$
 $\therefore t = \frac{\log\left(\frac{10}{3}\right)}{0.005 \times \log 2}$
 $\therefore t \div 347$

EXERCISE 4E

1 $W_t = 20 \times 2^{0.15t}$ grams

a When $W_t = 30$,
 $20 \times 2^{0.15t} = 30$
 $\therefore 2^{0.15t} = 1.5$
 $\therefore \log 2^{0.15t} = \log(1.5)$
 $\therefore 0.15t \log 2 = \log(1.5)$
 $\therefore t = \frac{\log(1.5)}{0.15 \times \log 2}$
 $\therefore t \doteq 3.90$ hours

b When $W_t = 100$,
 $20 \times 2^{0.15t} = 100$
 $\therefore 2^{0.15t} = 5$
 $\therefore \log 2^{0.15t} = \log 5$
 $\therefore 0.15t \log 2 = \log 5$
 $\therefore t = \frac{\log 5}{0.15 \times \log 2}$
 $\therefore t \doteq 15.5$ hours

2 $T = 100 \times 2^{-0.03t}$ °C

a When $T = 25$,
 $100 \times 2^{-0.03t} = 25$
 $\therefore 2^{-0.03t} = \frac{1}{4} = 2^{-2}$
 $\therefore -0.03t = -2$
 $\therefore t = \frac{2}{0.03}$
 $\therefore t \doteq 66.7$ min

b When $T = 1$,
 $100 \times 2^{-0.03t} = 1$
 $\therefore 2^{-0.03t} = 0.01$
 $\therefore \log 2^{-0.03t} = \log(0.01)$
 $\therefore -0.03t \log 2 = \log(0.01)$
 $\therefore t = \frac{\log(0.01)}{-0.03 \times \log 2} \doteq 221$ min

3 $W_t = 1000 \times 2^{-0.04t}$ has $W_0 = 1000 \times 2^0 = 1000$ grams

a For the weight to halve,
 $W_t = 500$
 $\therefore 1000 \times 2^{-0.04t} = 500$
 $\therefore 2^{-0.04t} = \frac{1}{2} = 2^{-1}$
 $\therefore -0.04t = -1$
 $\therefore t = \frac{1}{0.04}$
 $\therefore t = 25$ years

b For $W_t = 20$,
 $1000 \times 2^{-0.04t} = 20$
 $\therefore 2^{-0.04t} = 0.02$
 $\therefore \log 2^{-0.04t} = \log(0.02)$
 $\therefore -0.04t \log 2 = \log(0.02)$
 $\therefore t = \frac{\log(0.02)}{-0.04 \times \log 2}$
 $\therefore t \doteq 141$ years

c When $W_t = 1\%$ of 1000 grams = 10 g,
 $1000 \times 2^{-0.04t} = 10$
 $\therefore 2^{-0.04t} = 0.01$
 $\therefore \log 2^{-0.04t} = \log(0.01)$

and so $-0.04 \log 2 = \log(0.01)$
 $\therefore t = \frac{\log(0.01)}{-0.04 \times \log 2}$
 $\therefore t \doteq 166$ years

4 $W = W_0 \times 2^{-0.0002t}$ grams

a When W is 25% of original,
 $W = \frac{1}{4}$ of W_0
 $\therefore \cancel{W_0} \times 2^{-0.0002t} = \frac{1}{4} \times \cancel{W_0}$
 $\therefore 2^{-0.0002t} = 2^{-2}$
 $\therefore 0.0002t = 2$
 $\therefore t = \frac{2}{0.0002}$
 $\therefore t = 10\,000$
 $\therefore$ it would take 10 000 years

b When W is 0.1% of original,
 $W = \frac{0.1}{100}$ of W_0
 $\therefore \cancel{W_0} \times 2^{-0.0002t} = \frac{1}{1000} \times \cancel{W_0}$
 $\therefore \log 2^{-0.0002t} = \log(0.001)$
 $\therefore -0.0002t \log 2 = \log(0.001)$
 $\therefore t = \frac{\log(0.001)}{-0.0002 \times \log 2}$
 $\therefore t \doteq 49\,829$
 i.e., it would take about 49 800 years

$$5 \quad V = V_0 \times 2^{0.1t}$$

$$\text{When } t = 0, \quad V = V_0 \times 2^0 = V_0$$

So, we need to find t when

$$V = 3V_0$$

$$\text{i.e., } V_0 \times 2^{0.1t} = 3V_0$$

$$\therefore \log 2^{0.1t} = \log 3$$

$$\therefore 0.1t \log 2 = \log 3$$

$$\therefore t = \frac{\log 3}{0.1 \times \log 2} \doteq 15.8$$

$\therefore$ the temperature would be 15.8°C .

$$7 \quad V = 50(1 - 2^{-0.2t})$$

$$\text{So when } V = 40, \quad 50(1 - 2^{-0.2t}) = 40$$

$$\therefore 1 - 2^{-0.2t} = 0.8$$

$$\therefore 2^{-0.2t} = 0.2$$

$$\therefore \log 2^{-0.2t} = \log(0.2)$$

$$\therefore -0.2t \log 2 = \log(0.2)$$

$$\therefore t = \frac{\log(0.2)}{-0.2 \times \log 2} \doteq 11.6 \quad \text{i.e., it would take 11.6 sec.}$$

$$6 \quad I = I_0 \times 2^{-0.02t} \text{ amps}$$

$$\text{When } t = 0, \quad I = I_0 \times 2^0 = I_0 \text{ amps}$$

So, we need to find t when

$$I = 10\% \text{ of } I_0$$

$$\therefore I_0 \times 2^{-0.02t} = 0.1 \times I_0$$

$$\therefore \log 2^{-0.02t} = \log(0.1)$$

$$\therefore -0.02t \log 2 = \log(0.1)$$

$$\therefore t = \frac{\log(0.1)}{-0.02 \times \log 2} \doteq 166$$

i.e., it would take 166 seconds.

EXERCISE 4F

$$1 \quad r = 107.5\%,$$

$$= 1.075$$

$$u_1 = 160\,000$$

$$u_{n+1} = 250\,000$$

$$u_{n+1} = u_1 \times r^n$$

$$\therefore 250\,000 = 160\,000 \times (1.075)^n$$

$$\therefore (1.075)^n = \frac{25}{16}$$

$$\therefore \log(1.075)^n = \log\left(\frac{25}{16}\right)$$

$$\therefore n \log(1.075) = \log\left(\frac{25}{16}\right)$$

$$\therefore n = \frac{\log\left(\frac{25}{16}\right)}{\log(1.075)} \doteq 6.1709 \dots$$

i.e., it would take 6.17 years

($\doteq$ 6 years, 62 days)

$$2 \quad u_1 = 10\,000$$

$$u_{n+1} = 15\,000$$

$$r = 104.8\%$$

$$= 1.048$$

$$u_{n+1} = u_1 \times r^n$$

$$\therefore 15\,000 = 10\,000 \times (1.048)^n$$

$$\therefore (1.048)^n = 1.5$$

$$\therefore \log(1.048)^n = \log(1.5)$$

$$\therefore n \log(1.048) = \log(1.5)$$

$$\therefore n = \frac{\log(1.5)}{\log(1.048)}$$

$$\therefore n \doteq 8.648 \dots$$

i.e., it would take 8.65 years

($\doteq$ 8 years, 237 days)

$$3 \quad \text{a} \quad 8.4\% \text{ p.a. compounded monthly}$$

$$\text{is } \frac{8.4\%}{12} = 0.7\% \text{ a month}$$

$$\text{So } T = 100\% + 0.7\%$$

$$= 100.7\%$$

$$= 1.007$$

$$\text{b} \quad u_1 = 15\,000 \quad \text{and} \quad u_{n+1} = 25\,000$$

$$u_{n+1} = u_1 \times r^n$$

$$\therefore 25\,000 = 15\,000 \times (1.007)^n$$

$$\therefore (1.007)^n = \frac{25}{15} = \frac{5}{3}$$

$$\therefore \log(1.007)^n = \log\left(\frac{5}{3}\right)$$

$$\therefore n \log(1.007) = \log\left(\frac{5}{3}\right)$$

$$\therefore n = \frac{\log\left(\frac{5}{3}\right)}{\log(1.007)} \doteq 73.23 \dots$$

i.e., it would take 74 months.

EXERCISE 4G

$$\begin{array}{llll}
 \mathbf{1} \quad \mathbf{a} & \log_3 12 & \mathbf{b} & \log_{\frac{1}{2}} 1250 \\
 & = \frac{\log 12}{\log 3} & & = \frac{\log 1250}{\log(0.5)} \\
 & \div 2.26 & & \div -10.3 \\
 \mathbf{c} & \log_3(0.067) & \mathbf{d} & \log_{0.4}(0.006\,984) \\
 & = \frac{\log(0.067)}{\log 3} & & = \frac{\log(0.006\,984)}{\log(0.4)} \\
 & \div -2.46 & & \div 5.42
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{2} \quad \mathbf{a} & 2^x = 0.051 & \mathbf{b} & 4^x = 213.8 \\
 \therefore x = \log_2(0.051) & & \therefore x = \log_4 213.8 & \\
 \therefore x = \frac{\log(0.051)}{\log 2} & & \therefore x = \frac{\log(213.8)}{\log 4} & \\
 \therefore x \div -4.29 & & \therefore x \div 3.87 & \\
 \mathbf{c} & 3^{2x+1} = 4.069 & & \\
 \therefore 2x+1 = \log_3(4.069) & & & \\
 \therefore 2x+1 = \frac{\log(4.069)}{\log 3} & & & \\
 \therefore 2x+1 \div 1.2774 \dots & & & \\
 \therefore 2x \div 0.2774 \dots & & & \\
 \therefore x \div 0.139 & & &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{3} \quad \mathbf{a} & 25^x - 3 \times 5^x = 0 \\
 \therefore 5^{2x} - 3 \times 5^x = 0 & \\
 \therefore 5^x(5^x - 3) = 0 & \\
 \therefore 5^x = 3 & \\
 \quad \{\text{as } 5^x > 0 \text{ for all } x\} & \\
 \therefore x = \log_5 3 & \\
 \therefore x = \frac{\log 3}{\log 5} & \\
 \therefore x \div 0.683 & \\
 \mathbf{b} & 8 \times 9^x - 3^x = 0 \\
 \therefore 8 \times 3^{2x} - 3^x = 0 & \\
 \therefore 3^x(8 \times 3^x - 1) = 0 & \\
 \therefore 8 \times 3^x - 1 = 0 & \\
 \quad \{\text{as } 3^x > 0 \text{ for all } x\} & \\
 \therefore 3^x = \frac{1}{8} & \\
 \therefore x = \log_3\left(\frac{1}{8}\right) & \\
 \therefore x = \frac{\log(\frac{1}{8})}{\log 3} \div -1.89 &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{4} \quad \mathbf{a} & \log_4 x^3 + \log_2 \sqrt{x} = 8 \\
 \therefore \frac{\log x^3}{\log 4} + \frac{\log x^{\frac{1}{2}}}{\log 2} = 8 & \\
 \therefore \frac{3 \log x}{2 \log 2} + \frac{\frac{1}{2} \log x}{\log 2} = 8 & \\
 \therefore \frac{3 \log x}{2 \log 2} + \frac{\log x}{2 \log 2} = 8 & \\
 \therefore \frac{4 \log x}{2 \log 2} = 8 & \\
 \therefore \log x = 4 \log 2 & \\
 \therefore \log x = \log 2^4 & \\
 \therefore x = 16 & \\
 \mathbf{b} & \log_{16} x^5 = \log_{64} 125 - \log_4 \sqrt{x} \\
 \therefore \frac{\log x^5}{\log 16} = \frac{\log 125}{\log 64} - \frac{\log x^{\frac{1}{2}}}{\log 4} & \\
 \therefore \frac{5 \log x}{4 \log 2} = \frac{\log 125}{6 \log 2} - \frac{\frac{1}{2} \log x}{2 \log 2} & \\
 \therefore \frac{15 \log x}{12 \log 2} = \frac{2 \log 125}{12 \log 2} - \frac{3 \log x}{12 \log 2} & \\
 \therefore 15 \log x = 2 \log 125 - 3 \log x & \\
 \therefore 18 \log x = 2 \log 125 & \\
 \therefore \log x = \frac{1}{9} \log 125 & \\
 \therefore \log x = \log 125^{\frac{1}{9}} & \\
 \therefore x = 125^{\frac{1}{9}} \text{ or } x \div 1.71 &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{5} & 4^x \times 5^{4x+3} = 10^{2x+3} \\
 \therefore \log(4^x \times 5^{4x+3}) = \log 10^{2x+3} & \\
 \therefore x \log 4 + (4x+3) \log 5 = 2x+3 & \\
 \therefore x \log 4 + 4x \log 5 + 3 \log 5 = 2x+3 & \\
 \therefore x[\log 4 + 4 \log 5 - 2] = 3 - 3 \log 5 & \\
 \therefore x = \frac{3 - 3 \log 5}{\log 4 + 4 \log 5 - 2} & \\
 \therefore x = \frac{\log 10^3 - \log 5^3}{\log 4 + \log 5^4 - \log 10^2} & \\
 \therefore x = \frac{\log(\frac{1000}{125})}{\log(\frac{4 \times 5^4}{10^2})} & \\
 \therefore x = \frac{\log 8}{\log 25} \text{ or } \log_{25} 8 &
 \end{array}$$

REVIEW SET 4A

- 1**
- a** $\log_4 64$
 $= \log_4 4^3$
 $= 3$
- b** $\log_2 256$
 $= \log_2 2^8$
 $= 8$
- c** $\log_2 (0.25)$
 $= \log_2 \left(\frac{1}{4}\right)$
 $= \log_2 2^{-2}$
 $= -2$
- d** $\log_{25} 5$
 $= \log_{25} 25^{\frac{1}{2}}$
 $= \frac{1}{2}$
- e** $\log_8 1$
 $= \log_8 8^0$
 $= 0$
- f** $\log_6 6$
 $= \log_6 6^1$
 $= 1$
- g** $\log_{81} 3$
 $= \log_{81} 81^{\frac{1}{4}}$
 $= \frac{1}{4}$
- h** $\log_9 (0.\bar{1})$
 $= \log_9 \left(\frac{1}{9}\right)$
 $= \log_9 9^{-1}$
 $= -1$
- i** $\log_{27} 3$
 $= \log_{27} 27^{\frac{1}{3}}$
 $= \frac{1}{3}$
- j** $\log_k \sqrt{k}$
 $= \log_k k^{\frac{1}{2}}$
 $= \frac{1}{2}$
 $(k > 0, k \neq 1)$
- k** $\log_m \sqrt{m^5}$
 $= \log_m m^{\frac{5}{2}}$
 $= \frac{5}{2}$
 $(m > 0, m \neq 1)$
- l** $\log_n \left(\frac{1}{n^2}\right)$
 $= \log_n (n^{-2})$
 $= -2$
 $(n > 0, n \neq 1)$
- 2**
- a** $\log \sqrt{10}$
 $= \log 10^{\frac{1}{2}}$
 $= \frac{1}{2}$
- b** $\log \left(\frac{1}{\sqrt[3]{10}}\right)$
 $= \log 10^{-\frac{1}{3}}$
 $= -\frac{1}{3}$
- c** $\log(10^a \times 10^{b+1})$
 $= \log 10^{a+b+1}$
 $= a + b + 1$
- 3**
- a** $32 = 10^{\log 32}$
 $\div 10^{1.505}$
- b** 0.0013
 $= 10^{\log(0.0013)}$
 $\div 10^{-2.886}$
- c** 8.963×10^{-5}
 $= 10^{\log(8.963)} \times 10^{-5}$
 $\div 10^{0.952} \times 10^{-5}$
 $\div 10^{-4.048}$
- 4**
- a** $\log x = -3$
 $\therefore x = 10^{-3}$
 $\therefore x = 0.001$
- b** $\log x \div 2.743$
 $\therefore x \div 10^{2.743}$
 $\therefore x \div 553$
- c** $\log x \div -3.145$
 $\therefore x \div 10^{-3.145}$
 $\therefore x \div 0.000716$
- 5**
- a** $\log 4 + 2 \log 5$
 $= \log 4 + \log 5^2$
 $= \log(4 \times 25)$
 $= \log 100 \quad (\text{or } 2)$
- b** $\log 24 + \log 4 - \log 16$
 $= \log \left(\frac{24 \times 4}{16}\right)$
 $= \log 6$
- c** $2 - \log 25$
 $= \log 10^2 - \log 25$
 $= \log \left(\frac{100}{25}\right)$
 $= \log 4$
- 6**
- a** $P = 3 \times b^x$
 $\therefore \log P = \log(3 \times b^x)$
 $\therefore \log P = \log 3 + \log b^x$
 $\therefore \log P = \log 3 + x \log b$
- b** $m = \frac{n^3}{p^2} \therefore \log m = \log \left(\frac{n^3}{p^2}\right)$
 $\therefore \log m = \log n^3 - \log p^2$
 $\therefore \log m = 3 \log n - 2 \log p$
- 7**
- a** $\log k \div 1.699 + x$
 $\therefore k \div 10^{1.699+x}$
 $\therefore k \div 10^{1.699} \times 10^x$
 $\therefore k \div 50 \times 10^x$
- b** $\log Q = 3 \log P + \log R$
 $= \log(P^3 \times R)$
 $\therefore Q = P^3 R$
- c** $\log A \div 5 \log B - 2.602$
 $\therefore \log A - \log B^5 \div -2.602$
 $\therefore \log \left(\frac{A}{B^5}\right) \div -2.602$
 $\therefore \frac{A}{B^5} \div 10^{-2.602} \div 0.0025$
 $\therefore A \div \frac{B^5}{400}$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & 5^x = 7 \\
 & \therefore \log 5^x = \log 7 \\
 & \therefore x \log 5 = \log 7 \\
 & \therefore x = \frac{\log 7}{\log 5} \\
 & \therefore x \doteq 1.21
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 20 \times 2^{2x+1} = 500 \\
 & \therefore 2^{2x+1} = 25 \\
 & \therefore \log 2^{2x+1} = \log 25 \\
 & \therefore (2x+1) \log 2 = \log 25 \\
 & \therefore 2x+1 = \frac{\log 25}{\log 2} \doteq 4.6438 \dots \\
 & \therefore 2x \doteq 3.6438 \dots \\
 & \therefore x \doteq 1.82
 \end{aligned}$$

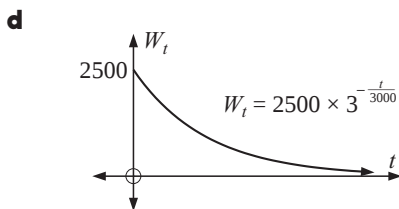
$$\mathbf{9} \quad W_t = 2500 \times 3^{-\frac{t}{3000}} \text{ grams}$$

$$\begin{aligned}
 \mathbf{a} \quad W_0 &= 2500 \times 3^0 \\
 &= 2500 \times 1 \\
 &= 2500 \text{ grams}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{We need } t \text{ when } W_t = 30\% \text{ of } 2500 \text{ g} \\
 \text{i.e., } & 2500 \times 3^{-\frac{t}{3000}} = 0.3 \times 2500 \\
 & \therefore \log 3^{-\frac{t}{3000}} = \log(0.3) \\
 & \therefore -\frac{t}{3000} \times \log 3 = \log(0.3) \\
 & \therefore t = \frac{-\log(0.3) \times 3000}{\log 3} \\
 & \therefore t \doteq 3287.7 \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \% \text{ loss} \\
 &= \left(\frac{W_{1500} - W_0}{W_0} \right) \times 100\% \\
 &= \left(\frac{2500 \times 3^{-\frac{1}{2}} - 2500}{2500} \right) \times 100\% \\
 &= (3^{-\frac{1}{2}} - 1) \times 100\% \\
 &\doteq -42.3\% \\
 &\text{i.e., a loss of } 42.3\%
 \end{aligned}$$

i.e., about 3290 years



REVIEW SET 4B

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \log \sqrt{1000} \\
 &= \log (10^3)^{\frac{1}{2}} \\
 &= \log 10^{\frac{3}{2}} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log \left(\frac{10}{\sqrt[3]{10}} \right) \\
 &= \log \left(\frac{10^1}{10^{\frac{1}{3}}} \right) \\
 &= \log 10^{\frac{2}{3}} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \log \left(\frac{10^a}{10^{-b}} \right) \\
 &= \log (10^{a-(-b)}) \\
 &= \log 10^{a+b} \\
 &= a + b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & 200 \\
 &= 10^{\log 200} \\
 &\doteq 10^{2.30}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 568\,000 \\
 &= 10^{\log(568\,000)} \\
 &\doteq 10^{5.75}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 3.69 \times 10^{-4} \\
 &= 10^{\log(3.69)} \times 10^{-4} \\
 &\doteq 10^{0.567} \times 10^{-4} \\
 &\doteq 10^{0.567-4} \\
 &\doteq 10^{-3.43}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \log x = 3 \\
 &\therefore x = 10^3 \\
 &\therefore x = 1000
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log(x+2) = 1.732 \\
 &\therefore x+2 = 10^{1.732} \\
 &\therefore x+2 \doteq 53.951 \\
 &\therefore x \doteq 51.951 \\
 &\therefore x \doteq 52.0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \log \left(\frac{x}{10} \right) = -0.671 \\
 &\therefore \frac{x}{10} = 10^{-0.671} \\
 &\therefore \frac{x}{10} \doteq 0.21330 \\
 &\therefore x \doteq 2.13
 \end{aligned}$$

- 4 a** $\log 16 + 2 \log 3$
 $= \log 16 + \log 3^2$
 $= \log(16 \times 9)$
 $= \log 144$
- b** $\log 16 - 2 \log 3$
 $= \log 16 - \log 3^2$
 $= \log\left(\frac{16}{9}\right)$
- c** $2 + \log 5$
 $= \log 10^2 + \log 5$
 $= \log(100 \times 5)$
 $= \log 500$
- 5 a** $M = ab^n$
 $\therefore \log M = \log(ab^n)$
 $\therefore \log M = \log a + \log b^n$
 $\therefore \log M = \log a + n \log b$
- b** $T = \frac{5}{\sqrt{l}}$
 $\therefore \log T = \log\left(\frac{5}{l^{\frac{1}{2}}}\right)$
 $\therefore \log T = \log 5 - \log l^{\frac{1}{2}}$
 $\therefore \log T = \log 5 - \frac{1}{2} \log l$
- c** $G = \frac{a^2 b}{c}$
 $\therefore \log G = \log\left(\frac{a^2 b}{c}\right)$
 $\therefore \log G = \log(a^2 b) - \log c$
 $\therefore \log G = \log a^2 + \log b - \log c$
 $\therefore \log G = 2 \log a + \log b - \log c$
- 6 a** $\log T = 2 \log x - \log y$
 $\therefore \log T = \log x^2 - \log y$
 $\therefore \log T = \log\left(\frac{x^2}{y}\right)$
 $\therefore T = \frac{x^2}{y}$
- b** $\log K = \log n + \frac{1}{2} \log t$
 $\therefore \log K = \log n + \log t^{\frac{1}{2}}$
 $\therefore \log K = \log(n \times \sqrt{t})$
 $\therefore K = n\sqrt{t}$
- 7 a** $3^x = 300$
 $\therefore \log 3^x = \log 300$
 $\therefore x \log 3 = \log 300$
 $\therefore x = \frac{\log 300}{\log 3}$
 $\therefore x \div 5.19$
- b** $30 \times 5^{1-x} = 0.15$
 $\therefore 5^{1-x} = 0.005$
 $\therefore \log 5^{1-x} = \log(0.005)$
 $\therefore (1-x) \log 5 = \log(0.005)$
 $\therefore 1-x = \frac{\log(0.005)}{\log 5}$
 $\therefore 1-x \div -3.292$
 $\therefore x \div 4.29$
- c** $3^{x+2} = 2^{1-x}$
 $\therefore \log 3^{x+2} = \log 2^{1-x}$
 $\therefore (x+2) \log 3 = (1-x) \log 2$
 $\therefore x \log 3 + 2 \log 3 = \log 2 - x \log 2$
 $\therefore x(\log 3 + \log 2) = \log 2 - 2 \log 3$
 $\therefore x \log 6 = \log\left(\frac{2}{9}\right)$
 $\therefore x = \frac{\log\left(\frac{2}{9}\right)}{\log 6} \div -0.839$
- 8 a** $\log 36$
 $= \log(2^2 \times 3^2)$
 $= \log 2^2 + \log 3^2$
 $= 2 \log 2 + 2 \log 3$
 $= 2A + 2B$
- b** $\log 54$
 $= \log(2 \times 3^3)$
 $= \log 2 + \log 3^3$
 $= \log 2 + 3 \log 3$
 $= A + 3B$
- c** $\log(8\sqrt{3})$
 $= \log(2^3 \times 3^{\frac{1}{2}})$
 $= \log 2^3 + \log 3^{\frac{1}{2}}$
 $= 3 \log 2 + \frac{1}{2} \log 3$
 $= 3A + \frac{1}{2}B$

$$\mathbf{d} \quad \log(20.25)$$

$$= \log(20\frac{1}{4})$$

$$= \log(\frac{81}{4})$$

$$= \log\left(\frac{3^4}{2^2}\right)$$

$$= \log 3^4 - \log 2^2$$

$$= 4\log 3 - 2\log 2$$

$$= 4B - 2A$$

$$\mathbf{e} \quad \log(0.\bar{8})$$

$$= \log(\frac{8}{9})$$

$$= \log\left(\frac{2^3}{3^2}\right)$$

$$= \log 2^3 - \log 3^2$$

$$= 3\log 2 - 2\log 3$$

$$= 3A - 2B$$

$$\mathbf{9} \quad P_n = P_0 \times 2^{\frac{t}{3}}$$

When $n = 0$, $P_0 = P_0 \times 2^0 = P_0$. So the initial population was P_0 .

$$\mathbf{a} \quad \text{If } P_n \text{ doubles, } P_n = 2P_0$$

$$\therefore P_0 2^{\frac{t}{3}} = 2P_0$$

$$\therefore 2^{\frac{t}{3}} = 2^1$$

$$\therefore \frac{t}{3} = 1$$

$$\therefore t = 3$$

So, it would take 3 years.

$$\mathbf{c} \quad \text{To increase by 200\%,}$$

P_0 becomes $3P_0$

$$\therefore 3P_0 = P_0 \times 2^{\frac{t}{3}}$$

$$\therefore 2^{\frac{t}{3}} = 3$$

$$\therefore \frac{t}{3} = \log_2 3$$

$$\therefore t = 3 \times \frac{\log 3}{\log 2} \doteq 4.75$$

So, it would take 4 years, 9 months.

$$\mathbf{b} \quad \% \text{ increase} = \left(\frac{P_4 - P_0}{P_0} \right) \times 100\%$$

$$= \left(\frac{P_0 \times 2^{\frac{4}{3}} - P_0}{P_0} \right) \times 100\%$$

$$= (2^{\frac{4}{3}} - 1) \times 100\%$$

$$\doteq 151.98 \dots$$

i.e., the % increase is 152%.

$$\text{or } P_4 = P_0 \times 2^{\frac{4}{3}}$$

$$\doteq P_0 \times 2.52$$

$$\doteq 252\% \text{ of } P_0$$

i.e., an increase of 152%.

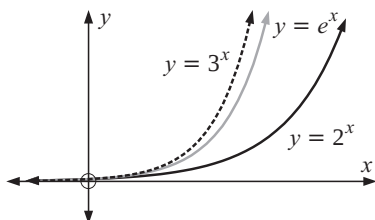
Chapter 5

NATURAL LOGARITHMS

EXERCISE 5A

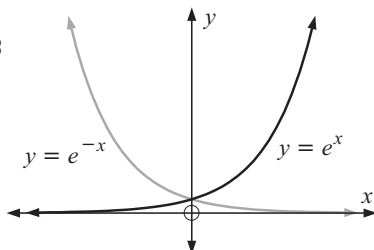
1 $e^1 \div 2.718\,281\,828\dots$

2



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

3



One is the other reflected in the y -axis.

4 When $x = 0$, $y = ae^0 = a \times 1 = a \quad \therefore$ the y -intercept is a

5 a The graph of $y = e^x$ is entirely above the x -axis.

So $y > 0$ for all x

i.e., $e^x > 0$ for all x

$\therefore 2e^x > 0$ for all x

$\therefore y$ cannot be negative if $y = 2e^x$

b i When $x = -20$, $y = 2e^{-20} \div 4.1 \times 10^{-9}$

ii When $x = 20$, $y = 2e^{20} \div 9.7 \times 10^8$

6 a $\div 7.39$ b $\div 20.1$ c $\div 2.01$ d $\div 1.65$ e $\div 0.368$

7 a $\sqrt{e} = e^{\frac{1}{2}}$

b $e\sqrt{e}$
 $= e^1 e^{\frac{1}{2}}$
 $= e^{\frac{3}{2}}$

c $\frac{1}{\sqrt{e}}$
 $= \frac{1}{e^{\frac{1}{2}}}$
 $= e^{-\frac{1}{2}}$

d $\frac{1}{e^2} = e^{-2}$

8 a $(e^{0.36})^{\frac{t}{2}}$
 $= e^{0.36 \times \frac{t}{2}}$
 $= e^{0.18t}$

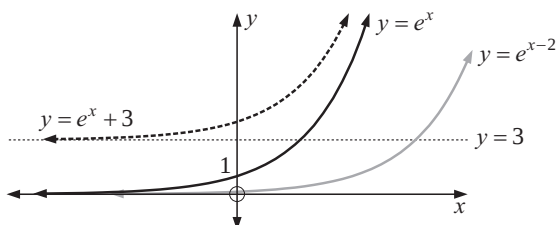
b $(e^{0.064})^{\frac{t}{16}}$
 $= e^{0.064 \times \frac{t}{16}}$
 $= e^{0.004t}$

c $(e^{-0.04})^{\frac{t}{8}}$
 $= e^{-0.04 \times \frac{t}{8}}$
 $= e^{-0.005t}$

d $(e^{-0.836})^{\frac{t}{5}}$
 $= e^{-0.836 \times \frac{t}{5}}$
 $= e^{-0.1672t}$

9 a 10.074 b 0.099 261 c 125.09 d 0.007 994 5 e 41.914 f 42.429 g 3540.3
 h 0.006 342 4

10



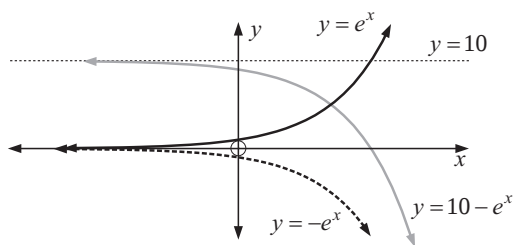
Domain of f , g and h is $\{x : x \in \mathcal{R}\}$

Range of f is $\{y : y > 0\}$

Range of g is $\{y : y > 0\}$

Range of h is $\{y : y > 3\}$

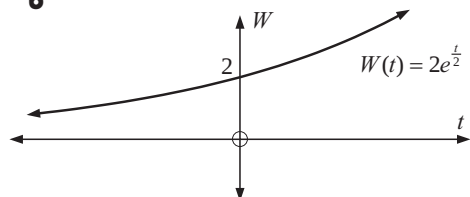
11

Domain of f, g and h is $\{x : x \in \mathbb{R}\}$ Range of f is $\{y : y > 0\}$ Range of g is $\{y : y < 0\}$ Range of h is $\{y : y < 10\}$

12 $W(t) = 2e^{\frac{t}{2}}$ grams

$$\begin{array}{llll} \text{a i} & W(0) = 2e^0 & \text{ii} & W\left(\frac{1}{2}\right) = 2e^{\frac{1}{4}} \\ & = 2 \times 1 & & \div 2.57 \text{ g} \\ & = 2 \text{ grams} & & \end{array} \quad \begin{array}{ll} \text{iii} & W\left(1\frac{1}{2}\right) = 2e^{\frac{3}{4}} \\ & \div 4.23 \text{ g} \end{array} \quad \begin{array}{ll} \text{iv} & W(6) = 2e^3 \\ & \div 40.2 \text{ g} \end{array}$$

b



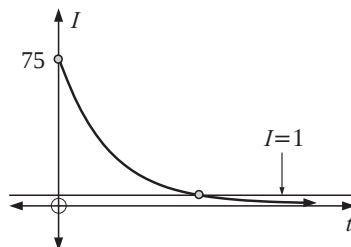
13 $I(t) = 75e^{-0.15t}$

$$\text{a i} \quad I(1) = 75e^{-0.15} = 64.6 \text{ amps}$$

$$\text{ii} \quad I(10) = 75e^{-1.5} \div 16.7 \text{ amps}$$

$$\text{c} \quad \text{We need to solve } 75e^{-0.15t} = 1. \\ \text{Using technology, } t \div 28.8 \text{ sec.}$$

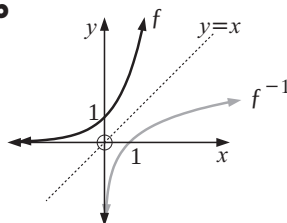
b



14 a $f(x) = e^x$

$$\begin{array}{l} \text{i.e., } y = e^x \text{ and has inverse } x = e^y \\ \therefore y = \log_e x \\ \text{i.e., } f^{-1}(x) = \log_e x \end{array}$$

b



EXERCISE 5B

1 a $\ln e^3$
 $= 3 \quad \{\ln e^a = a\}$

b $\ln 1$
 $= \ln e^0$
 $= 0$

c $\ln \sqrt[3]{e}$
 $= \ln e^{\frac{1}{3}}$
 $= \frac{1}{3}$

d $\ln \left(\frac{1}{e^2}\right)$
 $= \ln e^{-2}$
 $= -2$

3 $\ln x$ exists only when $x > 0$. {See the graph of $y = \ln x$.}

So, $\ln(-2)$ and $\ln(0)$ do not exist.Note: If $\ln(-2) = a$

then $-2 = e^a$

and $e^a = -2$ has no solutions as $e^a > 0$ for all a .

4 **a** 0.693 **b** 2.30 **c** 7.68 **d** -6.39

5 **a** $\ln e^a$
 $= a$

b $\ln(e \times e^a)$
 $= \ln e^{1+a}$
 $= 1 + a$

c $\ln\left(\frac{e^n}{e}\right)$
 $= \ln e^{n-1}$
 $= n - 1$

d $\ln(e^a \times e^b)$
 $= \ln(e^{a+b})$
 $= a + b$

e $\ln(e^a)^b$
 $= \ln e^{ab}$
 $= ab$

f $\ln\left(\frac{e^a}{e^b}\right)$
 $= \ln(e^{a-b})$
 $= a - b$

6 **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$ **e** $e^{-5.1160}$

f $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$ **i** $e^{-1.8971}$ **j** $e^{-8.8049}$

7 **a** $\ln x = 3$
 $\therefore x = e^3$
 $\therefore x \div 20.1$

b $\ln x = 1$
 $\therefore x = e^1$
 $\therefore x \div 2.72$

c $\ln x = 0$
 $\therefore x = e^0$
 $\therefore x = 1$

d $\ln x = -1$
 $\therefore x = e^{-1}$
 $\therefore x \div 0.368$

e $\ln x = -5$
 $\therefore x = e^{-5}$
 $\therefore x \div 0.00674$

f $\ln x \div 0.835$
 $\therefore x \div e^{0.835}$
 $\therefore x \div 2.30$

g $\ln x \div 2.145$
 $\therefore x \div e^{2.145}$
 $\therefore x \div 8.54$

h $\ln x \div -3.2971$
 $\therefore x \div e^{-3.2971}$
 $\therefore x \div 0.0370$

EXERCISE 5C

1 **a** $\ln 8 + \ln 2$
 $= \ln(8 \times 2)$
 $= \ln 16$

b $\ln 8 - \ln 2$
 $= \ln\left(\frac{8}{2}\right)$
 $= \ln 4$

c $\ln 40 - \ln 5$
 $= \ln\left(\frac{40}{5}\right)$
 $= \ln 8$

d $\ln 4 + \ln 5$
 $= \ln(4 \times 5)$
 $= \ln 20$

e $\ln 5 + \ln(0.4)$
 $= \ln(5 \times 0.4)$
 $= \ln 2$

f $\ln 2 + \ln 3 + \ln 4$
 $= \ln(2 \times 3 \times 4)$
 $= \ln 24$

g $1 + \ln 3$
 $= \ln e^1 + \ln 3$
 $= \ln(e \times 3)$
 $= \ln 3e$

h $\ln 4 - 1$
 $= \ln 4 - \ln e^1$
 $= \ln\left(\frac{4}{e}\right)$

i $\ln 5 + \ln 4 - \ln 2$
 $= \ln(5 \times 4) - \ln 2$
 $= \ln\left(\frac{20}{2}\right)$
 $= \ln 10$

j $2 + \ln 2$
 $= \ln e^2 + \ln 2$
 $= \ln(e^2 \times 2)$
 $= \ln(2e^2)$

k $\ln 40 - 2$
 $= \ln 40 - \ln e^2$
 $= \ln\left(\frac{40}{e^2}\right)$

l $\ln 6 - \ln 2 - \ln 3$
 $= \ln\left(\frac{6}{2}\right) - \ln 3$
 $= \ln 3 - \ln 3$
 $= \ln\left(\frac{3}{3}\right) = \ln 1$

2 **a** $5 \ln 2 + \ln 3$
 $= \ln 2^5 + \ln 3$
 $= \ln(2^5 \times 3)$
 $= \ln 96$

b $2 \ln 3 + 3 \ln 2$
 $= \ln 3^2 + \ln 2^3$
 $= \ln(9 \times 8)$
 $= \ln 72$

c $3 \ln 4 - \ln 8$
 $= \ln 4^3 - \ln 8$
 $= \ln\left(\frac{64}{8}\right)$
 $= \ln 8$

d $2 \ln 5 - 3 \ln 2$
 $= \ln 5^2 - \ln 2^3$
 $= \ln\left(\frac{25}{8}\right)$

e $\frac{1}{2} \ln 4 + \ln 3$
 $= \ln 4^{\frac{1}{2}} + \ln 3$
 $= \ln 2 + \ln 3$
 $= \ln(2 \times 3)$
 $= \ln 6$

f $\frac{1}{3} \ln\left(\frac{1}{8}\right)$
 $= \frac{1}{3} \ln(2^{-3})$
 $= \frac{1}{3} \times -3 \ln 2$
 $= -\ln 2 \quad (\text{or } \ln\left(\frac{1}{2}\right))$

g $-\ln 2$
 $= -1 \ln 2$
 $= \ln 2^{-1}$
 $= \ln\left(\frac{1}{2}\right)$

h $-\ln\left(\frac{1}{3}\right)$
 $= -1 \ln(3^{-1})$
 $= \ln(3^{-1})^{-1}$
 $= \ln 3$

i $-2 \ln\left(\frac{1}{4}\right)$
 $= \ln(2^{-2})^{-2}$
 $= \ln 2^4$
 $= \ln 16 \text{ or } 4 \ln 2$

$$\begin{array}{llll}
 \mathbf{3} \quad \mathbf{a} & \ln 9 & \mathbf{b} & \ln \sqrt{2} & \mathbf{c} & \ln\left(\frac{1}{8}\right) & \mathbf{d} & \ln\left(\frac{1}{5}\right) & \mathbf{e} & \ln\left(\frac{1}{\sqrt{2}}\right) \\
 & = \ln 3^2 & & = \ln 2^{\frac{1}{2}} & & = \ln\left(\frac{1}{2^3}\right) & & = \ln 5^{-1} & & = \ln 2^{-\frac{1}{2}} \\
 & = 2 \ln 3 & & = \frac{1}{2} \ln 2 & & = \ln(2^{-3}) & & = -1 \ln 5 & & = -\frac{1}{2} \ln 2 \\
 & & & & & = -3 \ln 2 & & = -\ln 5 & &
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{f} & \ln\left(\frac{e}{5}\right) & \mathbf{g} & \ln \sqrt[3]{5} & \mathbf{h} & \ln\left(\frac{1}{32}\right) & \mathbf{i} & \ln\left(\frac{1}{\sqrt[5]{2}}\right) = \ln\left(\frac{1}{2^{\frac{1}{5}}}\right) \\
 & = \ln e^1 - \ln 5 & & = \ln 5^{\frac{1}{3}} & & = \ln 2^{-5} & & = \ln 2^{-\frac{1}{5}} \\
 & = 1 - \ln 5 & & = \frac{1}{3} \ln 5 & & = -5 \ln 2 & & = \ln 2^{-\frac{1}{5}} \\
 & & & & & & & = -\frac{1}{5} \ln 2
 \end{array}$$

$$\mathbf{4} \quad \ln\left(\frac{e^2}{8}\right) = \ln e^2 - \ln 8 = 2 - \ln 2^3 = 2 - 3 \ln 2$$

$$\begin{array}{lll}
 \mathbf{5} \quad \mathbf{a} & \ln D = \ln x + 1 & \mathbf{b} \quad \ln F = -\ln p + 2 & \mathbf{c} \quad \ln P = \frac{1}{2} \ln x \\
 & \therefore \ln D - \ln x = 1 & & \therefore \ln P = \ln x^{\frac{1}{2}} \\
 & \therefore \ln\left(\frac{D}{x}\right) = 1 & & \therefore P = \sqrt{x} \\
 & \therefore \frac{D}{x} = e^1 & & \\
 & \therefore D = ex & &
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{d} & \ln M = 2 \ln y + 3 & \mathbf{e} & \ln B = 3 \ln t - 1 & \mathbf{f} & \ln N = -\frac{1}{3} \ln g \\
 & \therefore \ln M - 2 \ln y = 3 & & \therefore \ln B - \ln t^3 = -1 & & \therefore \ln N = \ln g^{-\frac{1}{3}} \\
 & \therefore \ln\left(\frac{M}{y^2}\right) = 3 & & \therefore \ln\left(\frac{B}{t^3}\right) = -1 & & \therefore N = g^{-\frac{1}{3}} \\
 & \therefore \frac{M}{y^2} = e^3 & & \therefore \frac{B}{t^3} = e^{-1} & & \therefore N = \frac{1}{\sqrt[3]{g}} \\
 & \therefore M = e^3 y^2 & & \therefore B = \frac{t^3}{e} & &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{g} & \ln Q \div 3 \ln x + 2.159 \\
 & \therefore \ln Q - 3 \ln x \div 2.159 \\
 & \therefore \ln\left(\frac{Q}{x^3}\right) \div 2.159 \\
 & \therefore \frac{Q}{x^3} \div e^{2.159} \\
 & \therefore \frac{Q}{x^3} \div 8.66 \\
 & \therefore Q \div 8.66x^3 \\
 \mathbf{h} & \ln D \div 0.4 \ln n - 0.6582 \\
 & \therefore \ln D - \ln n^{0.4} \div -0.6582 \\
 & \therefore \ln\left(\frac{D}{n^{0.4}}\right) \div -0.6582 \\
 & \therefore \frac{D}{n^{0.4}} \div e^{-0.6582} \\
 & \therefore \frac{D}{n^{0.4}} \div 0.518 \\
 & \therefore D \div 0.518n^{0.4}
 \end{array}$$

EXERCISE 5D

$$\begin{array}{lll}
 \mathbf{1} \quad \mathbf{a} & e^x = 10 & \mathbf{b} & e^x = 1000 & \mathbf{c} & e^x = 0.00862 \\
 & \therefore x = \ln 10 & & \therefore x = \ln 1000 & & \therefore x = \ln(0.00862) \\
 & \therefore x \div 2.303 & & \therefore x \div 6.908 & & \therefore x \div -4.754 \\
 \mathbf{d} & e^{\frac{x}{2}} = 5 & \mathbf{e} & e^{\frac{x}{3}} = 157.8 & \mathbf{f} & e^{\frac{x}{10}} = 0.01682 \\
 & \therefore \frac{x}{2} = \ln 5 & & \therefore \frac{x}{3} = \ln(157.8) & & \therefore \frac{x}{10} = \ln(0.01682) \\
 & \therefore x = 2 \ln 5 & & \therefore x = 3 \ln(157.8) & & \therefore x = 10 \ln(0.01682) \\
 & \therefore x \div 3.219 & & \therefore x \div 15.18 & & \therefore x \div -40.85
 \end{array}$$

$$\begin{array}{lll}
 \text{g} & 20 \times e^{0.06x} = 8.312 & \text{h} \quad 50e^{-0.03x} = 0.816 \\
 & \therefore e^{0.06x} = 0.4156 & \therefore e^{-0.03x} = 0.01632 \\
 & \therefore 0.06x = \ln(0.4156) & \therefore -0.03x = \ln(0.01632) \\
 & \therefore x = \frac{\ln(0.4156)}{0.06} & \therefore x = \frac{\ln(0.01632)}{-0.03} \\
 & \therefore x \doteq -14.63 & \therefore x \doteq 137.2 \\
 \text{i} & 41.83e^{0.652x} = 1000 & \\
 & \therefore e^{0.652x} \doteq 23.91 & \therefore 0.652x \doteq \ln(23.91) \\
 & & \therefore x \doteq \frac{\ln(23.91)}{0.652} \\
 & & \therefore x \doteq 4.868
 \end{array}$$

EXERCISE 5E

1 $M_t = 20e^{0.15t}$

a When $M_t = 25$,

$$20e^{0.15t} = 25$$

$$\therefore e^{0.15t} = \frac{25}{20}$$

$$\therefore 0.15t = \ln\left(\frac{25}{20}\right)$$

$$\therefore t = \frac{\ln(1.25)}{0.15}$$

$$\therefore t \doteq 1.488$$

i.e., after 1 hour 29 min

b When $M_t = 100$,

$$20e^{0.15t} = 100$$

$$\therefore e^{0.15t} = 5$$

$$\therefore 0.15t = \ln 5$$

$$\therefore t = \frac{\ln 5}{0.15}$$

$$\therefore t \doteq 10.73$$

i.e., after 10 hours 44 min

2 $T = 100e^{-0.03t} \text{ } ^\circ\text{C}$

a When $T = 30$,

$$100e^{-0.03t} = 30$$

$$\therefore e^{-0.03t} = 0.3$$

$$\therefore -0.03t = \ln(0.3)$$

$$\therefore t = \frac{\ln(0.3)}{-0.03}$$

$$\therefore t \doteq 40.13 \text{ min}$$

b When $T = 1$,

$$100e^{-0.03t} = 1$$

$$\therefore e^{-0.03t} = 0.01$$

$$\therefore -0.03t = \ln(0.01)$$

$$\therefore t = \frac{\ln(0.01)}{-0.03}$$

$$\therefore t \doteq 153\frac{1}{2} \text{ min}$$

3 $M_t = 1000e^{-0.04t} \therefore M_0 = 1000e^0 = 1000 \text{ g}$

a For M_t to halve,

$$M_t = 500$$

$$\therefore 1000e^{-0.04t} = 500$$

$$\therefore e^{-0.04t} = 0.5$$

$$\therefore -0.04t = \ln(0.5)$$

$$\therefore t = \frac{\ln(0.5)}{-0.04}$$

$$\therefore t \doteq 17.3 \text{ years}$$

b For $M_t = 25 \text{ g}$,

$$\therefore 1000e^{-0.04t} = 25$$

$$\therefore e^{-0.04t} = 0.025$$

$$\therefore -0.04t = \ln(0.025)$$

$$\therefore t = \frac{\ln(0.025)}{-0.04}$$

$$\therefore t \doteq 92.2 \text{ years}$$

c For $M_t = 1\% \text{ of } M_0$

$$\therefore 1000e^{-0.04t} = 0.01 \times 1000$$

$$\therefore e^{-0.04t} = 0.01$$

$$\therefore -0.04t = \ln(0.01)$$

$$\therefore t = \frac{\ln(0.01)}{-0.04}$$

$$\therefore t \doteq 115 \text{ years}$$

4 $V = 50(1 - e^{-0.2t})$ m/s

So, when $V = 40$,

$$50(1 - e^{-0.2t}) = 40$$

$$\therefore 1 - e^{-0.2t} = 0.8$$

$$\therefore e^{-0.2t} = 0.2$$

$$\therefore -0.2t = \ln(0.2)$$

$$\therefore t = \frac{\ln(0.2)}{-0.2}$$

$$\therefore t \div 8.05 \text{ sec}$$

i.e., it would take 8.05 sec

5 $T_m = (225 \times e^{-0.17m} - 6)^\circ \text{C}$

So, when $T_m = 0$,

$$225e^{-0.17m} - 6 = 0$$

$$\therefore 225e^{-0.17m} = 6$$

$$\therefore e^{-0.17m} = \frac{6}{225}$$

$$\therefore -0.17m = \ln\left(\frac{6}{225}\right)$$

$$\therefore m = \frac{\ln\left(\frac{6}{225}\right)}{-0.17}$$

$$\therefore m \div 21.3$$

i.e., it would take 21.3 minutes

EXERCISE 5F

1 a $f(x) = e^x + 5$

i.e., $y = e^x + 5$

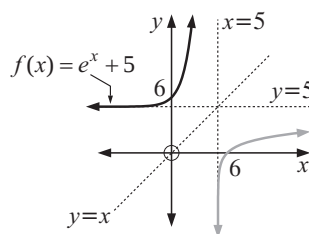
has inverse function

$$x = e^y + 5$$

$$\therefore x - 5 = e^y$$

$$\therefore y = \ln(x - 5)$$

b



c domain of f is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 5\}$

domain of f^{-1} is $\{x : x > 5\}$, range is $\{y : y \in \mathcal{R}\}$

2 a $f(x) = e^{x+1} - 3$

i.e., $y = e^{x+1} - 3$

has inverse function

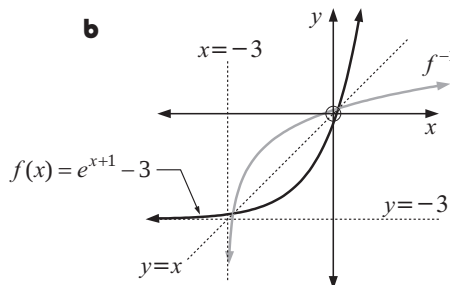
$$x = e^{y+1} - 3$$

$$\text{i.e., } x + 3 = e^{y+1}$$

$$\text{i.e., } y + 1 = \ln(x + 3)$$

$$\therefore f^{-1}(x) = \ln(x + 3) - 1$$

b



c domain of f is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > -3\}$

domain of f^{-1} is $\{x : x > -3\}$, range is $\{y : y \in \mathcal{R}\}$

3 a $f(x) = \ln x - 4$

i.e., $y = \ln x - 4$ and

has inverse function

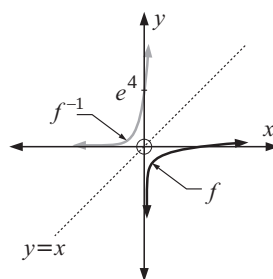
$$x = \ln y - 4$$

$$\text{i.e., } x + 4 = \ln y$$

$$\therefore y = e^{x+4}$$

$$\text{i.e., } f^{-1}(x) = e^{x+4}$$

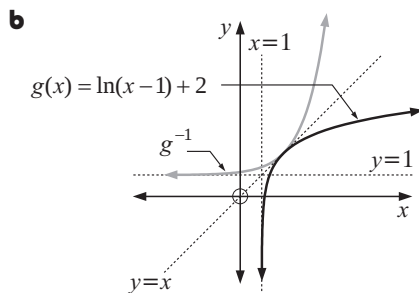
b



c domain of f is $\{x : x > 0\}$, range is $\{y : y \in \mathcal{R}\}$

domain of f^{-1} is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 0\}$

- 4 a** $g(x) = \ln(x-1) + 2, x > 1$
 i.e., $y = \ln(x-1) + 2$ and
 has inverse function
 $x = \ln(y-1) + 2$
 i.e., $\ln(y-1) = x-2$
 $\therefore y-1 = e^{x-2}$
 $\therefore y = e^{x-2} + 1$
 or $g^{-1}(x) = e^{x-2} + 1$



- c** domain of g is $\{x : x > 1\}$, range is $\{y : y \in \mathcal{R}\}$
 domain of g^{-1} is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 1\}$

- 5** $f(x) = e^{2x}$ and $g(x) = 2x - 1$

- a** Now for $f(x) = e^{2x}$, i.e., $y = e^{2x}$
 the inverse function is

$$\begin{aligned} x &= e^{2y} \\ \therefore 2y &= \ln x \\ \therefore y &= \frac{1}{2} \ln x \\ \text{i.e., } f^{-1}(x) &= \frac{1}{2} \ln x \\ \therefore (f^{-1} \circ g)(x) &= f^{-1}(g(x)) \\ &= \frac{1}{2} \ln(g(x)) \\ &= \frac{1}{2} \ln(2x - 1) \end{aligned}$$

- b** $(g \circ f)(x)$
 $= g(f(x))$
 $= 2f(x) - 1$
 $= 2e^{2x} - 1$ i.e., $y = 2e^{2x} - 1$

which has inverse function

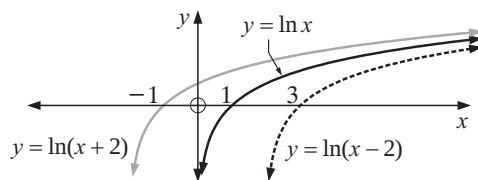
$$\begin{aligned} x &= 2e^{2y} - 1 \\ \text{i.e., } x+1 &= 2e^{2y} \\ \therefore \frac{x+1}{2} &= e^{2y} \\ \therefore 2y &= \ln\left(\frac{x+1}{2}\right) \\ \therefore y &= \frac{1}{2} \ln\left(\frac{x+1}{2}\right) \\ \therefore (g \circ f)^{-1}(x) &= \frac{1}{2} \ln\left(\frac{x+1}{2}\right) \end{aligned}$$

- 6 a** $y = \ln x$ cuts the x -axis when $y = 0$
 $\therefore \ln x = 0$
 $\therefore x = e^0 = 1$

So, graph A is that of $y = \ln x$.

Note: x -intercept of $y = \ln(x-2)$
 is when $x-2 = e^0 = 1$
 i.e., $x = 3$

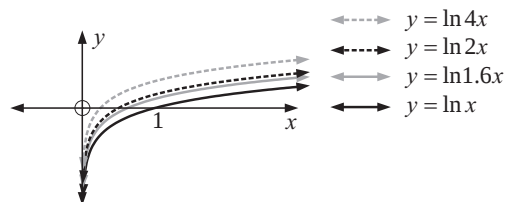
- b** The x -intercept of $y = \ln(x+2)$
 occurs when $x+2 = e^0 = 1$
 $\therefore x = -1$



- 7** Since $y = \ln x^2$ then $y = 2 \ln x$ {log law}
 i.e., new y -values are $2 \times$ old y -values. So, she is correct.

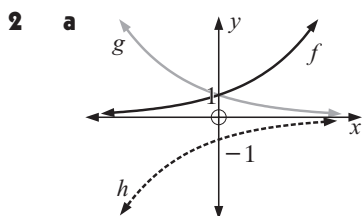
- 8** $\ln(2x) = \ln 2 + \ln x$
 $\ln(4x) = \ln 4 + \ln x$
 $\ln(1.6x) = \ln(1.6) + \ln x$

$$\begin{aligned} \therefore y = \ln(2x) &\text{ is } \ln 2 \text{ units above } y = \ln x \\ y = \ln(4x) &\text{ is } \ln 4 \text{ units above } y = \ln x \\ y = \ln(1.6x) &\text{ is } \ln(1.6) \text{ units above } y = \ln x \end{aligned}$$

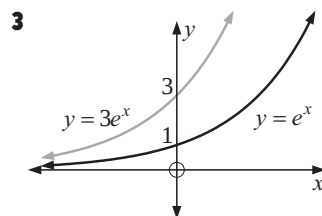


REVIEW SET 5A

1 a $\div 54.6$ b $\div 22.2$ c $\div 0.0613$ d $\div 6.07$

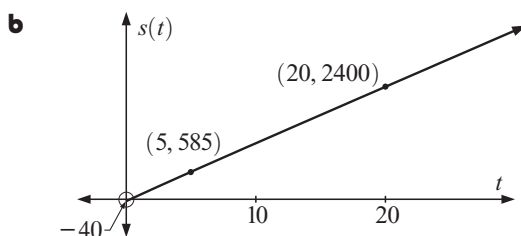


- b i g is the reflection of f in the y -axis
 ii h is the reflection of g in the x -axis



4 $s(t) = 120t - 40e^{-\frac{t}{5}}$ metres

- a i $s(0) = 0 - 40e^0 = -40$ m
 ii $s(5) = 600 - 40e^{-1} \div 585$ m
 iii $s(20) = 2400 - 40e^{-4} \div 2399.3$ m



5 a $\ln e^5 = 5$
 {as $\ln e^a = a$ }

b $\ln(\sqrt{e}) = \ln e^{\frac{1}{2}} = \frac{1}{2}$

c $\ln\left(\frac{1}{e}\right) = \ln e^{-1} = -1$

6 a $\ln(e^{2x}) = 2x$

b $\ln(e^2 e^x) = \ln(e^{2+x}) = 2 + x$

c $\ln\left(\frac{e}{e^x}\right) = \ln(e^{1-x}) = 1 - x$

7 a $\ln x = 5$
 $\therefore x = e^5$
 $\therefore x \div 148$

b $3 \ln x + 2 = 0$
 $\therefore 3 \ln x = -2$
 $\therefore \ln x = -\frac{2}{3}$
 $\therefore x = e^{-\frac{2}{3}}$
 $\therefore x \div 0.513$

8 a $\ln 6 + \ln 4 = \ln(6 \times 4) = \ln 24$

b $\ln 60 - \ln 20 = \ln\left(\frac{60}{20}\right) = \ln 3$

c $\ln 4 + \ln 1 = \ln 4 + 0 = \ln 4$

d $\ln 200 - \ln 8 + \ln 5 = \ln\left(\frac{200}{8}\right) + \ln 5 = \ln\left(\frac{200}{8} \times 5\right) = \ln 125$

9 a $\ln 32 = \ln 2^5 = 5 \ln 2$

b $\ln 125 = \ln 5^3 = 3 \ln 5$

c $\ln 729 = \ln 3^6 = 6 \ln 3$

10 a $e^x = 400$
 $\therefore x = \ln 400$
 $\therefore x \div 5.99$

b $e^{2x+1} = 11$
 $\therefore 2x + 1 = \ln 11$
 $\therefore 2x = \ln 11 - 1$
 $\therefore x = \frac{\ln 11 - 1}{2}$
 $\therefore x \div 0.699$

c $25e^{\frac{x}{2}} = 750$
 $\therefore e^{\frac{x}{2}} = 30$
 $\therefore \frac{x}{2} = \ln 30$
 $\therefore x = 2 \ln 30$
 $\therefore x \div 6.80$

d $e^{2x} = 7e^x - 12$ $\therefore e^{2x} - 7e^x + 12 = 0$ $\therefore e^x = 3$ or $e^x = 4$
 $\therefore (e^x - 3)(e^x - 4) = 0$ $\therefore x = \ln 3$ or $\ln 4$
 $\therefore x \div 1.10$ or 1.39

REVIEW SET 5B

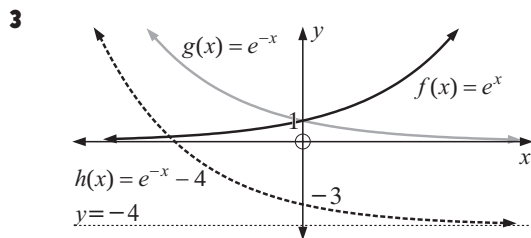
$$1 \quad a \quad \frac{1}{e^3} \\ = e^{-3}$$

$$b \quad \frac{\sqrt{e}}{e^2} \\ = e^{\frac{1}{2}} \\ = e^{\frac{1}{2}-2} \\ = e^{-\frac{3}{2}}$$

$$c \quad e^3 \sqrt{e^3} \\ = e^3 \times (e^3)^{\frac{1}{2}} \\ = e^{3+\frac{3}{2}} \\ = e^{\frac{9}{2}}$$

$$d \quad \sqrt{10e} \\ = (e^{\ln 10} e)^{\frac{1}{2}} \\ = e^{\frac{1}{2}(\ln 10 + 1)} \\ (\doteq e^{1.65})$$

$$2 \quad a \quad \doteq 26.9401 \quad b \quad \doteq 0.109447$$

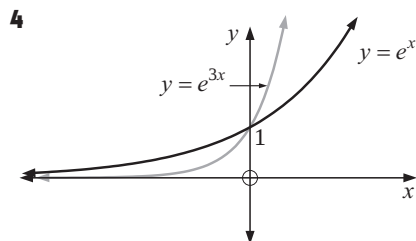


Each function has domain $\{x : x \in \mathcal{R}\}$

Range of f is $\{y : y > 0\}$

Range of g is $\{y : y > 0\}$

Range of h is $\{y : y > -4\}$



$$5 \quad a \quad \ln(e\sqrt{e}) \\ = \ln(e^1 e^{\frac{1}{2}}) \\ = \ln e^{1\frac{1}{2}} \\ = 1\frac{1}{2}$$

$$b \quad \ln\left(\frac{1}{e^3}\right) \\ = \ln e^{-3} \\ = -3$$

$$c \quad \ln\left(\frac{e}{\sqrt{e^5}}\right) \\ = \ln\left(\frac{e^1}{e^{\frac{5}{2}}}\right) \\ = \ln(e^{1-\frac{5}{2}}) \\ = \ln e^{-\frac{3}{2}} \\ = -\frac{3}{2}$$

$$6 \quad a \quad 20 = e^{\ln 20} \\ \doteq e^{3.00}$$

$$b \quad 3000 = e^{\ln 3000} \\ \doteq e^{8.01}$$

$$c \quad 0.075 = e^{\ln(0.075)} \\ \doteq e^{-2.59}$$

$$7 \quad a \quad 4 \ln 2 + 2 \ln 3 \\ = \ln 2^4 + \ln 3^2 \\ = \ln(16 \times 9) \\ = \ln 144$$

$$b \quad \frac{1}{2} \ln 9 - \ln 2 \\ = \ln 9^{\frac{1}{2}} - \ln 2 \\ = \ln 3 - \ln 2 \\ = \ln\left(\frac{3}{2}\right)$$

$$c \quad 2 \ln 5 - 1 \\ = \ln 5^2 - \ln e^1 \\ = \ln\left(\frac{25}{e}\right)$$

$$d \quad \frac{1}{4} \ln 81 \\ = \ln(3^4)^{\frac{1}{4}} \\ = \ln 3^1 \\ = \ln 3$$

$$8 \quad a \quad \ln P = 1.5 \ln Q + \ln T \\ \therefore \ln P = \ln Q^{\frac{3}{2}} + \ln T \\ = \ln\left(Q^{\frac{3}{2}} T\right) \\ \therefore P = Q^{\frac{3}{2}} T$$

$$b \quad \ln M = 1.2 - 0.5 \ln N \\ \therefore \ln M + \ln N^{\frac{1}{2}} = 1.2 \\ \therefore \ln(M\sqrt{N}) = 1.2 \\ \therefore M\sqrt{N} = e^{1.2} \\ \therefore M \doteq \frac{3.32}{\sqrt{N}}$$

9 a $g(x) = 2e^x - 5$ has inverse

function $x = 2e^y - 5$

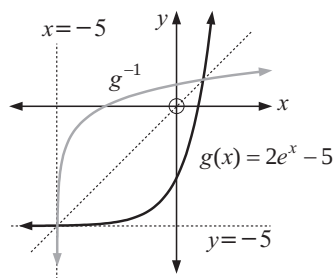
$$\therefore 2e^y = x + 5$$

$$\therefore e^y = \frac{x+5}{2}$$

$$\text{i.e., } y = \ln\left(\frac{x+5}{2}\right)$$

$$\therefore g^{-1}(x) = \ln\left(\frac{x+5}{2}\right)$$

b



c domain of g is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > -5\}$
 domain of g^{-1} is $\{x : x > -5\}$, range is $\{y : y \in \mathcal{R}\}$

10

$$W_t = 8000 \times e^{-\frac{t}{20}} \text{ grams}$$

$$W_0 = 8000e^0$$

$$= 8000 \times 1$$

$$= 8000 \text{ grams}$$

a When $W_t = \frac{1}{2} \times 8000$ grams,

$$8000e^{-\frac{t}{20}} = 4000$$

$$\therefore e^{-\frac{t}{20}} = 0.5$$

$$\therefore -\frac{t}{20} = \ln(0.5)$$

$$\therefore t = -20 \ln(0.5)$$

$$\therefore t \doteq 13.9 \text{ weeks}$$

b When $W_t = 1000$,

$$8000e^{-\frac{t}{20}} = 1000$$

$$\therefore e^{-\frac{t}{20}} = \frac{1}{8}$$

$$\therefore -\frac{t}{20} = \ln\left(\frac{1}{8}\right)$$

$$\therefore t = -20 \ln\left(\frac{1}{8}\right)$$

$$\therefore t \doteq 41.6 \text{ weeks}$$

c When $W_t = 0.1\%$ of W_0

$$= \frac{1}{1000} \text{ of } 8000 \text{ g}$$

$$= 8 \text{ g,}$$

$$8000e^{-\frac{t}{20}} = 8$$

$$\therefore e^{-\frac{t}{20}} = 0.001$$

$$\therefore -\frac{t}{20} = \ln(0.001)$$

$$\therefore t = -20 \ln(0.001)$$

$$\therefore t \doteq 138 \text{ weeks}$$

Chapter 6

GRAPHING AND TRANSFORMING FUNCTIONS

EXERCISE 6A

1 $f(x) = x$

a $f(2x) = 2x$

b $f(x) + 2$
 $= x + 2$

c $\frac{1}{2}f(x) = \frac{1}{2}x$

d $2f(x) + 3$
 $= 2x + 3$

2 $f(x) = x^2$

a $f(3x)$
 $= (3x)^2$
 $= 9x^2$

b $f\left(\frac{x}{2}\right)$
 $= \left(\frac{x}{2}\right)^2$
 $= \frac{x^2}{4}$

c $3f(x)$
 $= 3x^2$

d $2f(x-1) + 5$
 $= 2(x-1)^2 + 5$
 $= 2(x^2 - 2x + 1) + 5$
 $= 2x^2 - 4x + 7$

3 $f(x) = x^3$

a $f(4x)$
 $= (4x)^3$
 $= 64x^3$

b $\frac{1}{2}f(2x)$
 $= \frac{1}{2}(2x)^3$
 $= \frac{1}{2} \times 8x^3$
 $= 4x^3$

c $f(x+1)$
 $= (x+1)^3$
 $= x^3 + 3x^2 + 3x + 1$

d $2f(x+1) - 3$
 $= 2(x+1)^3 - 3$
 $= 2(x^3 + 3x^2 + 3x + 1) - 3$
 $= 2x^3 + 6x^2 + 6x - 1$

4 $f(x) = |x|$

a $f(2x) + 3$
 $= |2x| + 3$

b $f(-x)$
 $= |-x|$
 $= |x|$

c $f(x-2)$
 $= |x-2|$

d $f(x+1) + 2$
 $= |x+1| + 2$

5 $f(x) = 2^x$

a $f(2x) = 2^{2x}$
 $= 4^x$

b $f(-x) + 1$
 $= 2^{-x} + 1$

c $f(x-2) + 3$
 $= 2^{x-2} + 3$

d $2f(x) + 3$
 $= 2 \times 2^x + 3$
 $= 2^{1+x} + 3$

6 $f(x) = \frac{1}{x}$

a $f(-x)$
 $= \frac{1}{(-x)}$
 $= -\frac{1}{x}$

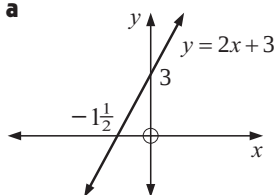
b $f\left(\frac{1}{2}x\right)$
 $= \frac{1}{\frac{1}{2}x}$
 $= \frac{2}{x}$

c $2f(x) + 3$
 $= 2\left(\frac{1}{x}\right) + 3$
 $= \frac{2}{x} + 3$
 $\left(\text{or } \frac{2+3x}{x}\right)$

d $3f(x-1) + 2$
 $= 3\left(\frac{1}{x-1}\right) + 2$
 $= \frac{3}{x-1} + 2$
 $\left(\text{or } \frac{2x+1}{x-1}\right)$

EXERCISE 6B

1 a



b i When $y = 0$, $2x + 3 = 0$

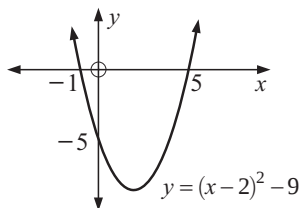
$$\therefore x = -\frac{3}{2}$$

$\therefore$ x -intercept is $-1\frac{1}{2}$

ii When $x = 0$, $y = 0 + 3 = 3$

$\therefore$ y -intercept is 3

iii As $y = 2x + 3$, the slope is 2 {the coefficient of x }

2 a**b i** When $y = 0$,

$$(x-2)^2 - 9 = 0$$

$$\therefore (x-2)^2 = 9$$

$$\therefore x-2 = \pm 3$$

$$\therefore x = 2+3 \text{ or } 2-3$$

$$\therefore x = 5 \text{ or } -1$$

$\therefore$ x -intercepts are 5 and -1

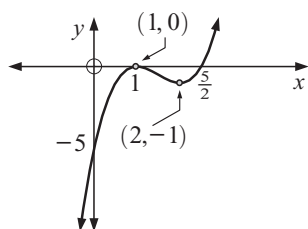
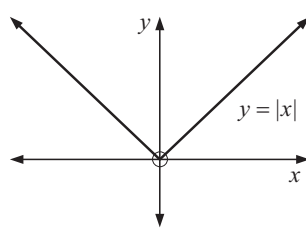
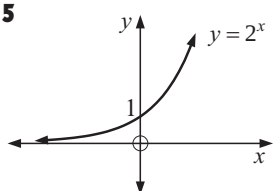
ii When $x = 0$,

$$y = (-2)^2 - 9$$

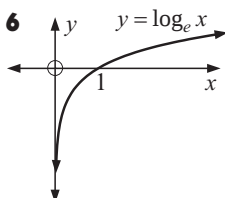
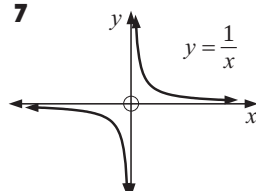
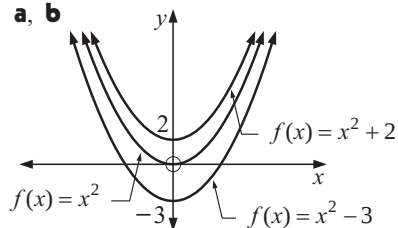
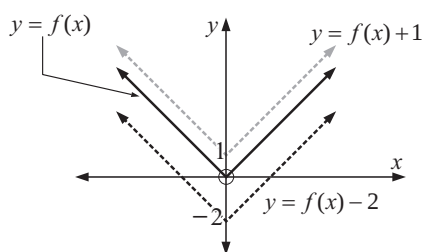
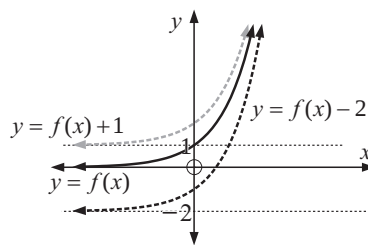
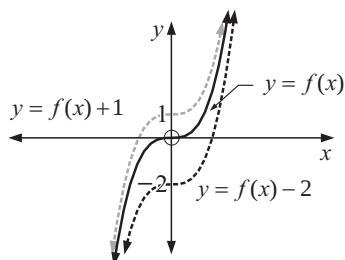
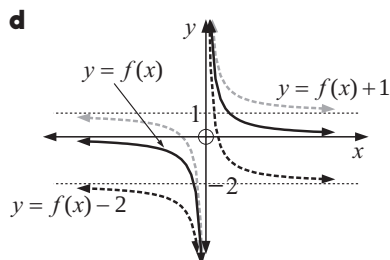
$$= 4 - 9$$

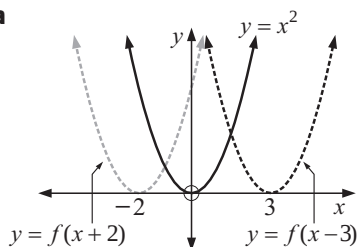
$$= -5$$

$\therefore$ y -intercept is -5

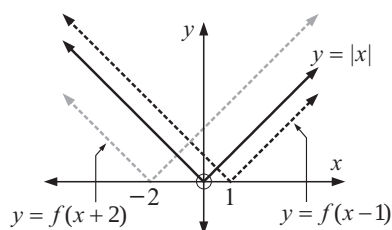
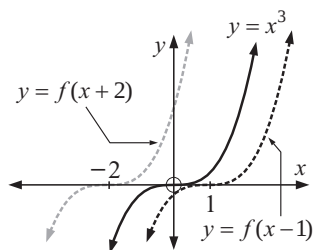
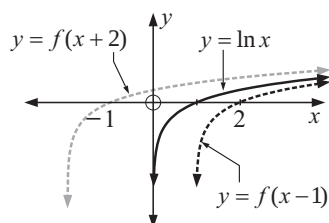
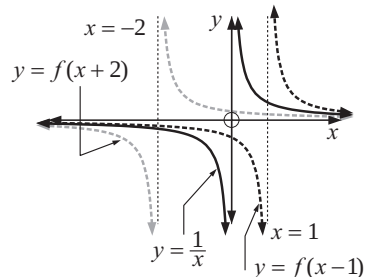
3 a, b**4****5**

When $x = 0$,
 $y = 2^0 = 1$ ✓
 $2^x > 0$ for all
 x as the graph
is always above
the y -axis. ✓

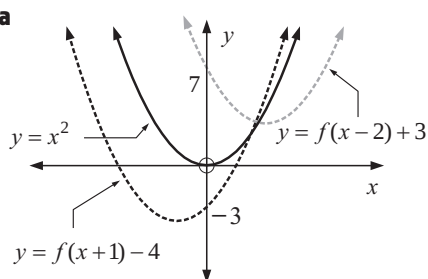
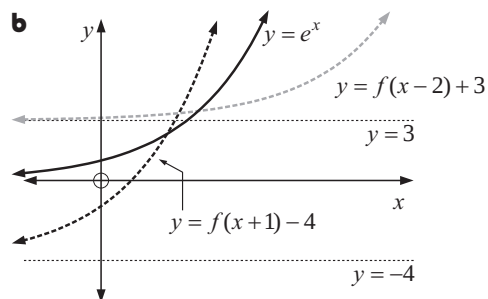
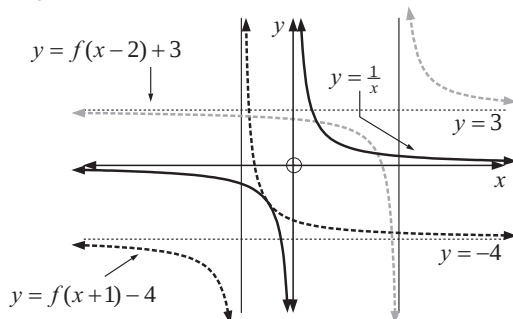
6**7****EXERCISE 6C.1****1 a, b****c i** If $b > 0$, the function is translated vertically upwards through b units.**ii** If $b < 0$, the function is translated vertically downwards $|b|$ units.**2 a****b****c****d**

3 a


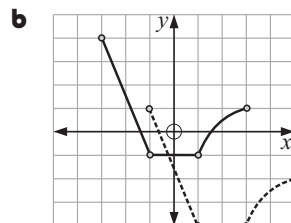
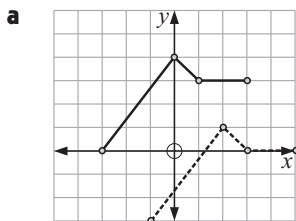
- b i** If $a > 0$, the graph is translated a units right.
ii If $a < 0$, the graph is translated $|a|$ units left.

4 a

b

c

d


$y = f(x - a)$ is a horizontal translation of $y = f(x)$ through $\begin{bmatrix} a \\ 0 \end{bmatrix}$.

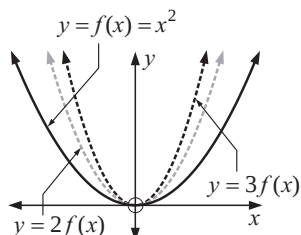
5 a

b

c


6 A translation of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

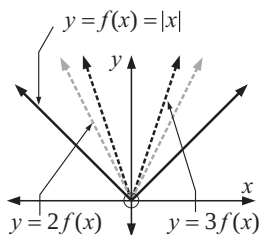


EXERCISE 6C.2

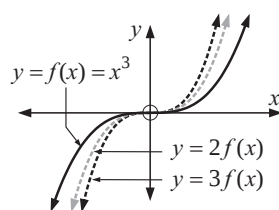
1 **a**



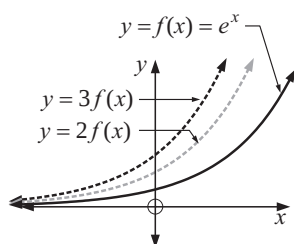
b



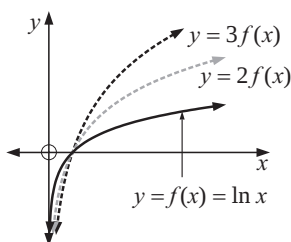
c



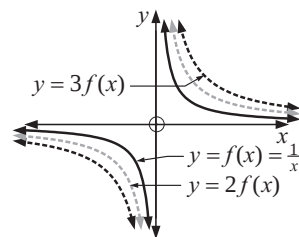
d



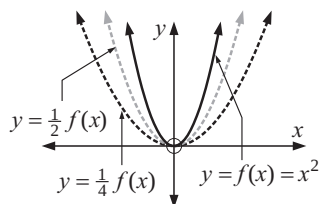
e



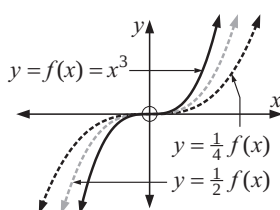
f



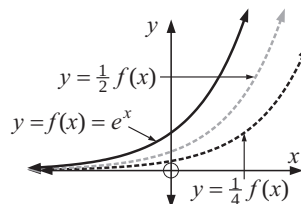
2 **a**



b

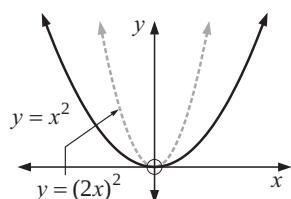


c

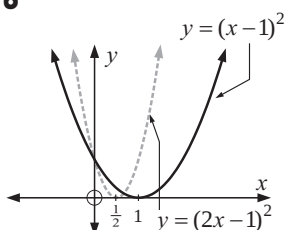


3 p affects the vertical stretching or compression of the graph of $y = f(x)$ by a factor of p . If $p > 1$, stretching occurs. If $0 < p < 1$, compression occurs.

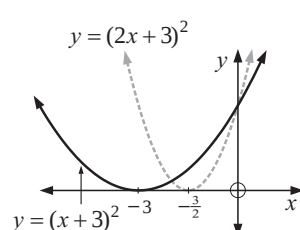
4 **a**

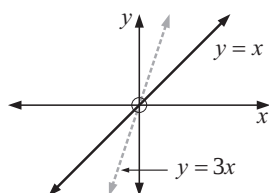
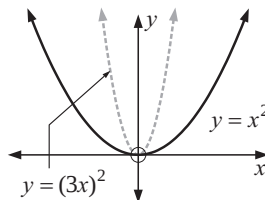
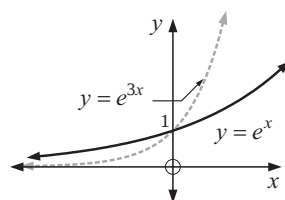
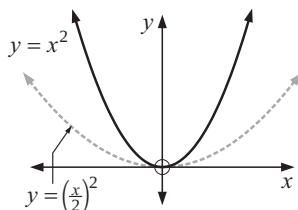
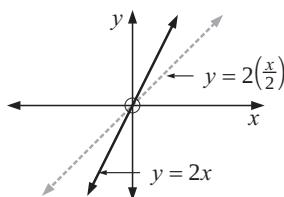
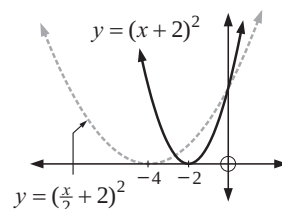


b



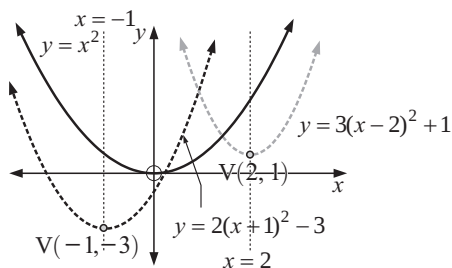
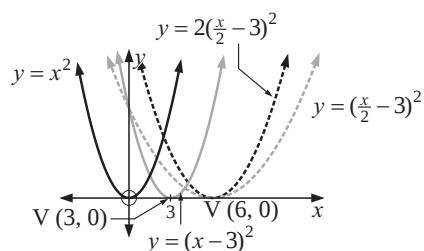
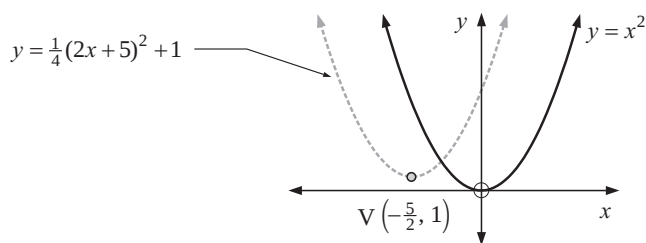
c



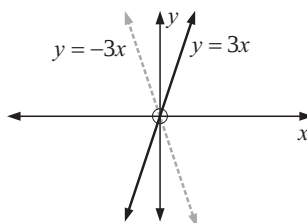
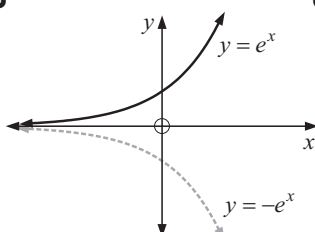
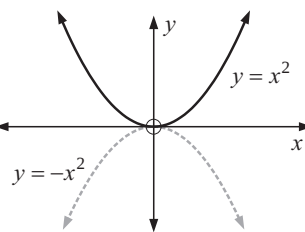
5 a

b

c

6 a

b

c


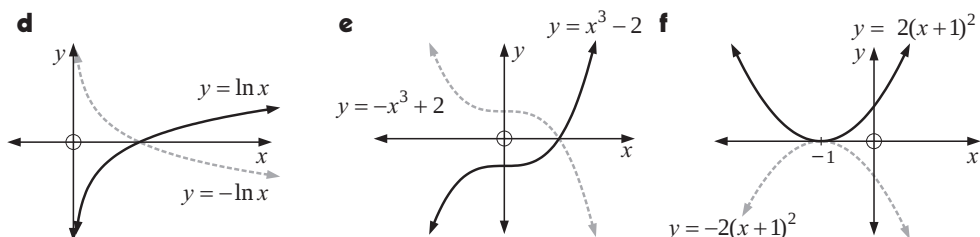
7 k affects the horizontal stretching or compression of $y = f(x)$ by a factor of $\frac{1}{k}$.

If $k > 1$, it moves closer to the y -axis. If $0 < k < 1$, it moves further from the y -axis.

8 a

b

c


EXERCISE 6C.3

1 a

b

c


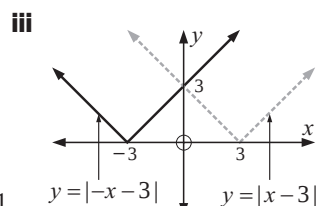
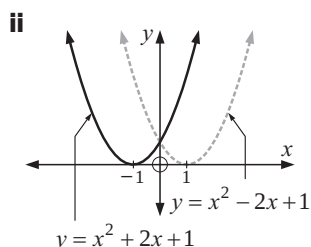
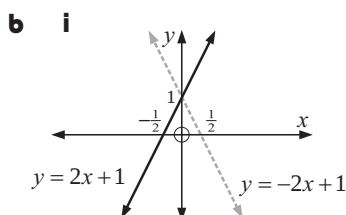


2 $y = -f(x)$ is the reflection of $y = f(x)$ in the x -axis.

3 a i $f(x) = 2x + 1$
 $\therefore f(-x)$
 $= 2(-x) + 1$
 $= -2x + 1$

ii $f(x) = x^2 + 2x + 1$
 $\therefore f(-x)$
 $= (-x)^2 + 2(-x) + 1$
 $= x^2 - 2x + 1$

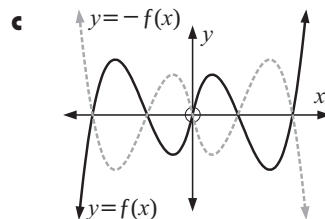
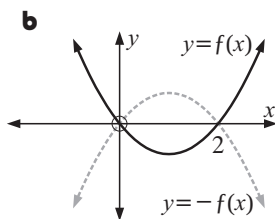
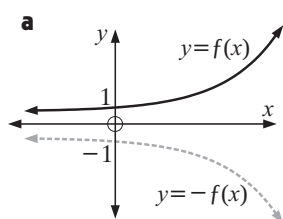
iii $f(x) = |x - 3|$
 $\therefore f(-x) = |-x - 3|$
 $= |-(x + 3)|$
 $= |x + 3|$



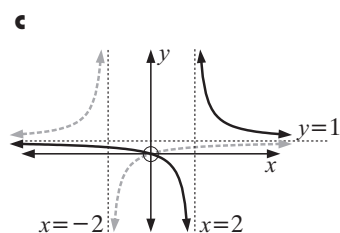
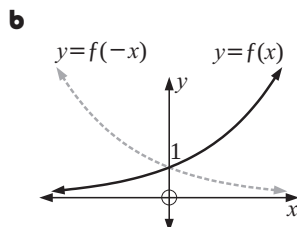
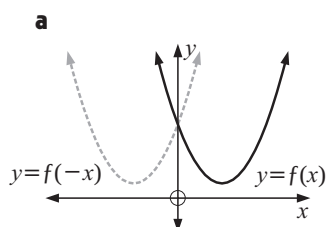
4 $y = f(-x)$ is the reflection of $y = f(x)$ in the y -axis.

EXERCISE 6D

1 $y = -f(x)$ is obtained from $y = f(x)$ by reflecting it in the x -axis.

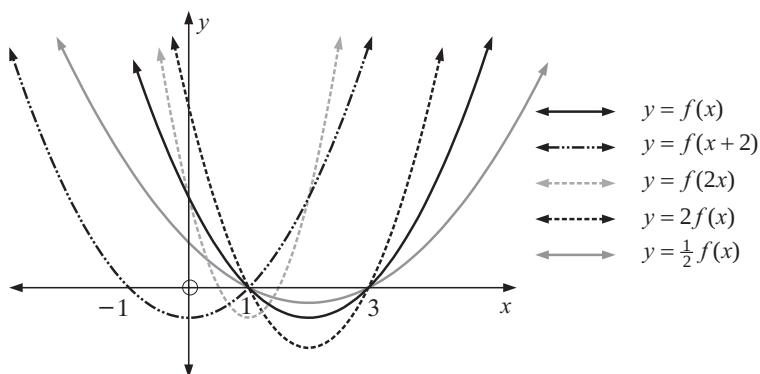


2 $y = f(-x)$ is obtained from $y = f(x)$ by reflecting it in the y -axis.

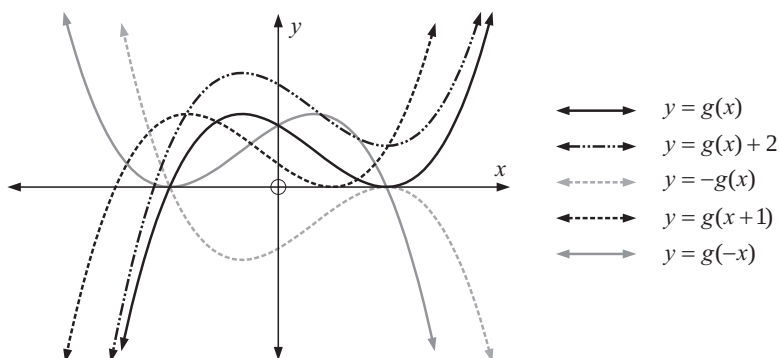


3 $y = 2x^4$ and $y = 6x^4$ are 'thinner' than $y = x^4$ and $y = \frac{1}{2}x^4$ is 'fatter'
 $\therefore$ **a** is **A**, **b** is **B**, **c** is **D** and **d** is **C**

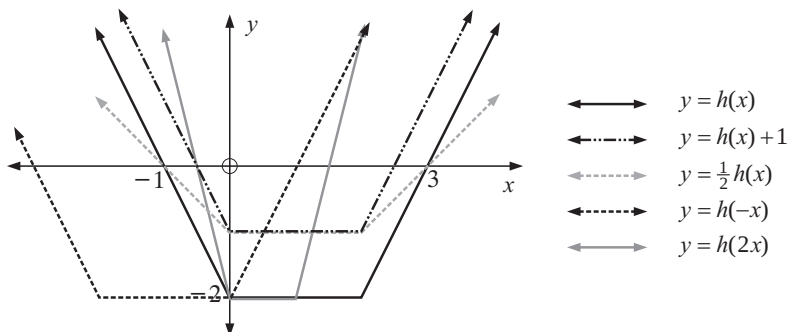
4



5



6



REVIEW SET 6

1 a linear b exponential c reciprocal d modulus e quadratic f logarithmic

2 $f(x) = x^2 - 2x$

$$\begin{aligned}
 \text{a} \quad & f(3) \\
 &= 3^2 - 2(3) \\
 &= 9 - 6 \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & f(-2) \\
 &= (-2)^2 - 2(-2) \\
 &= 4 + 4 \\
 &= 8
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & f(2x) \\
 &= (2x)^2 - 2(2x) \\
 &= 4x^2 - 4x
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & f(-x) \\
 &= (-x)^2 - 2(-x) \\
 &= x^2 + 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 3f(x) - 2 \\
 &= 3(x^2 - 2x) - 2 \\
 &= 3x^2 - 6x - 2
 \end{aligned}$$

3 $f(x) = 5 - x - x^2$

a $f(4)$
 $= 5 - 4 - 4^2$
 $= 1 - 16$
 $= -15$

b $f(-1)$
 $= 5 - (-1) - (-1)^2$
 $= 5 + 1 - 1$
 $= 5$

c $f(x-1)$
 $= 5 - (x-1) - (x-1)^2$
 $= 5 - x + 1 - [x^2 - 2x + 1]$
 $= 6 - x - x^2 + 2x - 1$
 $= -x^2 + x + 5$

d $f\left(\frac{x}{2}\right)$
 $= 5 - \left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2$
 $= 5 - \frac{x}{2} - \frac{x^2}{4}$

e $2f(x) - f(-x)$
 $= 2(5 - x - x^2) - [5 - (-x) - (-x)^2]$
 $= 10 - 2x - 2x^2 - [5 + x - x^2]$
 $= 10 - 2x - 2x^2 - 5 - x + x^2$
 $= -x^2 - 3x + 5$

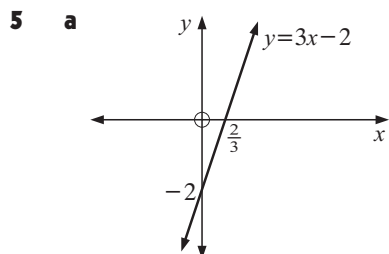
4 $f(x) = \frac{4}{x}$

a $f(-4)$
 $= \frac{4}{-4}$
 $= -1$

b $f(2x)$
 $= \frac{4}{2x}$
 $= \frac{2}{x}$

c $f\left(\frac{x}{2}\right)$
 $= \frac{4}{\frac{x}{2}}$
 $= 4 \times \frac{2}{x}$
 $= \frac{8}{x}$

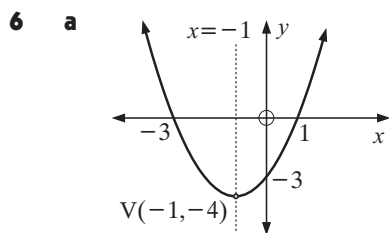
d $4f(x+2) - 3$
 $= 4\left(\frac{4}{x+2}\right) - 3$
 $= \frac{16}{x+2} - 3$
 $\left(\text{or } \frac{16 - 3(x+2)}{x+2} = \frac{10 - 3x}{x+2}\right)$



- b i** When $y = 0$,
 $3x - 2 = 0$
 $\therefore x = \frac{2}{3}$
 $\therefore x\text{-intercept is } \frac{2}{3}$
- ii** When $x = 0$,
 $y = 0 - 2 = -2$
 $\therefore y\text{-intercept is } -2$
- iii** As $y = 3x - 2$, the slope is 3 {coefficient of x }

c i When $x = 0.3$,
 $y = 3(0.3) - 2$
 $= 0.9 - 2$
 $= -1.1$

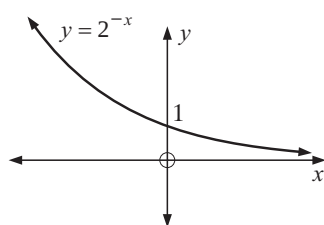
ii When $y = 0.7$,
 $3x - 2 = 0.7$
 $\therefore 3x = 2.7$
 $\therefore x = 0.9$



- b i** When $y = 0$,
 $(x+1)^2 - 4 = 0$
 $\therefore (x+1)^2 = 4$
 $\therefore x+1 = \pm 2$
 $\therefore x = 2 - 1 \text{ or } -2 - 1$
 $\text{i.e., } x = 1 \text{ or } -3$
 $\therefore x\text{-intercepts are } 1, -3$
- ii** When $x = 0$,
 $y = 1^2 - 4$
 $= -3$
 $\therefore y\text{-intercept is } -3$

- c** Since $y = (x+1)^2 - 4$ then $y + 4 = (x+1)^2$ which is obtained from $y = x^2$ under a translation of $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$. So, the vertex must be $(-1, -4)$.

7 a


 b i $x \rightarrow \infty$ means x is very large and positive.

 We see the graph approaching the x -axis

 i.e., $y \rightarrow 0 \therefore$ **true**.

 ii $x \rightarrow -\infty$ means x is very large and negative.

 We see the graph heading for ∞ and positive

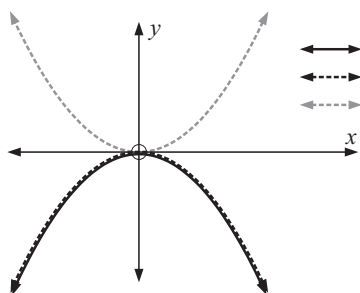
 $\therefore$ statement is **false**.

 iii The y -intercept is 1, when $x = 0$, $y = 2^0 = 1 \therefore$ **false**.

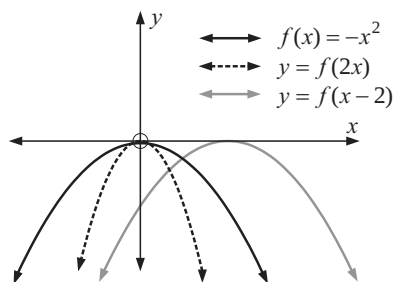
 iv The graph is above the x -axis for all $x \therefore 2^{-x} > 0$ for all $x \therefore$ **true**.

So that you can see the answers more easily in questions 8, 9 and 11, they have been drawn on two graphs.

8

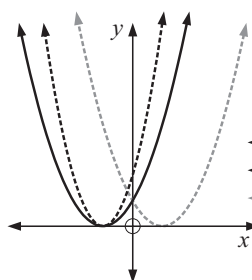


$$\begin{aligned} \longleftrightarrow & f(x) = -x^2 \\ \dashrightarrow & y = f(-x) \\ \dashrightarrow & y = -f(x) \end{aligned}$$

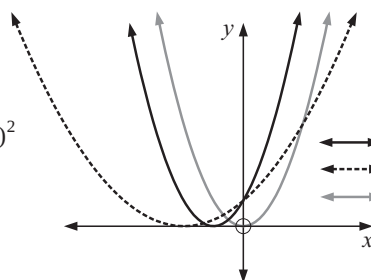


$$\begin{aligned} \longleftrightarrow & f(x) = -x^2 \\ \dashrightarrow & y = f(2x) \\ \dashrightarrow & y = f(x-2) \end{aligned}$$

9

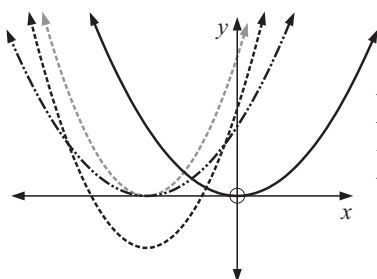


$$\begin{aligned} \longleftrightarrow & g(x) = (x+1)^2 \\ \dashrightarrow & y = 2g(x) \\ \dashrightarrow & y = g(-x) \end{aligned}$$



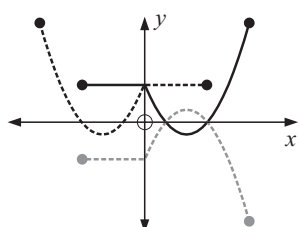
$$\begin{aligned} \longleftrightarrow & g(x) = (x+1)^2 \\ \dashrightarrow & y = g\left(\frac{x}{2}\right) \\ \dashrightarrow & y = g(x-1) \end{aligned}$$

10

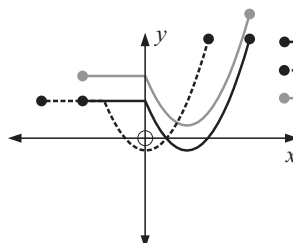


$$\begin{aligned} \longleftrightarrow & f(x) = x^2 \\ \dashrightarrow & y = f(x+2) \\ \dashrightarrow & y = 2f(x+2) \\ \dashrightarrow & y = 2f(x+2) - 3 \end{aligned}$$

11



$$\begin{aligned} \bullet \text{---} \bullet & y = f(x) \\ \bullet \text{---} \bullet & y = f(-x) \\ \bullet \text{---} \bullet & y = -f(x) \end{aligned}$$



$$\begin{aligned} \bullet \text{---} \bullet & y = f(x) \\ \bullet \text{---} \bullet & y = f(x+2) \\ \bullet \text{---} \bullet & y = f(x)+2 \end{aligned}$$

Chapter 7

COORDINATE GEOMETRY

EXERCISE 7A

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad AB &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\
 &= \sqrt{(4 - 1)^2 + (5 - 3)^2} \\
 &= \sqrt{9 + 4} \\
 &= \sqrt{13} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad PQ &= \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} \\
 &= \sqrt{(1 - 5)^2 + (4 - 2)^2} \\
 &= \sqrt{16 + 4} \\
 &= \sqrt{20} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad OC &= \sqrt{(x_C - x_O)^2 + (y_C - y_O)^2} \\
 &= \sqrt{(3 - 0)^2 + (-5 - 0)^2} \\
 &= \sqrt{9 + 25} \\
 &= \sqrt{34} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad ST &= \sqrt{(x_T - x_S)^2 + (y_T - y_S)^2} \\
 &= \sqrt{(-1 - 0)^2 + (0 - -3)^2} \\
 &= \sqrt{1 + 9} \\
 &= \sqrt{10} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad x_M &= \frac{x_A + x_B}{2} = \frac{3 + 1}{2} = 2 \quad \text{and} \\
 \therefore \text{ the midpoint is at } (2, 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad x_M &= \frac{x_A + x_B}{2} = \frac{5 - 1}{2} = 2 \quad \text{and} \\
 \therefore \text{ the midpoint is at } (2, -1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad x_M &= \frac{x_A + x_B}{2} = \frac{7 + 0}{2} = \frac{7}{2} \quad \text{and} \\
 \therefore \text{ the midpoint is at } \left(\frac{7}{2}, \frac{3}{2}\right)
 \end{aligned}$$

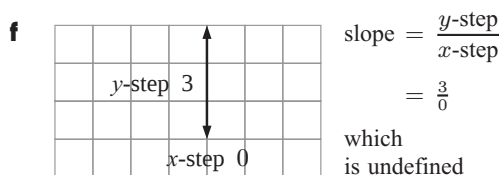
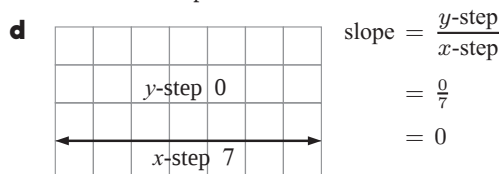
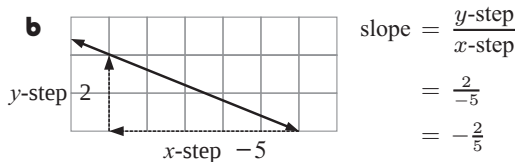
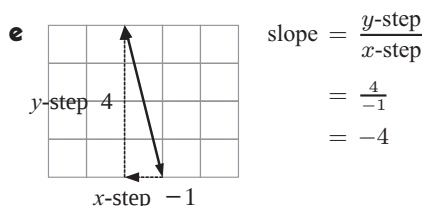
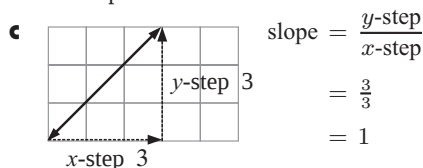
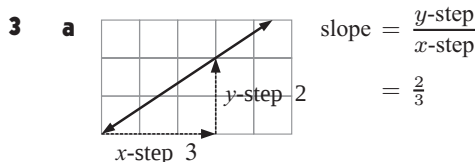
$$\begin{aligned}
 \mathbf{d} \quad x_M &= \frac{x_A + x_B}{2} = \frac{5 - 1}{2} = 2 \quad \text{and} \\
 \therefore \text{ the midpoint is at } (2, -\frac{5}{2})
 \end{aligned}$$

$$y_M = \frac{y_A + y_B}{2} = \frac{6 + 0}{2} = 3$$

$$y_M = \frac{y_A + y_B}{2} = \frac{2 - 4}{2} = -1$$

$$y_M = \frac{y_A + y_B}{2} = \frac{0 + 3}{2} = \frac{3}{2}$$

$$y_M = \frac{y_A + y_B}{2} = \frac{-2 - 3}{2} = -\frac{5}{2}$$



$$\begin{aligned} \text{4 a gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7 - 3}{4 - 2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{b gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8 - 2}{5 - 3} \\ &= 3 \end{aligned}$$

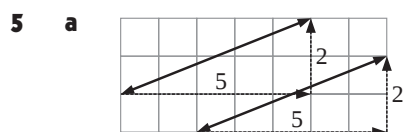
$$\begin{aligned} \text{c gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 2}{-1 - (-1)} \\ &= \frac{3}{0} \end{aligned}$$

which is undefined

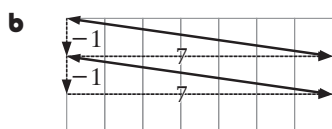
$$\begin{aligned} \text{d gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-3 - (-3)}{-1 - 4} \\ &= \frac{0}{-5} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{e gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 0}{-1 - 0} \\ &= -4 \end{aligned}$$

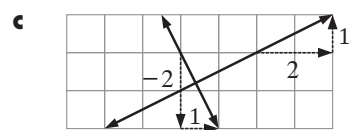
$$\begin{aligned} \text{f gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - (-1)}{-1 - 3} \\ &= \frac{-1}{-4} \\ &= \frac{1}{4} \end{aligned}$$



The lines have the same slope ($\frac{2}{5}$), so are parallel.

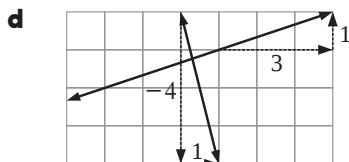


The lines have the same slope ($-\frac{1}{7}$), so are parallel.



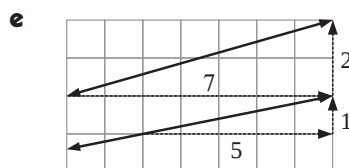
The two lines have slopes $\frac{1}{2}$ and $-\frac{2}{1}$, so the product of the slopes is -1 .

Hence the lines are perpendicular.



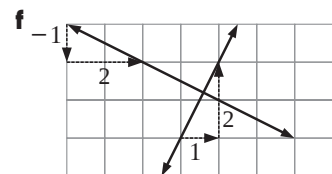
The two lines have slopes $\frac{1}{3}$ and $-\frac{4}{1}$, so the product of the slopes is $-\frac{4}{3}$.

Hence the lines are neither parallel nor perpendicular.



The two lines have slopes $\frac{2}{7}$ and $\frac{1}{5}$, so the product of the slopes is $\frac{2}{35}$.

Hence they are neither parallel nor perpendicular.



The two lines have slopes $-\frac{1}{2}$ and $\frac{2}{1}$, so the product of the slopes is -1 .

Hence the lines are perpendicular.

6 a gradient $= -\frac{1}{(\frac{3}{4})} = -\frac{4}{3}$

b gradient $= -\frac{1}{(\frac{11}{3})} = -\frac{3}{11}$

c gradient $= -\frac{1}{4}$

d gradient $= -\frac{1}{(-\frac{1}{3})} = 3$

e gradient $= -\frac{1}{(-5)} = \frac{1}{5}$

f gradient $= -\frac{1}{0}$

which is undefined

- 7 a**
- The equation of the line is

$$\frac{y-1}{x-4} = 2$$

$$\therefore y-1 = 2(x-4)$$

$$\therefore y-1 = 2x-8$$

$$\therefore y = 2x-7$$

- c**
- The equation of the line is

$$\frac{y-0}{x-5} = 3$$

$$\therefore y = 3(x-5)$$

$$\therefore y = 3x-15$$

- e**
- The equation of the line is

$$\frac{y-5}{x-1} = -4$$

$$\therefore y-5 = -4(x-1)$$

$$\therefore y-5 = -4x+4$$

$$\therefore y = -4x+9$$

- 8 a**
- The equation of the line is

$$\frac{y-1}{x-2} = \frac{3}{2}$$

$$\therefore 2(y-1) = 3(x-2)$$

$$\therefore 2y-2 = 3x-6$$

$$\therefore 3x-2y = 4$$

- c**
- The equation of the line is

$$\frac{y-0}{x-4} = \frac{1}{3}$$

$$\therefore 3y = x-4$$

$$\therefore x-3y = 4$$

- e**
- The equation of the line is

$$\frac{y-(-3)}{x-(-1)} = 3$$

$$\therefore y+3 = 3(x+1)$$

$$\therefore y+3 = 3x+3$$

$$\therefore 3x-y = 0$$

- b**
- The equation of the line is

$$\frac{y-2}{x-1} = -2$$

$$\therefore y-2 = -2(x-1)$$

$$\therefore y-2 = -2x+2$$

$$\therefore y = -2x+4$$

- d**
- The equation of the line is

$$\frac{y-7}{x-(-1)} = -3$$

$$\therefore y-7 = -3(x+1)$$

$$\therefore y-7 = -3x-3$$

$$\therefore y = -3x+4$$

- f**
- The equation of the line is

$$\frac{y-7}{x-2} = 1$$

$$\therefore y-7 = x-2$$

$$\therefore y = x+5$$

- b**
- The equation of the line is

$$\frac{y-4}{x-1} = -\frac{3}{2}$$

$$\therefore -2(y-4) = 3(x-1)$$

$$\therefore -2y+8 = 3x-3$$

$$\therefore 3x+2y = 11$$

- d**
- The equation of the line is

$$\frac{y-6}{x-0} = -4$$

$$\therefore y-6 = -4x$$

$$\therefore 4x+y = 6$$

- f**
- The equation of the line is

$$\frac{y-(-2)}{x-4} = -\frac{4}{9}$$

$$\therefore -9(y+2) = 4(x-4)$$

$$\therefore -9y-18 = 4x-16$$

$$\therefore 4x+9y = -2$$

- 9 a**
- The slope of the line is
- $\frac{2-1}{3-0} = \frac{1}{3}$
- $\therefore$
- its equation is
- $\frac{y-1}{x-0} = \frac{1}{3}$

$$\therefore 3(y-1) = x$$

$$\therefore 3y-3 = x$$

$$\therefore x-3y = -3$$

- b**
- The slope of the line is
- $\frac{-1-4}{0-1} = 5$
- $\therefore$
- its equation is
- $\frac{y-(-1)}{x-0} = 5$

$$\therefore y+1 = 5x$$

$$\therefore 5x-y = 1$$

- c** The slope of the line is $\frac{-4 - (-1)}{-1 - 2} = \frac{-3}{-3} = 1$ $\therefore$ its equation is $\frac{y - (-1)}{x - 2} = 1$
 $\therefore y + 1 = x - 2$
 $\therefore x - y = 3$
- d** The slope of the line is $\frac{2 - (-2)}{5 - 0} = \frac{4}{5}$ $\therefore$ its equation is $\frac{y - (-2)}{x - 0} = \frac{4}{5}$
 $\therefore 5(y + 2) = 4x$
 $\therefore 5y + 10 = 4x$
 $\therefore 4x - 5y = 10$
- e** The slope of the line is $\frac{2 - 0}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$ $\therefore$ its equation is $\frac{y - 0}{x - (-1)} = \frac{1}{2}$
 $\therefore 2y = x + 1$
 $\therefore x - 2y = -1$
- f** The slope of the line is $\frac{-3 - (-1)}{2 - (-1)} = \frac{-2}{3}$ $\therefore$ its equation is $\frac{y - (-1)}{x - (-1)} = \frac{-2}{3}$
 $\therefore -3(y + 1) = 2(x + 1)$
 $\therefore -3y - 3 = 2x + 2$
 $\therefore 2x + 3y = -5$

- 10 a** Since both points on the line have y -coordinate -2 , it must be horizontal.

$\therefore$ its equation is $y = -2$.

- b** Since both points on the line have x -coordinate 6 , it must be vertical.

$\therefore$ its equation is $x = 6$.

- c** Since both points on the line have x -coordinate -3 , it must be vertical.

$\therefore$ its equation is $x = -3$.

- 11 a** $2x - 3y = 6$

$$\therefore 3y = 2x - 6$$

$$\therefore y = \frac{2}{3}x - 2$$

$$\therefore \text{slope} = \frac{2}{3}, \text{ } y\text{-intercept} = -2$$

$$\text{When } y = 0, \quad 2x = 6$$

$$\therefore x\text{-intercept} = 3$$

- c** $y = -2x + 5$

$$\therefore \text{slope} = -2, \text{ } y\text{-intercept} = 5$$

$$\text{When } y = 0, \quad -2x + 5 = 0$$

$$\therefore x = \frac{5}{2}$$

i.e., the x -intercept $= \frac{5}{2}$

- e** $y = 5$

$\therefore$ the line is horizontal

$$\therefore \text{slope} = 0,$$

y -intercept $= 5$, no x -intercept

- b** $4x + 5y = 20$

$$\therefore 5y = -4x + 20$$

$$\therefore y = -\frac{4}{5}x + 4$$

$$\therefore \text{slope} = -\frac{4}{5}, \text{ } y\text{-intercept} = 4$$

$$\text{When } y = 0, \quad 4x = 20$$

$$\therefore x\text{-intercept} = 5$$

- d** $x = 8$

$\therefore$ the line is vertical

$\therefore$ slope is undefined,

no y -intercept, x -intercept $= 8$

- f** $x + y = 11$

$$\therefore y = 11 - x$$

$$\therefore \text{slope} = -1, \text{ } y\text{-intercept} = 11$$

$$\text{When } y = 0, \quad x = 11$$

$$\therefore x\text{-intercept} = 11$$

g $4x + y = 8$

$\therefore y = -4x + 8$

$\therefore$ slope = -4 , y -intercept = 8

When $y = 0$, $4x = 8$

$\therefore$ x -intercept = 2

h $x - 3y = 12$

$\therefore 3y = x - 12$

$\therefore y = \frac{1}{3}x - 4$

$\therefore$ slope = $\frac{1}{3}$, y -intercept = -4

When $y = 0$, $x = 12$

$\therefore$ x -intercept = 12

Summary of results:

	Equation of line	Slope	x -intercept	y -intercept
a	$2x - 3y = 6$	$\frac{2}{3}$	3	-2
b	$4x + 5y = 20$	$-\frac{4}{5}$	5	4
c	$y = -2x + 5$	-2	$\frac{5}{2}$	5
d	$x = 8$	undefined	8	none
e	$y = 5$	0	none	5
f	$x + y = 11$	-1	11	11
g	$4x + y = 8$	-4	2	8
h	$x - 3y = 12$	$\frac{1}{3}$	12	-4

12 a Substituting $(3, 4)$ into $3x - 2y = 1$ gives $3(3) - 2(4) = 1$

i.e., $1 = 1$ which is true

$\therefore (3, 4)$ lies on the line

b Substituting $(-2, 5)$ into $5x + 3y = -5$ gives $5(-2) + 3(5) = -5$

i.e., $5 = -5$ which is false

$\therefore (-2, 5)$ does not lie on the line

c Substituting $(6, -\frac{1}{2})$ into $3x - 8y = 22$ gives $3(6) - 8(-\frac{1}{2}) = 22$

i.e., $22 = 22$ which is true

$\therefore (6, -\frac{1}{2})$ lies on the line

13 a For $x + 2y = 8$,

when $x = 0$, $y = 4$

when $y = 0$, $x = 8$

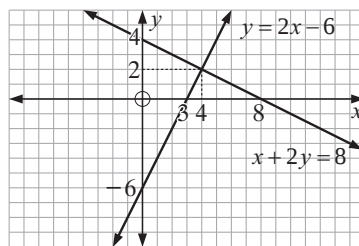
x	0	8
y	4	0

For $y = 2x - 6$,

when $x = 0$, $y = -6$

when $y = 0$, $x = 3$

x	0	3
y	-6	0

The lines meet at $(4, 2)$.

b For $y = -3x - 3$,

when $x = 0$, $y = -3$

when $y = 0$, $x = -1$

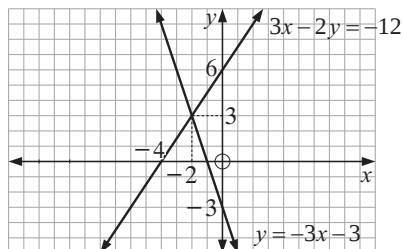
x	0	-1
y	-3	0

For $3x - 2y = -12$,

when $x = 0$, $y = 6$

when $y = 0$, $x = -4$

x	0	-4
y	6	0

The lines meet at $(-2, 3)$.

c For $3x + y = -3$,

when $x = 0$, $y = -3$

when $y = 0$, $x = -1$

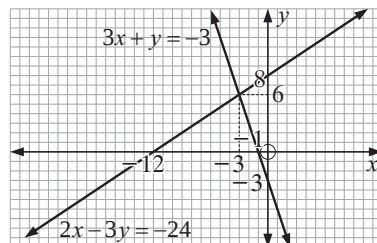
x	0	-1
y	-3	0

For $2x - 3y = -24$,

when $x = 0$, $y = 8$

when $y = 0$, $x = -12$

x	0	-12
y	8	0



The lines meet at $(-3, 6)$.

d For $2x - 3y = 8$,

when $x = 0$, $y = -\frac{8}{3}$

when $y = 0$, $x = 4$

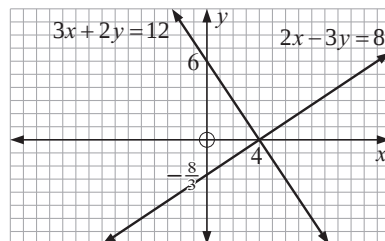
x	0	4
y	$-\frac{8}{3}$	0

For $3x + 2y = 12$,

when $x = 0$, $y = 6$

when $y = 0$, $x = 4$

x	0	4
y	6	0



The lines meet at $(4, 0)$.

e For $x + 3y = 10$,

when $x = 0$, $y = \frac{10}{3}$

when $y = 0$, $x = 10$

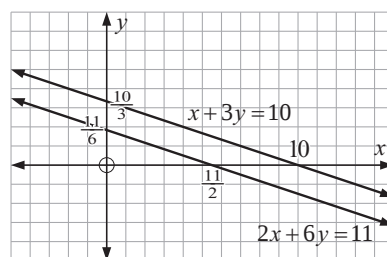
x	0	10
y	$\frac{10}{3}$	0

For $2x + 6y = 11$,

when $x = 0$, $y = \frac{11}{6}$

when $y = 0$, $x = \frac{11}{2}$

x	0	$\frac{11}{2}$
y	$\frac{11}{6}$	0



The lines are parallel, so never meet.

f For $5x + 3y = 10$,

when $x = 0$, $y = \frac{10}{3}$

when $y = 0$, $x = 2$

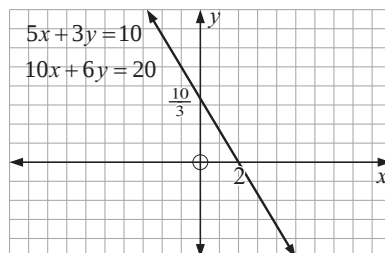
x	0	2
y	$\frac{10}{3}$	0

For $10x + 6y = 20$,

when $x = 0$, $y = \frac{20}{6} = \frac{10}{3}$

when $y = 0$, $x = 2$

x	0	2
y	$\frac{10}{3}$	0



The lines are coincident.

EXERCISE 7B.1

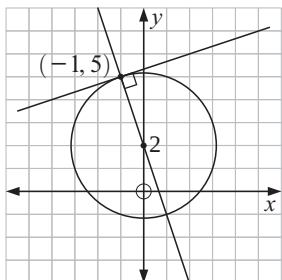
- 1 a** Since the line is horizontal, its equation is $y = -4$.
- b** Since the line is vertical, its equation is $x = 5$.
- c** Since the line is vertical, its equation is $x = -1$.
- d** Since the line is horizontal, its equation is $y = 2$.
- e** The x -axis corresponds to $y = 0$.
- f** The y -axis corresponds to $x = 0$.

- 2 a** The line has equation $\frac{y-4}{x-(-1)} = \frac{3}{4}$ $\therefore 4(y-4) = 3(x+1)$
 $\therefore 4y - 16 = 3x + 3$
 $\therefore 3x - 4y = -19$
- b** The line has slope $= \frac{0-(-5)}{7-2} = \frac{5}{5} = 1$ $\therefore$ its equation is $\frac{y-0}{x-7} = 1$
 $\therefore y = x - 7$
 $\therefore x - y = 7$
- c** $y = 3x - 2$ has slope 3, so this line has slope 3 also.
 It passes through (0, 0), so its equation is $\frac{y-0}{x-0} = 3$ i.e., $y = 3x$
- d** Now a line parallel to $2x + 3y = 8$ has equation $2x + 3y = k$, where k is a constant.
 Since $(-1, 7)$ lies on the line, $2(-1) + 3(7) = k$
 $\therefore k = 19$
 $\therefore$ the line is $2x + 3y = 19$
- e** $y = -2x + 5$ has slope -2 $\therefore$ lines perpendicular to it have slope $\frac{1}{2}$.
 But this line must pass through (3, -1). $\therefore \frac{y-(-1)}{x-3} = \frac{1}{2}$
 $\therefore 2(y+1) = x-3$
 $\therefore 2y+2 = x-3$
 $\therefore x-2y = 5$
- f** If $3x - y = 11$, then $y = 3x - 11$
 $\therefore$ this line has slope 3 $\therefore$ lines perpendicular to it have slope $-\frac{1}{3}$.
 But this line must pass through $(-2, 5)$. $\therefore \frac{y-5}{x-(-2)} = -\frac{1}{3}$
 $\therefore -3(y-5) = x+2$
 $\therefore -3y+15 = x+2$
 $\therefore x+3y = 13$
- 3 a** Keach Avenue passes through (5, 11) and (13, 12)
 $\therefore$ its line has slope $\frac{12-11}{13-5} = \frac{1}{8}$
 $\therefore$ its equation is $\frac{y-12}{x-13} = \frac{1}{8}$ $\therefore 8(y-12) = x-13$
 $\therefore 8y-96 = x-13$
 $\therefore x-8y = -83$
- b** Peacock Street is perpendicular to Keach Avenue, so its line has slope -8 .
 But this line also passes through (3, 17). $\therefore \frac{y-17}{x-3} = -8$
 $\therefore y-17 = -8(x-3)$
 $\therefore y-17 = -8x+24$
 $\therefore 8x+y = 41$
- c** Diagonal Road runs from (5, 11) to (7, 20), but ends at these points.
 Hence there is a restricted domain $5 \leq x \leq 7$.
 Its slope is $\frac{20-11}{7-5} = \frac{9}{2}$

$$\begin{aligned}\therefore \text{ its equation is } \frac{y-11}{x-5} &= \frac{9}{2} & \therefore 2(y-11) &= 9(x-5) \\ & & \therefore 2y-22 &= 9x-45 \\ & & \therefore 9x-2y &= 23, \quad 5 \leq x \leq 7\end{aligned}$$

- d** Plunkit Street has $x = 8$, so it meets Keach Avenue when $8 - 8y = -83$ {using **a**}
- $$\begin{aligned}\therefore 8y &= 91 \\ \therefore y &= \frac{91}{8}\end{aligned}$$
- $\therefore$ they intersect at $(8, \frac{91}{8})$

4 a



The line through $(-1, 5)$ and $(0, 2)$

$$\text{has slope} = \frac{5-2}{-1-0} = -3$$

$\therefore$ the slope of the tangent is $\frac{1}{3}$

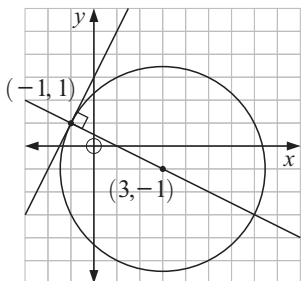
$$\therefore \text{ its equation is } \frac{y-5}{x-(-1)} = \frac{1}{3}$$

$$\therefore 3(y-5) = x+1$$

$$\therefore 3y-15 = x+1$$

$$\therefore x-3y = -16$$

b



The line through $(-1, 1)$ and $(3, -1)$

$$\text{has slope} = \frac{-1-1}{3-(-1)} = -\frac{1}{2}$$

$\therefore$ the slope of the tangent is 2

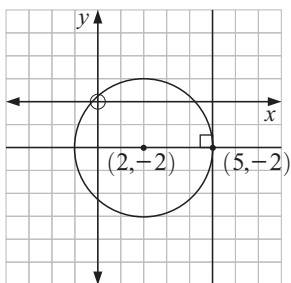
$$\therefore \text{ its equation is } \frac{y-1}{x-(-1)} = 2$$

$$\therefore y-1 = 2(x+1)$$

$$\therefore y-1 = 2x+2$$

$$\therefore 2x-y = -3$$

c



The line through $(2, -2)$ and $(5, -2)$

$$\text{has slope} = \frac{-2-(-2)}{5-2} = 0$$

i.e., it is horizontal

$\therefore$ the tangent must be vertical

$\therefore$ its equation is $x = 5$

EXERCISE 7B.2

- 1 a** The equation is $x-2y = (4)-2(1)$
i.e., $x-2y = 2$
- b** The equation is $2x-3y = 2(-2)-3(5)$
i.e., $2x-3y = -19$
- c** The equation is $3x-4y = 3(5)-4(0)$
i.e., $3x-4y = 15$
- d** The equation is $3x-y = 3(3)-(-2)$
i.e., $3x-y = 11$
- e** The equation is $x+3y = (1)+3(4)$
i.e., $x+3y = 13$
- f** The equation is $3x+y = 3(0)+(4)$
i.e., $3x+y = 4$

$$\begin{array}{lll} \mathbf{2} \quad \mathbf{a} & \text{slope} = -\frac{2}{3} & \mathbf{b} \quad \text{slope} = -\frac{3}{(-7)} = \frac{3}{7} \quad \mathbf{c} \quad \text{slope} = -\frac{6}{(-11)} = \frac{6}{11} \\ \mathbf{d} & \text{slope} = -\frac{5}{6} & \mathbf{e} \quad \text{slope} = -\frac{3}{6} = -\frac{1}{2} \quad \mathbf{f} \quad \text{slope} = -\frac{15}{(-5)} = 3 \end{array}$$

3 a Any line with equation $3x + 5y = \dots$ has slope $= -\frac{3}{5}$.

$\therefore$ since any parallel line must also have this slope,
its equation must also have the form $3x + 5y = \dots$

b Any line with equation $3x + 5y = \dots$ has slope $= -\frac{3}{5}$.

$\therefore$ any perpendicular line has slope $= -\frac{1}{(-\frac{3}{5})} = \frac{5}{3}$

$\therefore$ its equation has the form $5x - 3y = \dots$

4 a The line $3x + 4y = 6$ has slope $= -\frac{3}{4}$

$\therefore$ the new line has slope $= -\frac{3}{4}$ also

$\therefore$ since it passes through $(2, 1)$, its equation is $3x + 4y = 3(2) + 4(1)$
i.e., $3x + 4y = 10$

b The line $5x + 2y = 10$ has slope $= -\frac{5}{2}$

$\therefore$ the new line has slope $= -\frac{1}{(-\frac{5}{2})} = \frac{2}{5}$

$\therefore$ since it passes through $(-1, -1)$, its equation is $2x - 5y = 2(-1) - 5(-1)$
i.e., $2x - 5y = 3$

c The line $x - 3y + 6 = 0$ has slope $= -\frac{1}{(-3)} = \frac{1}{3}$

$\therefore$ the new line has slope $= -3$

$\therefore$ since it passes through $(-4, 0)$, its equation is $3x + y = 3(-4) + 0$
i.e., $3x + y = -12$

d The line $x - 3y = 11$ has slope $= -\frac{1}{(-3)} = \frac{1}{3}$

$\therefore$ the new line has slope $= \frac{1}{3}$ also

$\therefore$ since it passes through $(0, 0)$, its equation is $x - 3y = 0$

5 a $2x - 3y = 6$ has slope $= -\frac{2}{(-3)} = \frac{2}{3}$

$6x + ky = 4$ has slope $= -\frac{6}{k}$

b The lines are parallel if their slopes are equal.

$$\therefore \frac{2}{3} = -\frac{6}{k}$$

$$\therefore k = -6 \times \frac{3}{2}$$

$$\therefore k = -9$$

c The lines are perpendicular if the product of their slopes is -1

$$\therefore \frac{2}{3} \times \left(-\frac{6}{k}\right) = -1$$

$$\therefore -\frac{4}{k} = -1$$

$$\therefore k = 4$$

EXERCISE 7C

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad AB &= \sqrt{(-11-3)^2 + (3-8)^2} \\ &= \sqrt{196+25} \\ &= \sqrt{221} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-8-3)^2 + (-2-8)^2} \\ &= \sqrt{121+100} \\ &= \sqrt{221} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-8-(-11))^2 + (-2-3)^2} \\ &= \sqrt{9+25} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

Since only two sides of the triangle are equal in length, the triangle is isosceles.

$$\begin{aligned} \mathbf{c} \quad KL &= \sqrt{(-2-6)^2 + (-3-(-5))^2} \\ &= \sqrt{64+4} \\ &= \sqrt{68} \text{ units} \end{aligned}$$

$$\begin{aligned} KM &= \sqrt{(-1-6)^2 + (1-(-5))^2} \\ &= \sqrt{49+36} \\ &= \sqrt{85} \text{ units} \end{aligned}$$

$$\begin{aligned} LM &= \sqrt{(-1-(-2))^2 + (1-(-3))^2} \\ &= \sqrt{1+16} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{Now } (\sqrt{68})^2 + (\sqrt{17})^2 &= (\sqrt{85})^2 \\ \therefore KL^2 + LM^2 &= KM^2 \end{aligned}$$

$\therefore$ the triangle is right angled at L.

$$\begin{aligned} \mathbf{e} \quad AB &= \sqrt{(2-7)^2 + (3-5)^2} \\ &= \sqrt{25+4} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(6-7)^2 + (-7-5)^2} \\ &= \sqrt{1+144} \\ &= \sqrt{145} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(6-2)^2 + (-7-3)^2} \\ &= \sqrt{16+100} \\ &= \sqrt{116} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{Now } (\sqrt{29})^2 + (\sqrt{116})^2 &= (\sqrt{145})^2 \\ \therefore AB^2 + BC^2 &= AC^2 \end{aligned}$$

$\therefore$ the triangle is right angled at B.

$$\begin{aligned} \mathbf{b} \quad PQ &= \sqrt{(0-(-1))^2 + (\sqrt{3}-0)^2} \\ &= \sqrt{1+3} \\ &= 2 \text{ units} \end{aligned}$$

$$\begin{aligned} PR &= \sqrt{(1-(-1))^2 + 0^2} \\ &= \sqrt{4} \\ &= 2 \text{ units} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(1-0)^2 + (0-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= 2 \text{ units} \end{aligned}$$

Since all three sides of the triangle are equal in length, the triangle is equilateral.

$$\begin{aligned} \mathbf{d} \quad AB &= \sqrt{(3-1)^2 + (5-(-1))^2} \\ &= \sqrt{4+36} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-4-1)^2 + (-16-(-1))^2} \\ &= \sqrt{25+225} \\ &= \sqrt{250} \\ &= 5\sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-16-5)^2 + (-4-3)^2} \\ &= \sqrt{441+49} \\ &= \sqrt{490} \\ &= 7\sqrt{10} \text{ units} \end{aligned}$$

Now $AB + AC = BC$, so the points are collinear.

2 Suppose the outlet is at $T(x, 8)$.

$$\begin{aligned}
 \text{Then } AT = BT, \text{ so } \sqrt{(x-5)^2 + (8-5)^2} &= \sqrt{(x-7)^2 + (8-10)^2} \\
 \therefore (x-5)^2 + (8-5)^2 &= (x-7)^2 + (8-10)^2 \\
 \therefore x^2 - 10x + 25 + 9 &= x^2 - 14x + 49 + 4 \\
 \therefore -10x + 34 &= -14x + 53 \\
 \therefore 4x &= 19 \\
 \therefore x &= \frac{19}{4} \quad \therefore \text{the outlet is at } (4\frac{3}{4}, 8)
 \end{aligned}$$

3 a Suppose C is at $(x, 7)$

$$\begin{aligned}
 \text{Then } AC = BC, \text{ so } \sqrt{(x-2)^2 + (7-3)^2} &= \sqrt{(x-5)^2 + (7-4)^2} \\
 \therefore (x-2)^2 + (7-3)^2 &= (x-5)^2 + (7-4)^2 \\
 \therefore x^2 - 4x + 4 + 16 &= x^2 - 10x + 25 + 9 \\
 \therefore -4x + 20 &= -10x + 34 \\
 \therefore 6x &= 14 \\
 \therefore x &= \frac{7}{3} \quad \therefore \text{the pumping station is at } (\frac{7}{3}, 7)
 \end{aligned}$$

$$\begin{aligned}
 \text{b The length of each pipe} &= \sqrt{(x-2)^2 + (7-3)^2} \\
 &= \sqrt{(\frac{7}{3}-2)^2 + 16} \\
 &= \sqrt{16\frac{1}{9}} \quad \therefore \text{the total length} = 2\sqrt{16\frac{1}{9}} \div 8.03 \text{ km}
 \end{aligned}$$

c Now CD is horizontal, so BD is vertical $\therefore$ the x -coordinate of D is 5

$\therefore$ D is at $(5, 7)$

$$\begin{aligned}
 \text{Then } AB + BD &= \sqrt{(5-2)^2 + (4-3)^2} + \sqrt{(5-5)^2 + (7-4)^2} \\
 &= \sqrt{9+1} + \sqrt{0+9} \\
 &= \sqrt{10} + 3 \\
 &\div 6.16 \text{ km} \quad \text{Hence yes, it would be much cheaper.}
 \end{aligned}$$

$$\text{4 a} \quad PQ = \sqrt{(-3-3)^2 + (6-a)^2}$$

$$\begin{aligned}
 \therefore \sqrt{(-6)^2 + (6-a)^2} &= 10 \\
 \therefore 36 + (6-a)^2 &= 100 \\
 \therefore (6-a)^2 &= 64 \\
 \therefore 6-a &= \pm 8 \\
 \text{i.e., } 6-a &= -8 \quad \text{or} \quad 6-a = 8 \\
 \therefore a &= 14 \quad \text{or} \quad a = -2 \\
 \text{i.e., } a &= -2 \text{ or } 14
 \end{aligned}$$

$$\text{b} \quad AB = \sqrt{(-1-a)^2 + (5-2)^2}$$

$$\begin{aligned}
 \therefore \sqrt{(-1-a)^2 + 9} &= \sqrt{34} \\
 \therefore (-1-a)^2 + 9 &= 34 \\
 \therefore (-1-a)^2 &= 25 \\
 \therefore -1-a &= \pm 5 \\
 \text{i.e., } -1-a &= 5 \quad \text{or} \quad -1-a = -5 \\
 \therefore a &= -6 \quad \text{or} \quad a = 4 \\
 \text{i.e., } a &= -6 \text{ or } 4
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \quad \quad \text{QR} &= \sqrt{(-1-a)^2 + (1-2a)^2} \\
 \therefore \sqrt{(-1-a)^2 + (1-2a)^2} &= \sqrt{5} \\
 \therefore (-1-a)^2 + (1-2a)^2 &= 5 \\
 \therefore 1 + 2a + a^2 + 1 - 4a + 4a^2 &= 5 \\
 \therefore 5a^2 - 2a - 3 &= 0 \\
 \therefore (5a+3)(a-1) &= 0 \\
 \therefore a &= -\frac{3}{5} \text{ or } 1
 \end{aligned}$$

5 Suppose Jason's girlfriend lives at $G(x, 8)$

$$\begin{aligned}
 \text{Then } \text{GJ} &= \sqrt{(x-4)^2 + (8-1)^2} = 11.73 \\
 \therefore (x-4)^2 + 49 &= 11.73^2 \\
 \therefore (x-4)^2 &= 88.5929 \\
 \therefore x-4 &\doteq \pm 9.412 \\
 \therefore x &\doteq -5.412 \text{ or } 13.412
 \end{aligned}$$

i.e., Jason's girlfriend lives at $(-5.412, 8)$ or $(13.41, 8)$

6 a Any point on the x -axis can be represented by $P(a, 0)$.

$$\begin{aligned}
 \therefore \text{ given that } \text{AP} &= \text{BP}, \\
 \sqrt{(a-(-3))^2 + (0-2)^2} &= \sqrt{(a-5)^2 + (0-(-1))^2} \\
 \therefore (a+3)^2 + 4 &= (a-5)^2 + 1 \\
 \therefore a^2 + 6a + 13 &= a^2 - 10a + 26 \\
 \therefore 16a &= 13 \\
 \therefore a &= \frac{13}{16} \quad \text{Hence the point is } \left(\frac{13}{16}, 0\right)
 \end{aligned}$$

b Any point on the y -axis can be represented by $P(0, b)$.

$$\begin{aligned}
 \therefore \text{ given that } \text{CP} &= \text{DP}, \\
 \sqrt{(0-5)^2 + (b-0)^2} &= \sqrt{(0-(-1))^2 + (b-(-3))^2} \\
 \therefore 25 + b^2 &= 1 + (b+3)^2 \\
 \therefore 25 + b^2 &= 1 + b^2 + 6b + 9 \\
 \therefore 6b &= 15 \\
 \therefore b &= \frac{5}{2}
 \end{aligned}$$

Hence the point is $(0, \frac{5}{2})$.

c Let $(a, 3a)$ lie on the line $y = 3x$.

Since this point is 5 units from $(2, 1)$,

$$\begin{aligned}
 \sqrt{(a-2)^2 + (3a-1)^2} &= 5 \\
 \therefore (a-2)^2 + (3a-1)^2 &= 25 \\
 \therefore a^2 - 4a + 4 + 9a^2 - 6a + 1 &= 25 \\
 \therefore 10a^2 - 10a - 20 &= 0 \\
 \therefore a^2 - a - 2 &= 0 \\
 \therefore (a-2)(a+1) &= 0 \\
 \therefore a &= -1 \text{ or } 2
 \end{aligned}$$

$\therefore$ the points are $(-1, -3)$ and $(2, 6)$

EXERCISE 7D.1

1 B is the midpoint of PQ, so it lies at $\left(\frac{3+11}{2}, \frac{-2-6}{2}\right)$ i.e., B is at (7, -4)

A is the midpoint of BP, so it lies at $\left(\frac{3+7}{2}, \frac{-2-4}{2}\right)$ i.e., A is at (5, -3)

C is the midpoint of BQ, so it lies at $\left(\frac{7+11}{2}, \frac{-4-6}{2}\right)$ i.e., C is at (9, -5)

2 B is the midpoint of AC.

$\therefore$ if C is at (a, b), then B is at $\left(\frac{-1+a}{2}, \frac{4+b}{2}\right)$

But B is at (1, 1) $\therefore \frac{-1+a}{2} = 1$ and $\frac{4+b}{2} = 1$

$$\therefore a - 1 = 2 \quad \text{and} \quad b + 4 = 2$$

$$\therefore a = 3 \quad \text{and} \quad b = -2 \quad \therefore \text{C is at (3, -2)}$$

C is the midpoint of AE.

$\therefore$ if E is at (c, d), then C is at $\left(\frac{-1+c}{2}, \frac{4+d}{2}\right)$ $\therefore \frac{-1+c}{2} = 3$ and $\frac{4+d}{2} = -2$

$$\therefore c - 1 = 6 \quad \text{and} \quad d + 4 = -4$$

$$\therefore c = 7 \quad \text{and} \quad d = -8$$

$\therefore$ E is at (7, -8)

D is the midpoint of CE, so it lies at $\left(\frac{3+7}{2}, \frac{-2-8}{2}\right)$ i.e., D is at (5, -5)

3 a The midpoint of AC is $\left(\frac{-1-3}{2}, \frac{-2+2}{2}\right)$, i.e., (-2, 0)

The midpoint of BD is $\left(\frac{0-4}{2}, \frac{1-1}{2}\right)$, i.e., (-2, 0)

Hence the diagonals bisect one another.

b AB has slope $\frac{1-(-2)}{0-(-1)} = 3$ and CD has slope $\frac{-1-2}{-4-(-3)} = \frac{-3}{-1} = 3$

Hence AB is parallel to CD.

BC has slope $\frac{2-1}{-3-0} = -\frac{1}{3}$ and AD has slope $\frac{-1-(-2)}{-4-(-1)} = \frac{1}{-3}$

Hence BC is parallel to AD.

Hence ABCD is a parallelogram.

4 a AB has slope $= \frac{-2-6}{-1-3} = \frac{-8}{-4} = 2$

AC has slope $= \frac{4-6}{7-3} = \frac{-2}{4} = -\frac{1}{2}$

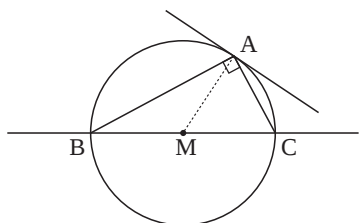
The product of the slopes is -1, so AB and AC are perpendicular.

Hence $\angle BAC$ is a right angle.

b The midpoint of BC is $\left(\frac{-1+7}{2}, \frac{-2+4}{2}\right)$ i.e., (3, 1)

c Since triangle ABC is right angled at A, BC must be a diameter of the circle. (This makes $\angle BAC$ the angle in the semicircle.)

∴ the midpoint of BC found in **b** is the centre of the circle.



$$\text{Now the slope of } AM = \frac{1-6}{3-3}$$

which is undefined.

i.e., AM is vertical.

Hence the tangent must be horizontal.

∴ since it passes through A,
its equation is $y = 6$.

- 5 a** Since ABCD is a parallelogram, the diagonals bisect each other.

∴ the midpoint of DB is the same as the midpoint of AC.

Letting D have coordinates (a, b) ,

$$\frac{a+2}{2} = \frac{3+8}{2} \quad \text{and} \quad \frac{b-1}{2} = \frac{0-2}{2}$$

$$\therefore a+2 = 11 \quad \text{and} \quad b-1 = -2$$

$$\therefore a = 9 \quad \text{and} \quad b = -1$$

∴ D is at $(9, -1)$

- b** Since PQRS is a parallelogram, the diagonals bisect each other.

∴ the midpoint of PR is the same as the midpoint of QS.

Letting R have coordinates (a, b) ,

$$\frac{-1+a}{2} = \frac{4-2}{2} \quad \text{and} \quad \frac{4+b}{2} = \frac{0+5}{2}$$

$$\therefore a-1 = 2 \quad \text{and} \quad b+4 = 5$$

$$\therefore a = 3 \quad \text{and} \quad b = 1$$

∴ R is at $(3, 1)$

- c** Since WXYZ is a parallelogram, the diagonals bisect each other.

∴ the midpoint of WY is the same as the midpoint of XZ.

Letting X have coordinates (a, b) ,

$$\frac{-1+3}{2} = \frac{a+0}{2} \quad \text{and} \quad \frac{5-2}{2} = \frac{b+4}{2}$$

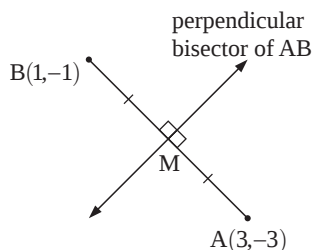
$$\therefore 2 = a \quad \text{and} \quad 3 = b+4$$

$$\therefore a = 2 \quad \text{and} \quad b = -1$$

∴ X is at $(2, -1)$

EXERCISE 7D.2

- 1 a**



The midpoint M of AB is at $\left(\frac{3+1}{2}, \frac{-3-1}{2}\right)$,
i.e., at $(2, -2)$

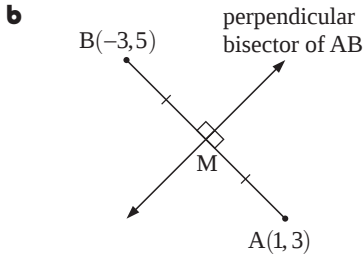
The slope of AB is $\frac{-1-(-3)}{1-3} = \frac{2}{-2} = -1$

∴ the slope of the perpendicular is 1

∴ the equation of the perpendicular bisector is

$$x - y = (2) - (-2)$$

$$\text{i.e., } x - y = 4$$



The midpoint M of AB is at $\left(\frac{1-3}{2}, \frac{3+5}{2}\right)$,
i.e., at $(-1, 4)$

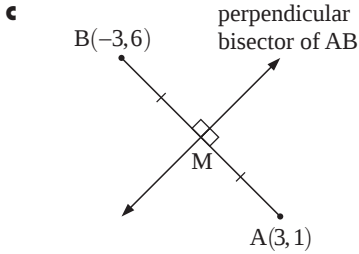
The slope of AB is $\frac{5-3}{-3-1} = -\frac{2}{4} = -\frac{1}{2}$

$\therefore$ the slope of the perpendicular is 2

$\therefore$ the equation of the perpendicular bisector is

$$2x - y = 2(-1) - 4$$

$$\text{i.e., } 2x - y = -6$$



The midpoint M of AB is at $\left(\frac{3-3}{2}, \frac{1+6}{2}\right)$,
i.e., at $(0, \frac{7}{2})$

The slope of AB is $\frac{6-1}{-3-3} = -\frac{5}{6}$

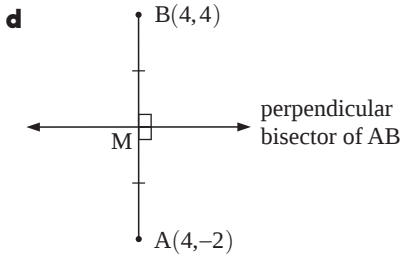
$\therefore$ the slope of the perpendicular is $\frac{6}{5}$

$\therefore$ the equation of the perpendicular bisector is

$$6x - 5y = 6(0) - 5(\frac{7}{2})$$

$$\text{i.e., } 6x - 5y = -\frac{35}{2}$$

$$\text{i.e., } 12x - 10y = -35$$



The midpoint M of AB is at $\left(\frac{4+4}{2}, \frac{-2+4}{2}\right)$,
i.e., at $(4, 1)$

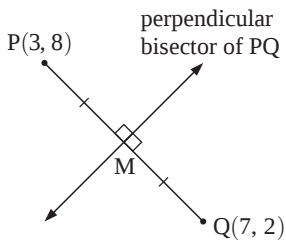
The slope of AB is $\frac{4-(-2)}{4-4}$ which is undefined

$\therefore$ AB is vertical

$\therefore$ its perpendicular bisector is horizontal

$\therefore$ it has equation $y = 1$

2



The midpoint M of PQ is at $\left(\frac{3+7}{2}, \frac{8+2}{2}\right)$,
i.e., at $(5, 5)$

The slope of PQ is $\frac{2-8}{7-3} = \frac{-6}{4} = -\frac{3}{2}$

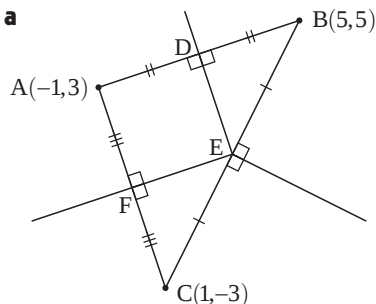
$\therefore$ the slope of the perpendicular is $\frac{2}{3}$

$\therefore$ the equation of the perpendicular bisector which is to be the boundary between the regions is

$$2x - 3y = 2(5) - 3(5)$$

$$\text{i.e., } 2x - 3y = -5$$

3



Suppose D, E and F are the midpoints of AB, BC and AC respectively.

Then D is at $\left(\frac{-1+5}{2}, \frac{3+5}{2}\right)$, i.e., $(2, 4)$

E is at $\left(\frac{5+1}{2}, \frac{5-3}{2}\right)$, i.e., $(3, 1)$

and F is at $\left(\frac{1-1}{2}, \frac{-3+3}{2}\right)$, i.e., $(0, 0)$

Now AB has slope $\frac{5-3}{5-(-1)} = \frac{2}{6} = \frac{1}{3}$

∴ the perpendicular bisector through D has slope -3

and hence equation $3x + y = 3(2) + 4$ i.e., $3x + y = 10$

BC has slope $\frac{5-(-3)}{5-1} = \frac{8}{4} = 2$

∴ the perpendicular bisector through E has slope $-\frac{1}{2}$,

and hence equation $x + 2y = (3) + 2(1)$ i.e., $x + 2y = 5$

AC has slope $\frac{-3-3}{1-(-1)} = \frac{-6}{2} = -3$

∴ the perpendicular bisector through F has slope $\frac{1}{3}$,

and hence equation $x - 3y = 0 - 3(0)$ i.e., $x - 3y = 0$

b The vertex should lie at the intersection of the three edges:

$$3x + y = 10, \quad x + 2y = 5 \quad \text{and} \quad x - 3y = 0.$$

For $3x + y = 10$,

when $x = 0$, $y = 10$

when $y = 0$, $x = \frac{10}{3}$

x	0	$\frac{10}{3}$
y	10	0

For $x + 2y = 5$,

when $x = 0$, $y = \frac{5}{2}$

when $y = 0$, $x = 5$

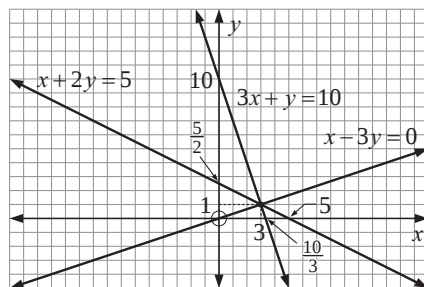
x	0	5
y	$\frac{5}{2}$	0

For $x - 3y = 0$,

when $x = 0$, $y = 0$

when $x = 3$, $y = 1$

x	0	3
y	0	1



All of the graphs meet at $(3, 1)$, so this is the location of the Voronoi vertex.

4 The midpoint of PQ is $\left(\frac{5+7}{2}, \frac{7+1}{2}\right)$, i.e., $(6, 4)$

PQ has slope $\frac{1-7}{7-5} = \frac{-6}{2} = -3$

∴ its bisector has slope $\frac{1}{3}$ and equation $x - 3y = (6) - 3(4)$

i.e., $x - 3y = -6$ (1)

The midpoint of QR is $\left(\frac{7-1}{2}, \frac{1+5}{2}\right)$, i.e., $(3, 3)$

QR has slope $\frac{5-1}{-1-7} = \frac{4}{-8} = -\frac{1}{2}$

∴ its bisector has slope 2 and equation $2x - y = 2(3) - (3)$

i.e., $2x - y = 3$ (2)

From (1), $x = 3y - 6$

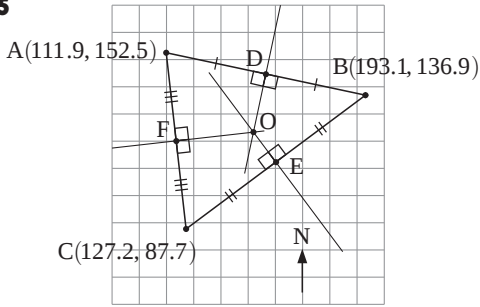
∴ in (2), $2(3y - 6) - y = 3$

∴ $6y - 12 - y = 3$

∴ $5y = 15$

∴ $y = 3$ and so $x = 9 - 6 = 3$

∴ the centre of the circle is $(3, 3)$

5

The oval must be located at the intersection of the perpendicular bisectors between the three towns.

Let D, E and F be the midpoints of AB, BC, and AC respectively.

Then D is at $\left(\frac{111.9 + 193.1}{2}, \frac{152.5 + 136.9}{2}\right)$, i.e., (152.5, 144.7)

E is at $\left(\frac{127.2 + 193.1}{2}, \frac{87.7 + 136.9}{2}\right)$, i.e., (160.15, 112.3)

F is at $\left(\frac{127.2 + 111.9}{2}, \frac{87.7 + 152.5}{2}\right)$, i.e., (119.55, 120.1)

Now we know that the three bisectors meet at a common point, so we need to find the equations of any two of them.

AB has slope $\frac{136.9 - 152.5}{193.1 - 111.9} = \frac{-15.6}{81.2}$, so DO has slope $\frac{81.2}{15.6}$

and equation $81.2x - 15.6y = 81.2(152.5) - 15.6(144.7)$
i.e., $81.2x - 15.6y \div 10\,126$ (1)

BC has slope $\frac{87.7 - 136.9}{127.2 - 193.1} = \frac{-49.2}{-65.9} = \frac{49.2}{65.9}$, so EO has slope $-\frac{65.9}{49.2}$

and equation $65.9x + 49.2y = 65.9(160.15) + 49.2(112.3)$
i.e., $65.9x + 49.2y \div 16\,079$ (2)

The location of the oval will be at the intersection of the lines (1) and (2).

We can find this point graphically, or using algebra as follows:

$$49.2 \times (1): \quad 3995.04x - 767.52y = 498\,199.2$$

$$15.6 \times (2): \quad 1028.04x + 767.52y = 250\,832.4$$

$$\hline 5023.08x \qquad \qquad = 749\,031.6$$

$$\therefore x \div 149.12$$

$$\text{Using (2),} \quad 49.2y \div 16\,079 - 65.9 \times 149.12$$

$$\therefore y \div 127.08$$

$\therefore$ the oval should be at (149.1, 127.1)

6 a P is the midpoint of OA, and lies at $\left(\frac{2a+0}{2}, \frac{2c+0}{2}\right)$, i.e., (a, c)

Q is the midpoint of AB, and lies at $\left(\frac{2a+2b}{2}, \frac{2c+0}{2}\right)$, i.e., (a + b, c)

Now OA has slope $\frac{2c}{2a} = \frac{c}{a}$, so the perpendicular bisector through P has slope $-\frac{a}{c}$

$\therefore$ it has equation $ax + cy = a(a) + c(c)$,

$$\text{i.e., } ax + cy = a^2 + c^2$$

AB has slope $\frac{0-2c}{2b-2a} = -\frac{c}{b-a}$, so the perpendicular bisector through Q has slope $\frac{b-a}{c}$

$$\therefore \text{ it has equation } (b-a)x - cy = (b-a)(a+b) - c(c) \\ = ba + b^2 - a^2 - ab - c^2$$

$$\text{i.e., } (b-a)x - cy = b^2 - a^2 - c^2$$

b From **a**, S lies at the intersection of $ax + cy = a^2 + c^2$ (1)

$$\text{and } (b-a)x - cy = b^2 - a^2 - c^2 \quad \text{..... (2)}$$

Adding (1) and (2) gives $(b-a+a)x + (c-c)y = a^2 + c^2 + b^2 - a^2 - c^2$

$$\therefore bx = b^2$$

$$\therefore x = b$$

c Since R and S have the same x -coordinate, RS is the vertical line $x = b$.

$\therefore$ RS is perpendicular to OB, which is part of the x -axis.

d The perpendicular bisectors of the sides of a triangle intersect at a point.

REVIEW SET 7A

$$\begin{aligned} \mathbf{1} \quad PQ &= \sqrt{(-1 - (-4))^2 + (3 - 7)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \text{The midpoint is at } &\left(\frac{3+7}{2}, \frac{5-2}{2}\right), \\ \text{i.e., at } &(5, \frac{3}{2}) \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \text{The equation is } &\frac{y - (-1)}{x - 2} = -3 \quad \therefore y + 1 = -3(x - 2) \\ &\therefore y + 1 = -3x + 6 \\ &\therefore y = -3x + 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{The line has slope } &\frac{4 - (-2)}{-1 - 3} = \frac{6}{-4} = -\frac{3}{2} \quad \therefore \text{ its equation is } \frac{y - (-2)}{x - 3} = -\frac{3}{2} \\ &\therefore y + 2 = -\frac{3}{2}(x - 3) \\ &\therefore y + 2 = -\frac{3}{2}x + \frac{9}{2} \\ &\therefore y = -\frac{3}{2}x + \frac{5}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \text{The equation is } &2x - 3y = 2(1) - 3(-5) \\ \text{i.e., } &2x - 3y = 17 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{The line has slope } &\frac{-5 - (-3)}{-4 - 2} = \frac{-2}{-6} = \frac{1}{3} \quad \therefore \text{ its equation is } x - 3y = (2) - 3(-3) \\ \text{i.e., } &x - 3y = 11 \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad \text{If } y = -\frac{3}{2}x + 7, \quad \text{when } x = 0, \quad y = 7 \quad \text{i.e., the } y\text{-intercept is } 7 \\ \text{and when } y = 0, \quad \frac{3}{2}x = 7 \quad \therefore x = \frac{14}{3} \quad \text{i.e., the } x\text{-intercept is } \frac{14}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{If } 5x - 3y = 12, \quad \text{when } x = 0, \quad y = -4 \quad \text{i.e., the } y\text{-intercept is } -4 \\ \text{and when } y = 0, \quad 5x = 12 \quad \therefore x = \frac{12}{5} \quad \text{i.e., the } x\text{-intercept is } \frac{12}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \text{If } (2, -5) \text{ lies on } 3x + 4y &= -14, \\ \text{then } 3(2) + 4(-5) &= -14 \\ \therefore 6 - 20 &= -14 \quad \text{which is true} \end{aligned}$$

Hence $(2, -5)$ lies on $3x + 4y = -14$

- 7 a** The line $4x - 3y = 6$ has slope $\frac{4}{3}$.
 $\therefore$ any line parallel to it also has slope $\frac{4}{3}$.
 But the line must pass through $P(1, -5)$.
 $\therefore$ its equation is $4x - 3y = 4(1) - 3(-5)$
 i.e., $4x - 3y = 19$
- b** The line $y = -4x + 7$ has slope -4 .
 $\therefore$ any line perpendicular to it has slope $\frac{1}{4}$.
 But the line must pass through $Q(-2, 1)$.
 $\therefore$ its equation is $x - 4y = (-2) - 4(1)$
 i.e., $x - 4y = -6$

- 8 a i** AE has slope $\frac{2-2}{7-0} = 0$ so its equation is $y = 2$.
ii CE is perpendicular to AE, so its equation is $x = 0$.
iii BX has slope $\frac{7-10}{7-3} = -\frac{3}{4}$ $\therefore$ Diagonal Rd has slope $\frac{4}{3}$
 $\therefore$ since it passes through $(3, 10)$, its equation is $4x - 3y = 4(3) - 3(10)$
 i.e., $4x - 3y = -18$
- b i** Diagonal Rd intersects East Ave when $y = 2$ $\therefore 4x - 6 = -18$
 $\therefore 4x = -12$
 $\therefore x = -3$ i.e., at $(-3, 2)$
- ii** Diagonal Rd intersects North St when $x = 0$ $\therefore 0 - 3y = -18$
 $\therefore y = 6$ i.e., at $(0, 6)$

- 9** $3x + ky = 7$ has slope $-\frac{3}{k}$ $y = 3 - 4x$ has slope -4

- a** The lines are parallel if the slopes are equal $\therefore -\frac{3}{k} = -4$ $\therefore k = \frac{3}{4}$
- b** The lines are perpendicular if the product of their slopes is -1 ,

$$\therefore \left(-\frac{3}{k}\right)(-4) = -1 \quad \therefore \frac{12}{k} = -1 \quad \therefore k = -12$$

- 10** The distance from $(0, 4)$ to (x, y) is three times the distance from $(4, 0)$ to (x, y)

$$\therefore \sqrt{(x-0)^2 + (y-4)^2} = 3\sqrt{(x-4)^2 + (y-0)^2}$$

$$\therefore (x-0)^2 + (y-4)^2 = 9((x-4)^2 + (y-0)^2)$$

$$\therefore x^2 + y^2 - 8y + 16 = 9x^2 - 72x + 144 + 9y^2$$

$$\therefore 8x^2 + 8y^2 - 72x + 8y + 128 = 0$$

$$\therefore x^2 + y^2 - 9x + y + 16 = 0$$

REVIEW SET 7B

- 1 a** $3x + 5y = 7$ has slope $-\frac{3}{5}$
 $\therefore$ any line parallel to it must also have slope $-\frac{3}{5}$
 But the line required passes through $(-1, 3)$
 $\therefore$ its equation is $3x + 5y = 3(-1) + 5(3)$ i.e., $3x + 5y = 12$
- b** $2x - 7y = 5$ has slope $\frac{2}{7}$
 $\therefore$ any line perpendicular to it must have slope $-\frac{7}{2}$
 But the line required passes through $(4, 2)$ $\therefore$ its equation is $7x + 2y = 7(4) + 2(2)$
 i.e., $7x + 2y = 32$

$$\mathbf{2} \quad 5x - 7y = 8 \quad \text{has slope } \frac{5}{7} \quad 3x + ky = -11 \quad \text{has slope } -\frac{3}{k}$$

$$\mathbf{a} \quad \text{If the lines are parallel, their slopes are equal} \quad \therefore \frac{5}{7} = -\frac{3}{k}$$

$$\therefore k = -\frac{21}{5}$$

b If the lines are perpendicular, the product of their slopes is -1 .

$$\therefore \frac{5}{7} \left(-\frac{3}{k} \right) = -1 \quad \therefore -\frac{15}{7k} = -1$$

$$\therefore 7k = 15$$

$$\therefore k = \frac{15}{7}$$

$$\mathbf{3} \quad \text{AB has slope } \frac{7-2}{1-(-3)} = \frac{5}{4} \quad \therefore \text{its equation is } \frac{y-7}{x-1} = \frac{5}{4} \quad \therefore y-7 = \frac{5}{4}(x-1)$$

$$\therefore y-7 = \frac{5}{4}x - \frac{5}{4}$$

$$\therefore y = \frac{5}{4}x + \frac{23}{4}$$

$$\therefore \text{this line meets } 3x + 2y = 6 \quad \text{when} \quad 3x + 2 \left(\frac{5}{4}x + \frac{23}{4} \right) = 6$$

$$\therefore 3x + \frac{5}{2}x + \frac{23}{2} = 6$$

$$\therefore \frac{11}{2}x = -\frac{11}{2}$$

$$\therefore x = -1$$

$$\therefore y = \frac{5}{4}(-1) + \frac{23}{4} = \frac{18}{4} = \frac{9}{2}$$

$\therefore$ the lines intersect at $(-1, \frac{9}{2})$

$$\mathbf{4} \quad \text{KL} = \sqrt{(0 - (-5))^2 + (1 - (-2))^2} \quad \text{KM} = \sqrt{(3 - (-5))^2 + (-4 - (-2))^2}$$

$$= \sqrt{25 + 9} \quad = \sqrt{64 + 4}$$

$$= \sqrt{34} \text{ units} \quad = \sqrt{68} \text{ units}$$

$$\text{LM} = \sqrt{(3 - 0)^2 + (-4 - 1)^2}$$

$$= \sqrt{9 + 25}$$

$$= \sqrt{34} \text{ units}$$

$\therefore$ since $\text{LM} = \text{KL}$, the triangle is isosceles
and since $\text{KL}^2 + \text{LM}^2 = \text{KM}^2$, the triangle
is also right angled at L.

i.e., KLM is a right angled isosceles triangle,
with the right angle being at L.

5 Suppose the power outlet is at $P(a, 5)$.

$$\text{Now } \text{AP} = \text{BP} \quad \therefore \sqrt{(a-3)^2 + (5-2)^2} = \sqrt{(a-5)^2 + (5-7)^2}$$

$$\therefore (a-3)^2 + (5-2)^2 = (a-5)^2 + (5-7)^2$$

$$\therefore a^2 - 6a + 9 + 9 = a^2 - 10a + 25 + 4$$

$$\therefore 4a = 11$$

$$\therefore a = \frac{11}{4} \quad \therefore \text{the point is } \left(\frac{11}{4}, 5 \right)$$

$$\mathbf{6} \quad \text{Now } \sqrt{(2 - (-3))^2 + (4 - k)^2} = 7 \quad \therefore 25 + (4 - k)^2 = 49$$

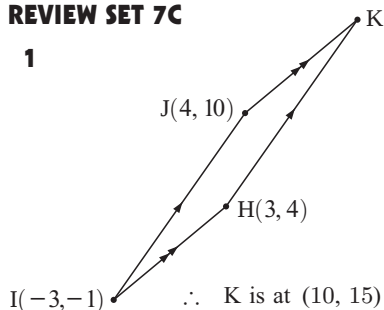
$$\therefore (4 - k)^2 = 24$$

$$\therefore 4 - k = \pm\sqrt{24}$$

$$\text{i.e., } 4 - k = 2\sqrt{6} \quad \text{or} \quad 4 - k = -2\sqrt{6}$$

$$\therefore k = 4 - 2\sqrt{6} \quad \text{or} \quad k = 4 + 2\sqrt{6}$$

$$\therefore k = 4 \pm 2\sqrt{6}$$

REVIEW SET 7C**1**

Since HIJK is a parallelogram, the diagonals bisect each other.

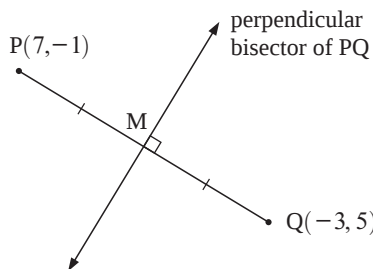
$\therefore$ the midpoint of IK is the same as the midpoint of HJ.

Letting K be at (a, b) ,

$$\frac{a-3}{2} = \frac{3+4}{2} \quad \text{and} \quad \frac{b-1}{2} = \frac{4+10}{2}$$

$$\therefore a-3 = 7 \quad \text{and} \quad b-1 = 14$$

$$\therefore a = 10 \quad \text{and} \quad b = 15$$

2

The midpoint M of PQ is at

$$\left(\frac{7-3}{2}, \frac{-1+5}{2} \right) \quad \text{i.e., at } (2, 2)$$

$$\text{The slope of PQ is } \frac{5-(-1)}{-3-7} = \frac{6}{-10} = -\frac{3}{5}$$

$$\therefore \text{the slope of the perpendicular is } \frac{5}{3}$$

$\therefore$ the equation of the perpendicular bisector is

$$5x - 3y = 5(2) - 3(2)$$

$$\text{i.e., } 5x - 3y = 4$$

3 If the three points are collinear, then the slope of AB equals the slope of AC.

$$\therefore \frac{8-2}{6-(-3)} = \frac{k-2}{2-(-3)} \quad \text{and so} \quad \frac{6}{9} = \frac{k-2}{5}$$

$$\therefore k-2 = \frac{30}{9} = \frac{10}{3}$$

$$\therefore k = \frac{16}{3}$$

4 Since AB is a diameter, the centre of the circle is the midpoint of AB.

$$\therefore \text{if A has coordinates } (a, b), \text{ then} \quad \frac{a+6}{2} = -3 \quad \text{and} \quad \frac{b+1}{2} = 4$$

$$\therefore a+6 = -6 \quad \text{and} \quad b+1 = 8$$

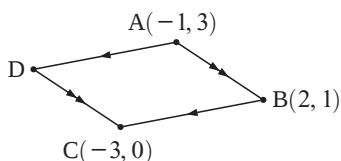
$$\therefore a = -12 \quad \text{and} \quad b = 7$$

$$\therefore A \text{ is at } (-12, 7)$$

5 a AB has slope $= \frac{1-3}{2-(-1)} = -\frac{2}{3}$

$$\therefore \text{the line through C must also have slope } -\frac{2}{3}$$

$$\therefore \text{its equation is } 2x + 3y = 2(-3) + 3(0) \quad \text{i.e., } 2x + 3y = -6$$

b

Since ABCD is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of AC is the same as the midpoint of BD.

Letting D be at (a, b) ,

$$\frac{-1-3}{2} = \frac{2+a}{2} \quad \text{and} \quad \frac{3+0}{2} = \frac{1+b}{2}$$

$$\therefore -4 = a+2 \quad \text{and} \quad 3 = b+1$$

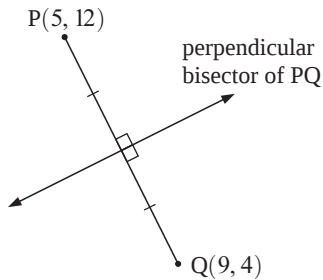
$$\therefore a = -6 \quad \text{and} \quad b = 2 \quad \therefore D \text{ is at } (-6, 2)$$

- 6** Suppose A has x -coordinate a , so its y -coordinate must be $a - 1$.

$$\begin{aligned}\text{Since it is } 2\sqrt{2} \text{ units from B(4, -1), } \quad & \sqrt{(a-4)^2 + ((a-1) - (-1))^2} = 2\sqrt{2} \\ \therefore \quad & \sqrt{(a-4)^2 + a^2} = 2\sqrt{2} \\ \therefore \quad & a^2 - 8a + 16 + a^2 = 8 \\ \therefore \quad & 2a^2 - 8a + 8 = 0 \\ \therefore \quad & a^2 - 4a + 4 = 0 \\ \therefore \quad & (a-2)^2 = 0 \\ \therefore \quad & a = 2\end{aligned}$$

Hence A is at (2, 1).

7



The midpoint M of PQ is at $\left(\frac{5+9}{2}, \frac{12+4}{2}\right)$

i.e., at (7, 8)

The slope of PQ is $\frac{4-12}{9-5} = \frac{-8}{4} = -2$

$\therefore$ the slope of the perpendicular bisector is $\frac{1}{2}$

$\therefore$ its equation is $x - 2y = (7) - 2(8)$

i.e., $x - 2y = -9$

Hence the equation of the boundary between the two regions is $x - 2y = -9$.

- 8 a** XY has slope $\frac{-3-5}{8-2} = \frac{-8}{6} = -\frac{4}{3}$

$\therefore$ its equation is $4x + 3y = 4(2) + 3(5)$

i.e., $4x + 3y = 23$ (1)

Now PT is perpendicular to XY, so its slope is $\frac{3}{4}$.

$\therefore$ its equation is $3x - 4y = 3(9) - 4(4)$

i.e., $3x - 4y = 11$ (2)

T lies at the intersection of (1) and (2).

$$(1) \times 4: \quad 16x + 12y = 92$$

$$(2) \times 3: \quad 9x - 12y = 33$$

$$\hline 25x = 125$$

$$\therefore x = 5$$

$\therefore$ using (1), $20 + 3y = 23$

$$\therefore 3y = 3$$

$$\therefore y = 1$$

Hence T is at (5, 1)

$$\begin{aligned}\mathbf{b} \quad PT &= \sqrt{(9-5)^2 + (4-1)^2} \\ &= \sqrt{16+9} \\ &= 5 \text{ units} \\ &\equiv 50 \text{ km}\end{aligned}$$

- 9 a** The radius equals the length of AB, so $\text{radius} = \sqrt{(-1-4)^2 + (2-(-2))^2}$
- $$= \sqrt{25+16}$$
- $$= \sqrt{41} \text{ units}$$

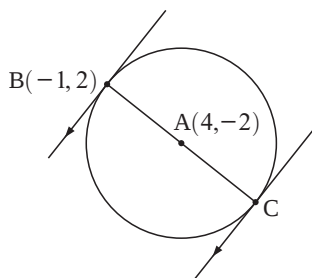
b slope of AB = $\frac{2-(-2)}{-1-4} = -\frac{4}{5}$

- c** Using **b**, the slope of the tangent at B is $\frac{5}{4}$.

$$\therefore \text{ its equation is } 5x - 4y = 5(-1) - 4(2)$$

$$\text{i.e., } 5x - 4y = -13$$

d



The other parallel tangent must pass through the point C at the other end of the diameter from B.

Now A is the midpoint of BC, so if C has coordinates (c, d) , then

$$\frac{c-1}{2} = 4 \quad \text{and} \quad \frac{d+2}{2} = -2$$

$$\therefore c-1 = 8 \quad \text{and} \quad d+2 = -4$$

$$\therefore c = 9 \quad \text{and} \quad d = -6$$

$$\therefore \text{ C is at } (9, -6)$$

$$\therefore \text{ since the tangent must have slope } \frac{5}{4}, \text{ its equation is } 5x - 4y = 5(9) - 4(-6)$$

$$\text{i.e., } 5x - 4y = 69$$

- 10** The distance from $(4, 3)$ to $(7, 6)$ is $\sqrt{(7-4)^2 + (6-3)^2}$
 $= \sqrt{9+9}$
 $\div 4.24 \text{ units}$
 $\div 84.9 \text{ km}$

Hence the radar will detect the other ship.

Chapter 8

QUADRATIC EQUATIONS AND FUNCTIONS

EXERCISE 8A

1 a $y = 3x^2 - 4x + 1$ is of the form $y = ax^2 + bx + c$, where $a = 3$, $b = -4$ and $c = 1$.
Hence it is a quadratic function.

b $y = 5x - 7$ is of the form $y = ax^2 + bx + c$, but requires $a = 0$.
Hence it is not a quadratic function.

c $y = -x^2$ is of the form $y = ax^2 + bx + c$, where $a = -1$, $b = 0$ and $c = 0$.
Hence it is a quadratic function.

d $y = \frac{2}{3}x^2 + 4$ is of the form $y = ax^2 + bx + c$, where $a = \frac{2}{3}$, $b = 0$ and $c = 4$.
Hence it is a quadratic function.

e If $2y + 3x^2 - 5 = 0$, then $2y = -3x^2 + 5 \quad \therefore y = -\frac{3}{2}x^2 + \frac{5}{2}$
This is of the form $f(x) = ax^2 + bx + c$, where $a = -\frac{3}{2}$, $b = 0$ and $c = \frac{5}{2}$.
Hence it is a quadratic function.

f $y = 5x^3 + x - 6$ contains the term x^3 , so is not of the form $y = ax^2 + bx + c$.
Hence it is not a quadratic function.

2 a When $x = 3$, $y = 3^2 + 5 \times 3 - 4$
 $= 9 + 15 - 4$
 $= 20$

b When $x = -3$, $y = 2 \times (-3)^2 + 9$
 $= 2 \times 9 + 9$
 $= 27$

c When $x = 1$, $y = -2 \times 1^2 + 3 \times 1 - 5$
 $= -2 + 3 - 5$
 $= -4$

d When $x = 4$, $y = 4 \times 4^2 - 7 \times 4 + 1$
 $= 64 - 28 + 1$
 $= 37$

3 a $f(2)$
 $= 2^2 - 2 \times 2 + 3$
 $= 4 - 4 + 3$
 $= 3$

b $f(-3)$
 $= 4 - (-3)^2$
 $= 4 - 9$
 $= -5$

c $f(0)$
 $= -\frac{1}{4} \times 0^2 + 3 \times 0 - 4$
 $= -4$

d $f(2)$
 $= \frac{1}{2} \times 2^2 + 3 \times 2$
 $= 2 + 6$
 $= 8$

4 a $f(x) = 5x^2 - 10$
 $\therefore f(0) = 5 \times 0 - 10$
 $= -10$
 $\therefore f(0) \neq 5$
 $\therefore$ the function is not satisfied by the ordered pair.

b $y = 2x^2 + 5x - 3$
when $x = 4$,
 $y = 2 \times 4^2 + 5 \times 4 - 3$
 $= 32 + 20 - 3$
 $= 49$
 $\therefore$ the function is not satisfied by the ordered pair.

c $y = -2x^2 + 3x$
when $x = -\frac{1}{2}$,
 $y = -2 \times (-\frac{1}{2})^2 + 3(-\frac{1}{2})$
 $= -2 \times \frac{1}{4} - \frac{3}{2}$
 $= -\frac{1}{2} - \frac{3}{2}$
 $= -2$

$\therefore$ the function is not satisfied by the ordered pair.

d $y = -7x^2 + 8x + 15$
when $x = -1$,
 $y = -7 \times (-1)^2 + 8 \times (-1) + 15$
 $= -7 - 8 + 15$
 $= 0$
 $\therefore$ the function is not satisfied by the ordered pair.

e $f(x) = 3x^2 - 13x + 4$
 $\therefore f(2) = 3 \times 2^2 - 13 \times 2 + 4$
 $= 12 - 26 + 4$
 $= -10$

$\therefore$ the function is satisfied by the ordered pair.

f $f(x) = -3x^2 + x + 2$
 $\therefore f(\frac{1}{3}) = -3 \times (\frac{1}{3})^2 + \frac{1}{3} + 2$
 $= -\frac{3}{9} + \frac{1}{3} + 2$
 $= 2$

$\therefore$ the function is satisfied by the ordered pair.

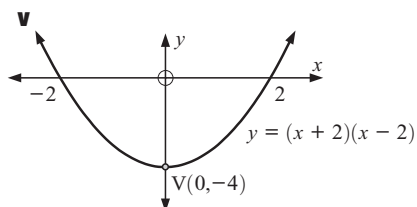
EXERCISE 8B.1

- 1 a i** $y = (x + 2)(x - 2)$ has x -intercepts -2 and 2 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = 0$.

iii When $x = 0$, $y = 2 \times (-2) = -4$
 $\therefore$ the vertex is at $(0, -4)$

iv From **iii**, when $x = 0$, $y = -4$
 $\therefore$ the y -intercept is -4 .

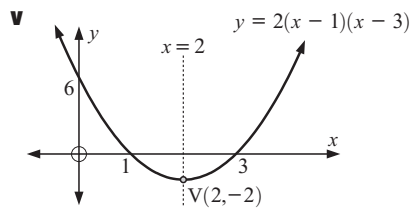


- b i** $y = 2(x - 1)(x - 3)$ has x -intercepts 1 and 3 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = 2 \times 1 \times (-1) = -2$
 $\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 2(-1)(-3) = 6$
 $\therefore$ the y -intercept is 6 .

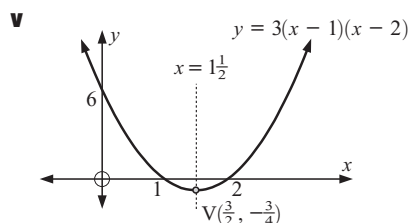


- c i** $y = 3(x - 1)(x - 2)$ has x -intercepts 1 and 2 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = \frac{3}{2}$.

iii When $x = \frac{3}{2}$, $y = 3 \times \frac{1}{2} \times (-\frac{1}{2}) = -\frac{3}{4}$
 $\therefore$ the vertex is at $(\frac{3}{2}, -\frac{3}{4})$

iv When $x = 0$, $y = 3(-1)(-2) = 6$
 $\therefore$ the y -intercept is 6 .

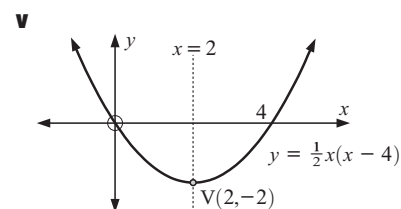


- d i** $y = \frac{1}{2}x(x - 4)$ has x -intercepts 0 and 4 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = \frac{1}{2} \times 2 \times (-2) = -2$
 $\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 0$ also.
 $\therefore$ the y -intercept is 0 .

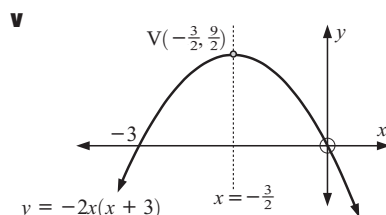


- e i** $y = -2x(x + 3)$ has x -intercepts 0 and -3 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = -\frac{3}{2}$.

iii When $x = -\frac{3}{2}$, $y = -2 \times (-\frac{3}{2}) \times (\frac{3}{2}) = \frac{9}{2}$
 $\therefore$ the vertex is at $(-\frac{3}{2}, \frac{9}{2})$

iv When $x = 0$, $y = 0$ also.
 $\therefore$ the y -intercept is 0 .

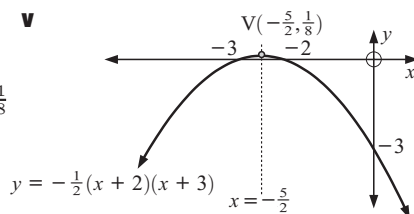


f i $y = -\frac{1}{2}(x+2)(x+3)$ has x -intercepts -2 and -3 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = -\frac{5}{2}$.

iii When $x = -\frac{5}{2}$, $y = -\frac{1}{2} \times (-\frac{1}{2}) \times (\frac{1}{2}) = \frac{1}{8}$
 $\therefore$ the vertex is at $(-\frac{5}{2}, \frac{1}{8})$

iv When $x = 0$, $y = -\frac{1}{2} \times 2 \times 3 = -3$
 $\therefore$ the y -intercept is -3 .



2 a $y = x(x-2)$ has x -intercepts 0 and 2 .

Its axis of symmetry is $x = 1$, so its vertex has $y = 1 \times (-1) = -1$.
Hence the graph is **B**.

b $y = 3x(x-2)$ also has x -intercepts 0 and 2 .

Its axis of symmetry is $x = 1$, so its vertex has $y = 3 \times 1 \times (-1) = -3$.
Hence the graph is **A**.

c $y = -x(x-2)$ also has x -intercepts 0 and 2 .

$\therefore$ its graph must be the remaining one like this, i.e., **F**.

Check: axis of symmetry is $x = 1$,

so its vertex has $y = -1 \times (-1) = 1$, which is true.

d $y = (x+2)(x-1)$ has x -intercepts -2 and 1 .

Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{4}$.
Hence the graph is **D**.

e $y = 2(x+2)(x-1)$ also has x -intercepts -2 and 1 .

Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = 2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{2}$.
Hence the graph is **E**.

f $y = -2(x+2)(x-1)$ also has x -intercepts -2 and 1 .

Its graph must be the only one remaining, i.e., **C**.

Check: axis of symmetry is $x = -\frac{1}{2}$,

so its vertex has $y = -2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = \frac{9}{2}$, which is true.

EXERCISE 8B.2

1 a i $y = (x-4)^2 + 3$ has axis of symmetry $x = 4$.

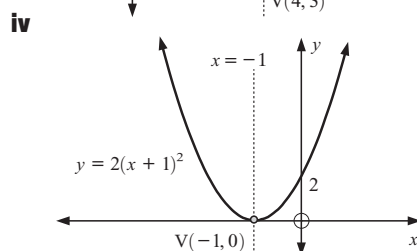
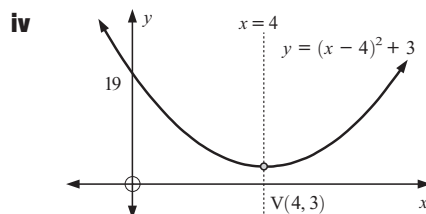
ii Its vertex is $(4, 3)$.

iii When $x = 0$, $y = (-4)^2 + 3 = 19$
 $\therefore$ the y -intercept is 19 .

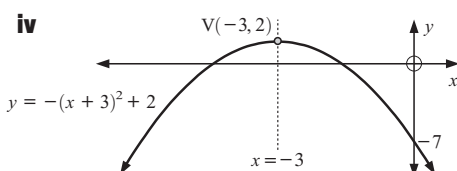
b i $y = 2(x+1)^2$ has axis of symmetry $x = -1$.

ii Its vertex is $(-1, 0)$.

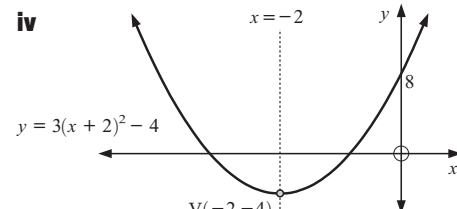
iii When $x = 0$, $y = 2 \times 1^2 = 2$
 $\therefore$ the y -intercept is 2 .



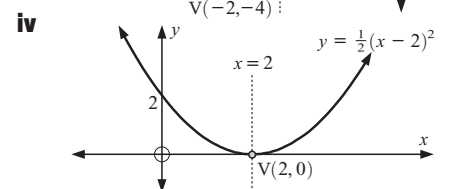
- c**
- i** $y = -(x+3)^2 + 2$ has axis of symmetry $x = -3$.
 - ii** Its vertex is $(-3, 2)$.
 - iii** When $x = 0$, $y = -3^2 + 2 = -7$
 $\therefore$ the y -intercept is -7 .



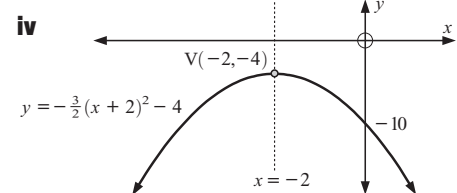
- d**
- i** $y = 3(x+2)^2 - 4$ has axis of symmetry $x = -2$.
 - ii** Its vertex is $(-2, -4)$.
 - iii** When $x = 0$, $y = 3 \times 2^2 - 4 = 8$
 $\therefore$ the y -intercept is 8 .



- e**
- i** $y = \frac{1}{2}(x-2)^2$ has axis of symmetry $x = 2$.
 - ii** Its vertex is $(2, 0)$.
 - iii** When $x = 0$, $y = \frac{1}{2} \times (-2)^2 = 2$
 $\therefore$ the y -intercept is 2 .



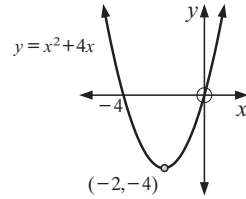
- f**
- i** $y = -\frac{3}{2}(x+2)^2 - 4$ has axis of symmetry $x = -2$.
 - ii** Its vertex is $(-2, -4)$.
 - iii** When $x = 0$, $y = -\frac{3}{2} \times 2^2 - 4 = -10$
 $\therefore$ the y -intercept is -10 .



- 2**
- a** $y = -(x+1)^2 + 3$ has vertex $(-1, 3)$, so its graph must be **G**.
 - b** $y = -2(x-3)^2 + 2$ has vertex $(3, 2)$, so its graph must be **A**.
 - c** $y = x^2 + 2$ has vertex $(0, 2)$, so its graph must be **E**.
 - d** $y = -(x-1)^2 + 1$ has vertex $(1, 1)$, so its graph must be **B**.
 - e** $y = (x-2)^2 - 2$ has vertex $(2, -2)$, so its graph must be **I**.
 - f** $y = \frac{1}{3}(x+3)^2 - 3$ has vertex $(-3, -3)$, so its graph must be **C**.
 - g** $y = -x^2$ has vertex $(0, 0)$, so its graph must be **D**.
 - h** $y = -\frac{1}{2}(x-1)^2 + 1$ has vertex $(1, 1)$, so its graph must be **F**.
 - i** $y = 2(x+2)^2 - 1$ has vertex $(-2, -1)$, so its graph must be **H**.

- 3**
- a** The graph has x -intercepts 0 and 4 ,
and its axis of symmetry is midway between them, i.e., $x = 2$.
 - b** The graph has x -intercepts -5 and 0 ,
and its axis of symmetry is midway between them, i.e., $x = -\frac{5}{2}$.
 - c** The graph has x -intercepts -1 and 3 ,
and its axis of symmetry is midway between them, i.e., $x = 1$.
 - d** The graph touches the x -axis at its vertex $(3, 0)$
 $\therefore$ its axis of symmetry is $x = 3$.
 - e** The graph touches the x -axis at its vertex $(-4, 0)$
 $\therefore$ its axis of symmetry is $x = -4$.
 - f** Both $(-8, -5)$ and $(0, -5)$ lie on the graph. Since they have the same y -coordinate, the axis of symmetry must lie midway between them, i.e., $x = -4$.

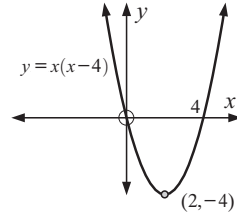
- 4 a i** $y = x^2 + 4x$ has x -intercepts -4 and 0 , and y -intercept 0 .



- ii** The axis of symmetry is midway between the x -intercepts, i.e., $x = -2$.

- iii** When $x = -2$, $y = (-2)^2 + 4(-2) = -4$
 $\therefore$ the vertex is $(-2, -4)$.

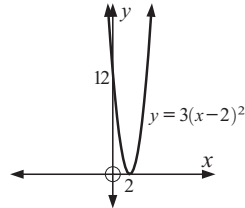
- b i** $y = x(x - 4)$ has x -intercepts 0 and 4 , and y -intercept 0 .



- ii** The axis of symmetry is midway between the x -intercepts, i.e., $x = 2$.

- iii** When $x = 2$, $y = 2(-2) = -4$
 $\therefore$ the vertex is $(2, -4)$.

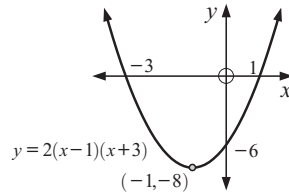
- c i** $y = 3(x - 2)^2$ has x -intercept 2 and y -intercept $= 3 \times (-2)^2 = 12$.



- ii** Since the vertex is when $x = 2$, $x = 2$ is the axis of symmetry.

- iii** The vertex is $(2, 0)$.

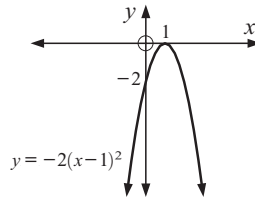
- d i** $y = 2(x - 1)(x + 3)$ has x -intercepts -3 and 1 , and y -intercept -6 .



- ii** The axis of symmetry is midway between the x -intercepts, i.e., $x = -1$.

- iii** When $x = -1$, $y = 2(-2)(2) = -8$
 $\therefore$ the vertex is $(-1, -8)$.

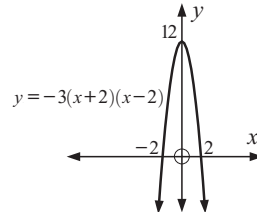
- e i** $y = -2(x - 1)^2$ has x -intercept 1 and y -intercept $= -2 \times (-1)^2 = -2$.



- ii** Since the vertex is when $x = 1$, $x = 1$ is the axis of symmetry.

- iii** The vertex is $(1, 0)$.

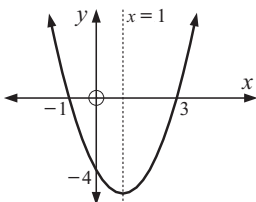
- f i** $y = -3(x + 2)(x - 2)$ has x -intercepts -2 and 2 , and y -intercept $= -3(2)(-2) = 12$.



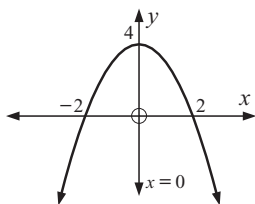
- ii** The axis of symmetry is midway between the x -intercepts, i.e., $x = 0$.

- iii** Since the y -intercept is 12 , the vertex is $(0, 12)$.

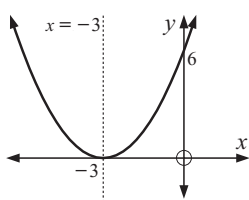
- 5 a i**



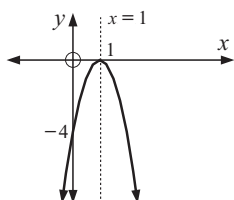
- ii** The axis of symmetry lies halfway between the x -intercepts, i.e., the axis of symmetry is $x = 1$.

b i

- ii** The axis of symmetry lies halfway between the x -intercepts,
i.e., the axis of symmetry is $x = 0$.

c i

- ii** Since the vertex is at $x = -3$,
the axis of symmetry is $x = -3$.

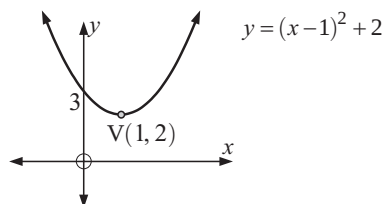
d i

- ii** Since the vertex is at $x = 1$,
the axis of symmetry is $x = 1$.

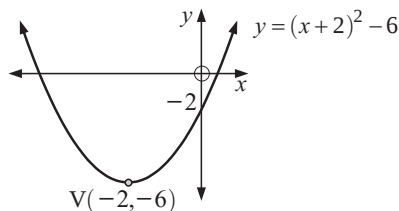
- 6 a** The axis of symmetry, $x = 4$, lies midway between the x -intercepts.
 $\therefore$ since one x -intercept is 2, the other is 6
 $\therefore$ the x -intercepts are 2 and 6.
- b** The axis of symmetry, $x = -3$, lies midway between the x -intercepts.
 $\therefore$ since one x -intercept is -1 , the other is -5
 $\therefore$ the x -intercepts are -1 and -5 .
- c** Since the graph touches the x -axis at 3 and it is a quadratic function,
the only x -intercept is 3 (touching).

EXERCISE 8C**1**

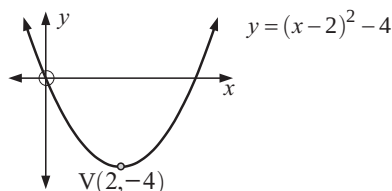
- a** $y = x^2 - 2x + 3$
 $\therefore y = x^2 - 2x + 1^2 + 3 - 1^2$
 $\therefore y = (x - 1)^2 + 2$
 $\therefore$ vertex is $(1, 2)$,
 y -intercept is 3



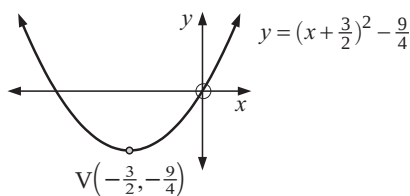
- b** $y = x^2 + 4x - 2$
 $\therefore y = x^2 + 4x + 2^2 - 2 - 2^2$
 $\therefore y = (x + 2)^2 - 6$
 $\therefore$ vertex is $(-2, -6)$,
 y -intercept is -2



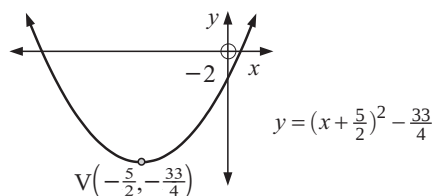
- c** $y = x^2 - 4x$
 $\therefore y = x^2 - 4x + 2^2 - 2^2$
 $\therefore y = (x - 2)^2 - 4$
 $\therefore$ vertex is $(2, -4)$,
 y -intercept is 0



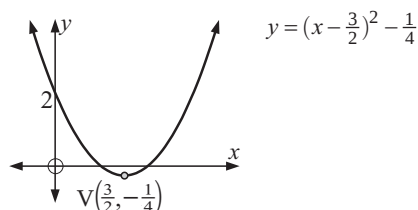
d $y = x^2 + 3x$
 $\therefore y = x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$
 $\therefore y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$
 $\therefore$ vertex is $\left(-\frac{3}{2}, -\frac{9}{4}\right)$,
 y -intercept is 0



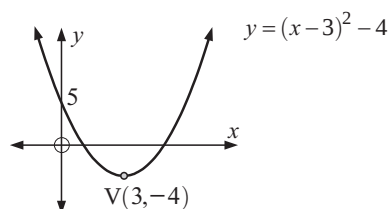
e $y = x^2 + 5x - 2$
 $\therefore y = x^2 + 5x + \left(\frac{5}{2}\right)^2 - 2 - \left(\frac{5}{2}\right)^2$
 $\therefore y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$
 $\therefore$ vertex is $\left(-\frac{5}{2}, -\frac{33}{4}\right)$,
 y -intercept is -2



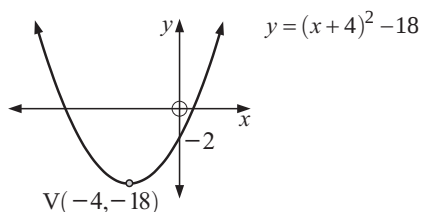
f $y = x^2 - 3x + 2$
 $\therefore y = x^2 - 3x + \left(\frac{3}{2}\right)^2 + 2 - \left(\frac{3}{2}\right)^2$
 $\therefore y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$
 $\therefore$ vertex is $\left(\frac{3}{2}, -\frac{1}{4}\right)$,
 y -intercept is 2



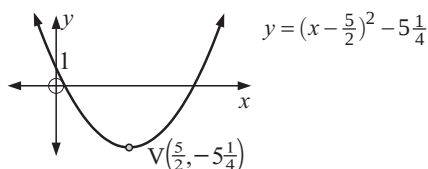
g $y = x^2 - 6x + 5$
 $\therefore y = x^2 - 6x + 3^2 + 5 - 3^2$
 $\therefore y = (x - 3)^2 - 4$
 $\therefore$ vertex is $(3, -4)$,
 y -intercept is 5



h $y = x^2 + 8x - 2$
 $\therefore y = x^2 + 8x + 4^2 - 2 - 4^2$
 $\therefore y = (x + 4)^2 - 18$
 $\therefore$ vertex is $(-4, -18)$,
 y -intercept is -2



i $y = x^2 - 5x + 1$
 $\therefore y = x^2 - 5x + \left(\frac{5}{2}\right)^2 + 1 - \left(\frac{5}{2}\right)^2$
 $\therefore y = \left(x - \frac{5}{2}\right)^2 - \frac{21}{4}$
 $\therefore$ vertex is $\left(\frac{5}{2}, -\frac{21}{4}\right)$,
 y -intercept is 1



2 a i $y = 2x^2 + 4x + 5$
 $= 2 \left[x^2 + 2x + \frac{5}{2} \right]$
 $= 2 \left[x^2 + 2x + 1^2 - 1^2 + \frac{5}{2} \right]$
 $= 2 \left[(x+1)^2 + \frac{3}{2} \right]$
 $= 2(x+1)^2 + 3$

ii The vertex is $(-1, 3)$.

b i $y = 2x^2 - 8x + 3$
 $= 2 \left[x^2 - 4x + \frac{3}{2} \right]$
 $= 2 \left[x^2 - 4x + 2^2 - 2^2 + \frac{3}{2} \right]$
 $= 2 \left[(x-2)^2 - \frac{5}{2} \right]$
 $= 2(x-2)^2 - 5$

ii The vertex is $(2, -5)$.

c i $y = 2x^2 - 6x + 1$
 $= 2 \left[x^2 - 3x + \frac{1}{2} \right]$
 $= 2 \left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{1}{2} \right]$
 $= 2 \left[\left(x - \frac{3}{2}\right)^2 - \frac{7}{4} \right]$
 $= 2\left(x - \frac{3}{2}\right)^2 - \frac{7}{2}$

ii The vertex is $\left(\frac{3}{2}, -\frac{7}{2}\right)$.

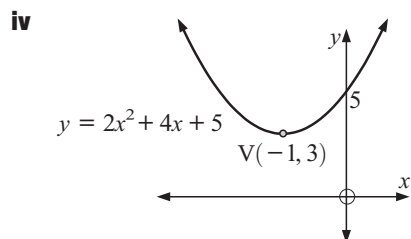
d i $y = 3x^2 - 6x + 5$
 $= 3 \left[x^2 - 2x + \frac{5}{3} \right]$
 $= 3 \left[x^2 - 2x + 1^2 - 1^2 + \frac{5}{3} \right]$
 $= 3 \left[(x-1)^2 + \frac{2}{3} \right]$
 $= 3(x-1)^2 + 2$

ii The vertex is $(1, 2)$.

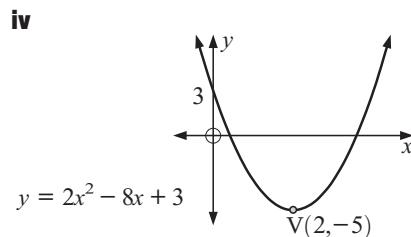
e i $y = -x^2 + 4x + 2$
 $= - \left[x^2 - 4x - 2 \right]$
 $= - \left[x^2 - 4x + 2^2 - 2^2 - 2 \right]$
 $= - \left[(x-2)^2 - 6 \right]$
 $= -(x-2)^2 + 6$

ii The vertex is $(2, 6)$.

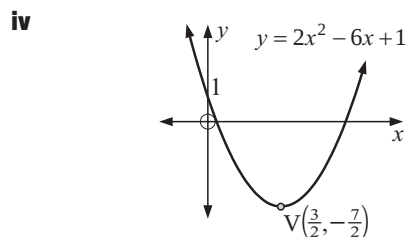
iii When $x = 0$, $y = 5$
 $\therefore$ the y -intercept is 5



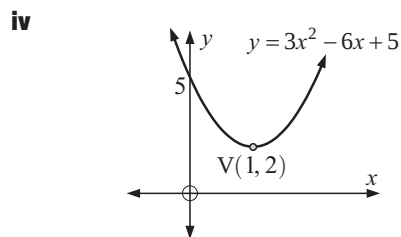
iii When $x = 0$, $y = 3$
 $\therefore$ the y -intercept is 3



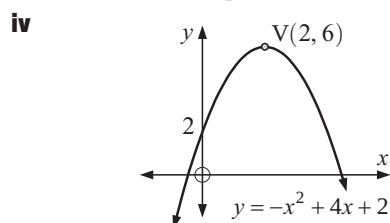
iii When $x = 0$, $y = 1$
 $\therefore$ the y -intercept is 1



iii When $x = 0$, $y = 5$
 $\therefore$ the y -intercept is 5



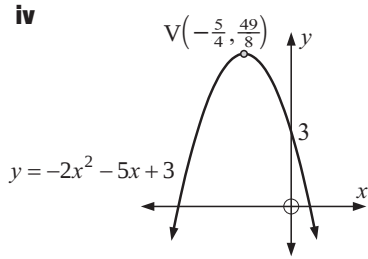
iii When $x = 0$, $y = 2$
 $\therefore$ the y -intercept is 2



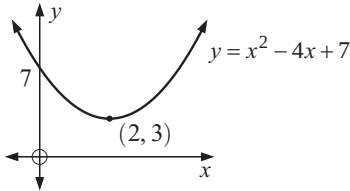
f i $y = -2x^2 - 5x + 3$
 $= -2 \left[x^2 + \frac{5}{2}x - \frac{3}{2} \right]$
 $= -2 \left[x^2 + \frac{5}{2}x + \left(\frac{5}{4} \right)^2 - \left(\frac{5}{4} \right)^2 - \frac{3}{2} \right]$
 $= -2 \left[\left(x + \frac{5}{4} \right)^2 - \frac{25}{16} - \frac{24}{16} \right]$
 $= -2 \left[\left(x + \frac{5}{4} \right)^2 - \frac{49}{16} \right]$
 $= -2 \left(x + \frac{5}{4} \right)^2 + \frac{49}{8}$

ii The vertex is $\left(-\frac{5}{4}, \frac{49}{8} \right)$.

iii When $x = 0$, $y = 3$
 $\therefore$ the y -intercept is 3



3 a Using technology, the graph is



Since the vertex is at $(2, 3)$,
the function must be of the form
 $y = a(x - 2)^2 + 3$ for some a .

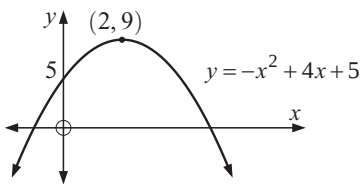
$\therefore$ when $x = 0$,
 $y = a(-2)^2 + 3 = 4a + 3$
but the y -intercept is 7

$$\therefore 4a + 3 = 7$$

$$\therefore a = 1$$

$\therefore$ the equation is $y = (x - 2)^2 + 3$

c Using technology, the graph is



Since the vertex is at $(2, 9)$,
the function must be of the form
 $y = a(x - 2)^2 + 9$ for some a .

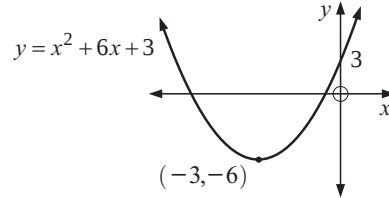
$\therefore$ when $x = 0$,
 $y = a(-2)^2 + 9 = 4a + 9$
but the y -intercept is 5

$$\therefore 4a + 9 = 5$$

$$\therefore a = -1$$

$\therefore$ the equation is $y = -(x - 2)^2 + 9$

b Using technology, the graph is



Since the vertex is at $(-3, -6)$,
the function must be of the form
 $y = a(x + 3)^2 - 6$ for some a .

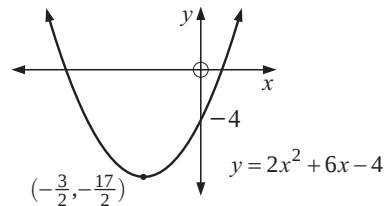
$\therefore$ when $x = 0$,
 $y = a \times 3^2 - 6 = 9a - 6$
but the y -intercept is 3

$$\therefore 9a - 6 = 3$$

$$\therefore a = 1$$

$\therefore$ the equation is $y = (x + 3)^2 - 6$

d Using technology, the graph is



Since the vertex is at $\left(-\frac{3}{2}, -\frac{17}{2} \right)$,
the function must be of the form
 $y = a \left(x + \frac{3}{2} \right)^2 - \frac{17}{2}$ for some a .

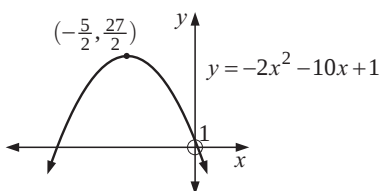
$\therefore$ when $x = 0$,
 $y = a \left(\frac{3}{2} \right)^2 - \frac{17}{2} = \frac{9}{4}a - \frac{17}{2}$
but the y -intercept is -4

$$\therefore \frac{9}{4}a - \frac{17}{2} = -4$$

$$\therefore \frac{9}{4}a = \frac{9}{2}$$

$$\therefore a = \frac{4}{9} \times \frac{9}{2} = 2$$

$\therefore$ the equation is $y = 2 \left(x + \frac{3}{2} \right)^2 - \frac{17}{2}$

e Using technology, the graph isSince the vertex is at $(-\frac{5}{2}, \frac{27}{2})$,

the function must be of the form

$$y = a(x + \frac{5}{2})^2 + \frac{27}{2} \quad \text{for some } a.$$

 $\therefore$ when $x = 0$,

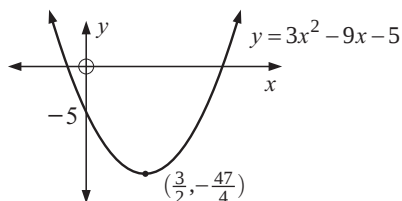
$$y = a(\frac{5}{2})^2 + \frac{27}{2} = \frac{25}{4}a + \frac{27}{2}$$

but the y -intercept is 1

$$\therefore \frac{25}{4}a + \frac{27}{2} = 1$$

$$\therefore \frac{25}{4}a = -\frac{25}{2}$$

$$\therefore a = -\frac{25}{2} \times \frac{4}{25} = -2$$

 $\therefore$ the equation is $y = -2(x + \frac{5}{2})^2 + \frac{27}{2}$ **f** Using technology, the graph isSince the vertex is at $(\frac{3}{2}, -\frac{47}{4})$,

the function must be of the form

$$y = a(x - \frac{3}{2})^2 - \frac{47}{4} \quad \text{for some } a.$$

 $\therefore$ when $x = 0$,

$$y = a(-\frac{3}{2})^2 - \frac{47}{4} = \frac{9}{4}a - \frac{47}{4}$$

but the y -intercept is -5

$$\therefore \frac{9}{4}a - \frac{47}{4} = -5$$

$$\therefore \frac{9}{4}a = \frac{27}{4}$$

$$\therefore a = 3$$

 $\therefore$ the equation is $y = 3(x - \frac{3}{2})^2 - \frac{47}{4}$ **EXERCISE 8D.1**

1 a $4x^2 + 7x = 0$

$\therefore x(4x + 7) = 0$

$\therefore x = 0 \text{ or } 4x + 7 = 0$

{Null Factor law}

$\therefore x = 0 \text{ or } -\frac{7}{4}$

b $6x^2 + 2x = 0$

$\therefore 2x(3x + 1) = 0$

$\therefore x = 0 \text{ or } 3x + 1 = 0$

{Null Factor law}

$\therefore x = 0 \text{ or } -\frac{1}{3}$

c $3x^2 - 7x = 0$

$\therefore x(3x - 7) = 0$

$\therefore x = 0 \text{ or } 3x - 7 = 0$

{Null Factor law}

$\therefore x = 0 \text{ or } \frac{7}{3}$

d $2x^2 - 11x = 0$

$\therefore x(2x - 11) = 0$

$\therefore x = 0 \text{ or } 2x - 11 = 0$

{Null Factor law}

$\therefore x = 0 \text{ or } \frac{11}{2}$

e $3x^2 = 8x$

$\therefore 3x^2 - 8x = 0$

$\therefore x(3x - 8) = 0$

$\therefore x = 0 \text{ or } 3x - 8 = 0$

{Null Factor law}

$\therefore x = 0 \text{ or } \frac{8}{3}$

f $9x = 6x^2$

$\therefore 6x^2 - 9x = 0$

$\therefore 3x(2x - 3) = 0$

$\therefore x = 0 \text{ or } 2x - 3 = 0$

{Null Factor law}

$\therefore x = 0 \text{ or } \frac{3}{2}$

g $x^2 - 5x + 6 = 0$

$\therefore (x - 2)(x - 3) = 0$

$\therefore x - 2 = 0 \text{ or } x - 3 = 0$

{Null Factor law}

$\therefore x = 2 \text{ or } 3$

h $x^2 = 2x + 8$

$\therefore x^2 - 2x - 8 = 0$

$\therefore (x - 4)(x + 2) = 0$

$\therefore x - 4 = 0 \text{ or } x + 2 = 0$

{Null Factor law}

$\therefore x = -2 \text{ or } 4$

i $x^2 + 21 = 10x$

$\therefore x^2 - 10x + 21 = 0$

$\therefore (x - 3)(x - 7) = 0$

$\therefore x - 3 = 0 \text{ or } x - 7 = 0$

{Null Factor law}

$\therefore x = 3 \text{ or } 7$

j $9 + x^2 = 6x$

$\therefore x^2 - 6x + 9 = 0$

$\therefore (x - 3)^2 = 0$

$\therefore x - 3 = 0$

$\therefore x = 3$

k $x^2 + x = 12$

$\therefore x^2 + x - 12 = 0$

$\therefore (x + 4)(x - 3) = 0$

$\therefore x + 4 = 0 \text{ or } x - 3 = 0$

{Null Factor law}

$\therefore x = -4 \text{ or } 3$

l $x^2 + 8x = 33$

$\therefore x^2 + 8x - 33 = 0$

$\therefore (x + 11)(x - 3) = 0$

$\therefore x + 11 = 0 \text{ or } x - 3 = 0$

{Null Factor law}

$\therefore x = -11 \text{ or } 3$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad 9x^2 - 12x + 4 &= 0 \\ \therefore (3x - 2)^2 &= 0 \\ \therefore x &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2x^2 - 13x - 7 &= 0 \\ \therefore (2x + 1)(x - 7) &= 0 \\ \therefore x &= -\frac{1}{2} \text{ or } 7 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 3x^2 &= 16x + 12 \\ \therefore 3x^2 - 16x - 12 &= 0 \\ \therefore (3x + 2)(x - 6) &= 0 \\ \therefore x &= -\frac{2}{3} \text{ or } 6 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 3x^2 + 5x &= 2 \\ \therefore 3x^2 + 5x - 2 &= 0 \\ \therefore (3x - 1)(x + 2) &= 0 \\ \therefore x &= \frac{1}{3} \text{ or } -2 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 2x^2 + 3 &= 5x \\ \therefore 2x^2 - 5x + 3 &= 0 \\ \therefore (2x - 3)(x - 1) &= 0 \\ \therefore x &= \frac{3}{2} \text{ or } 1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad 3x^2 &= 4x + 4 \\ \therefore 3x^2 - 4x - 4 &= 0 \\ \therefore (3x + 2)(x - 2) &= 0 \\ \therefore x &= -\frac{2}{3} \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad 3x^2 &= 10x + 8 \\ \therefore 3x^2 - 10x - 8 &= 0 \\ \therefore (3x + 2)(x - 4) &= 0 \\ \therefore x &= -\frac{2}{3} \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad 4x^2 + 4x &= 3 \\ \therefore 4x^2 + 4x - 3 &= 0 \\ \therefore (2x + 3)(2x - 1) &= 0 \\ \therefore x &= -\frac{3}{2} \text{ or } \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad 4x^2 &= 11x + 3 \\ \therefore 4x^2 - 11x - 3 &= 0 \\ \therefore (4x + 1)(x - 3) &= 0 \\ \therefore x &= -\frac{1}{4} \text{ or } 3 \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad 12x^2 &= 11x + 15 \\ \therefore 12x^2 - 11x - 15 &= 0 \\ \therefore (4x + 3)(3x - 5) &= 0 \\ \therefore x &= -\frac{3}{4} \text{ or } \frac{5}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad 7x^2 + 6x &= 1 \\ \therefore 7x^2 + 6x - 1 &= 0 \\ \therefore (7x - 1)(x + 1) &= 0 \\ \therefore x &= \frac{1}{7} \text{ or } -1 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad 15x^2 + 2x &= 56 \\ \therefore 15x^2 + 2x - 56 &= 0 \\ \therefore (15x - 28)(x + 2) &= 0 \\ \therefore x &= \frac{28}{15} \text{ or } -2 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad (x + 1)^2 &= 2x^2 - 5x + 11 \\ \therefore x^2 + 2x + 1 &= 2x^2 - 5x + 11 \\ \therefore x^2 - 7x + 10 &= 0 \\ \therefore (x - 2)(x - 5) &= 0 \\ \therefore x &= 2 \text{ or } 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (x + 2)(1 - x) &= -4 \\ \therefore x - x^2 + 2 - 2x &= -4 \\ \therefore x^2 + x - 6 &= 0 \\ \therefore (x + 3)(x - 2) &= 0 \\ \therefore x &= -3 \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 5 - 4x^2 &= 3(2x + 1) + 2 \\ \therefore 5 - 4x^2 &= 6x + 3 + 2 \\ \therefore 4x^2 + 6x &= 0 \\ \therefore 2x(2x + 3) &= 0 \\ \therefore x &= 0 \text{ or } -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad x + \frac{2}{x} &= 3 \\ \therefore x^2 + 2 &= 3x \\ \therefore x^2 - 3x + 2 &= 0 \\ \therefore (x - 1)(x - 2) &= 0 \\ \therefore x &= 1 \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 2x - \frac{1}{x} &= -1 \\ \therefore 2x^2 - 1 &= -x \\ \therefore 2x^2 + x - 1 &= 0 \\ \therefore (2x - 1)(x + 1) &= 0 \\ \therefore x &= \frac{1}{2} \text{ or } -1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{x + 3}{1 - x} &= -\frac{9}{x} \\ \therefore x(x + 3) &= -9(1 - x) \\ \therefore x^2 + 3x &= -9 + 9x \\ \therefore x^2 - 6x + 9 &= 0 \\ \therefore (x - 3)^2 &= 0 \\ \therefore x &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \text{If } y = 1, \text{ then } x^2 + 6x + 10 &= 1 \\ \therefore x^2 + 6x + 9 &= 0 \\ \therefore (x + 3)^2 &= 0 \\ \therefore x &= -3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{If } y = 2, \text{ then } x^2 + 5x + 8 &= 2 \\ \therefore x^2 + 5x + 6 &= 0 \\ \therefore (x + 3)(x + 2) &= 0 \\ \therefore x &= -3 \text{ or } -2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{If } y = -3, \text{ then } x^2 - 5x + 1 &= -3 \\ \therefore x^2 - 5x + 4 &= 0 \\ \therefore (x - 4)(x - 1) &= 0 \\ \therefore x &= 4 \text{ or } 1 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \text{If } y = -3, \text{ then } 3x^2 &= -3 \\ \therefore x^2 &= -1 \\ \therefore \text{there is no solution} \end{aligned}$$

5 a $f(x) = 5$

$$\therefore 3x^2 - 2x + 5 = 5$$

$$\therefore 3x^2 - 2x = 0$$

$$\therefore x(3x - 2) = 0$$

$$\therefore x = 0 \text{ or } \frac{2}{3}$$

c $f(x) = -4$

$$\therefore -2x^2 - 13x + 3 = -4$$

$$\therefore 2x^2 + 13x - 7 = 0$$

$$\therefore (2x - 1)(x + 7) = 0$$

$$\therefore x = \frac{1}{2} \text{ or } -7$$

b $f(x) = 1$

$$\therefore x^2 - x - 5 = 1$$

$$\therefore x^2 - x - 6 = 0$$

$$\therefore (x + 2)(x - 3) = 0$$

$$\therefore x = -2 \text{ or } 3$$

d $f(x) = -17$

$$\therefore 2x^2 - 12x + 1 = -17$$

$$\therefore 2x^2 - 12x + 18 = 0$$

$$\therefore 2(x^2 - 6x + 9) = 0$$

$$\therefore 2(x - 3)^2 = 0$$

$$\therefore x = 3$$

6 a i $h(1) = 30 - 5 = 25 \text{ m}$ **ii** $h(5) = 30 \times 5 - 5 \times 5^2 = 150 - 125 = 25 \text{ m}$ **iii** $h(3) = 30 \times 3 - 5 \times 3^2 = 90 - 45 = 45 \text{ m}$

b i When $h = 40 \text{ m}$,

$$30t - 5t^2 = 40$$

$$\therefore 5t^2 - 30t + 40 = 0$$

$$\therefore 5(t^2 - 6t + 8) = 0$$

$$\therefore 5(t - 4)(t - 2) = 0$$

$$\therefore t = 4 \text{ or } 2 \text{ seconds}$$

ii When $h = 0 \text{ m}$,

$$30t - 5t^2 = 0$$

$$\therefore 5t(6 - t) = 0$$

$$\therefore t = 0 \text{ or } 6 \text{ seconds}$$

c We get two answers in each case because the object is projected up, and gravity brings it back down.

7 a i $P(0) = -\frac{1}{4} \times 0 + 16 \times 0 - 30 = -\30 **b** If $P(x) = 57$, then $-\frac{1}{4}x^2 + 16x - 30 = 57$

$$\therefore \frac{1}{4}x^2 - 16x + 87 = 0$$

ii $P(10) = -\frac{1}{4} \times 10^2 + 16 \times 10 - 30$

$$= -25 + 160 - 30$$

$$= \$105$$

$$\therefore x^2 - 64x + 348 = 0$$

$$\therefore (x - 6)(x - 58) = 0$$

$$\therefore x = 6 \text{ or } 58$$

i.e., either 6 or 58 cakes are made.

EXERCISE 8D.2

1 a $(x + 5)^2 = 2$

$$\therefore x + 5 = \pm\sqrt{2}$$

$$\therefore x = -5 \pm \sqrt{2}$$

b $(x + 6)^2 = 11$

$$\therefore x + 6 = \pm\sqrt{11}$$

$$\therefore x = -6 \pm \sqrt{11}$$

c $(x - 4)^2 = 8$

$$\therefore x - 4 = \pm\sqrt{8}$$

$$\therefore x = 4 \pm 2\sqrt{2}$$

d $(x - 8)^2 = 7$

$$\therefore x - 8 = \pm\sqrt{7}$$

$$\therefore x = 8 \pm \sqrt{7}$$

e $2(x + 3)^2 = 10$

$$\therefore (x + 3)^2 = 5$$

$$\therefore x + 3 = \pm\sqrt{5}$$

$$\therefore x = -3 \pm \sqrt{5}$$

f $3(x - 2)^2 = 18$

$$\therefore (x - 2)^2 = 6$$

$$\therefore x - 2 = \pm\sqrt{6}$$

$$\therefore x = 2 \pm \sqrt{6}$$

g $(x + 1)^2 + 1 = 11$

$$\therefore (x + 1)^2 = 10$$

$$\therefore x + 1 = \pm\sqrt{10}$$

$$\therefore x = -1 \pm \sqrt{10}$$

h $(2x + 1)^2 = 3$

$$\therefore 2x + 1 = \pm\sqrt{3}$$

$$\therefore 2x = -1 \pm \sqrt{3}$$

$$\therefore x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & x^2 - 4x + 1 = 0 \\
 & \therefore x^2 - 4x = -1 \\
 \therefore & x^2 - 4x + (-2)^2 = -1 + (-2)^2 \\
 & \therefore (x-2)^2 = 3 \\
 & \therefore x-2 = \pm\sqrt{3} \\
 & \therefore x = 2 \pm \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & x^2 - 14x + 46 = 0 \\
 & \therefore x^2 - 14x = -46 \\
 \therefore & x^2 - 14x + (-7)^2 = -46 + (-7)^2 \\
 & \therefore (x-7)^2 = 3 \\
 & \therefore x-7 = \pm\sqrt{3} \\
 & \therefore x = 7 \pm \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & x^2 + 6x + 7 = 0 \\
 & \therefore x^2 + 6x = -7 \\
 \therefore & x^2 + 6x + 3^2 = -7 + 3^2 \\
 & \therefore (x+3)^2 = 2 \\
 & \therefore x+3 = \pm\sqrt{2} \\
 & \therefore x = -3 \pm \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & x^2 + 6x = 2 \\
 \therefore & x^2 + 6x + 3^2 = 2 + 3^2 \\
 & \therefore (x+3)^2 = 11 \\
 & \therefore x+3 = \pm\sqrt{11} \\
 & \therefore x = -3 \pm \sqrt{11}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & x^2 + 6x = -11 \\
 \therefore & x^2 + 6x + 3^2 = -11 + 3^2 \\
 & \therefore (x+3)^2 = -2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & 2x^2 + 4x + 1 = 0 \\
 \therefore & x^2 + 2x + \frac{1}{2} = 0 \\
 & \therefore x^2 + 2x = -\frac{1}{2} \\
 \therefore & x^2 + 2x + 1^2 = -\frac{1}{2} + 1^2 \\
 & \therefore (x+1)^2 = \frac{1}{2} \\
 & \therefore x+1 = \pm\frac{1}{\sqrt{2}} \\
 & \therefore x = -1 \pm \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 3x^2 + 12x + 5 = 0 \\
 \therefore & x^2 + 4x + \frac{5}{3} = 0 \\
 & \therefore x^2 + 4x = -\frac{5}{3} \\
 \therefore & x^2 + 4x + 2^2 = -\frac{5}{3} + 2^2 \\
 & \therefore (x+2)^2 = \frac{7}{3} \\
 & \therefore x+2 = \pm\sqrt{\frac{7}{3}} \\
 & \therefore x = -2 \pm \sqrt{\frac{7}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & x^2 + 6x + 2 = 0 \\
 & \therefore x^2 + 6x = -2 \\
 \therefore & x^2 + 6x + 3^2 = -2 + 3^2 \\
 & \therefore (x+3)^2 = 7 \\
 & \therefore x+3 = \pm\sqrt{7} \\
 & \therefore x = -3 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & x^2 = 4x + 3 \\
 & \therefore x^2 - 4x = 3 \\
 \therefore & x^2 - 4x + (-2)^2 = 3 + (-2)^2 \\
 & \therefore (x-2)^2 = 7 \\
 & \therefore x-2 = \pm\sqrt{7} \\
 & \therefore x = 2 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & x^2 = 2x + 6 \\
 & \therefore x^2 - 2x = 6 \\
 \therefore & x^2 - 2x + (-1)^2 = 6 + (-1)^2 \\
 & \therefore (x-1)^2 = 7 \\
 & \therefore x-1 = \pm\sqrt{7} \\
 & \therefore x = 1 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & x^2 + 10 = 8x \\
 & \therefore x^2 - 8x = -10 \\
 \therefore & x^2 - 8x + (-4)^2 = -10 + (-4)^2 \\
 & \therefore (x-4)^2 = 6 \\
 & \therefore x-4 = \pm\sqrt{6} \\
 & \therefore x = 4 \pm \sqrt{6}
 \end{aligned}$$

$\therefore x$ has no real solutions, since the perfect square cannot be negative.

$$\begin{aligned}
 \mathbf{b} \quad & 2x^2 - 10x + 3 = 0 \\
 \therefore & x^2 - 5x + \frac{3}{2} = 0 \\
 & \therefore x^2 - 5x = -\frac{3}{2} \\
 \therefore & x^2 - 5x + \left(-\frac{5}{2}\right)^2 = -\frac{3}{2} + \left(-\frac{5}{2}\right)^2 \\
 & \therefore \left(x - \frac{5}{2}\right)^2 = -\frac{3}{2} + \frac{25}{4} \\
 & \therefore \left(x - \frac{5}{2}\right)^2 = \frac{19}{4} \\
 & \therefore x - \frac{5}{2} = \pm\frac{\sqrt{19}}{2} \\
 & \therefore x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 3x^2 = 6x + 4 \\
 & \therefore x^2 = 2x + \frac{4}{3} \\
 & \therefore x^2 - 2x = \frac{4}{3} \\
 \therefore & x^2 - 2x + (-1)^2 = \frac{4}{3} + (-1)^2 \\
 & \therefore (x-1)^2 = \frac{7}{3} \\
 & \therefore x-1 = \pm\sqrt{\frac{7}{3}} \\
 & \therefore x = 1 \pm \sqrt{\frac{7}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 5x^2 - 15x + 2 = 0 \\
 \therefore & x^2 - 3x + \frac{2}{5} = 0 \\
 \therefore & x^2 - 3x = -\frac{2}{5} \\
 \therefore & x^2 - 3x + \left(-\frac{3}{2}\right)^2 = -\frac{2}{5} + \left(-\frac{3}{2}\right)^2 \\
 \therefore & \left(x - \frac{3}{2}\right)^2 = -\frac{2}{5} + \frac{9}{4} = \frac{37}{20} \\
 \therefore & x - \frac{3}{2} = \pm\sqrt{\frac{37}{20}} \\
 \therefore & x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 4x^2 + 4x = 5 \\
 \therefore & x^2 + x = \frac{5}{4} \\
 \therefore & x^2 + x + \left(\frac{1}{2}\right)^2 = \frac{5}{4} + \left(\frac{1}{2}\right)^2 \\
 \therefore & \left(x + \frac{1}{2}\right)^2 = \frac{6}{4} \\
 \therefore & x + \frac{1}{2} = \pm\frac{\sqrt{6}}{2} \\
 \therefore & x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}
 \end{aligned}$$

EXERCISE 8E

$$\begin{aligned}
 1 \quad \text{a} \quad & x^2 - 4x - 3 = 0 \\
 \text{has } & a = 1, \quad b = -4, \quad c = -3 \\
 \therefore & x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)} \\
 & = \frac{4 \pm \sqrt{28}}{2} \\
 & = \frac{4 \pm 2\sqrt{7}}{2} \\
 & = 2 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & x^2 + 1 = 4x \\
 \therefore & x^2 - 4x + 1 = 0 \\
 \text{which has } & a = 1, \quad b = -4, \quad c = 1 \\
 \therefore & x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \\
 & = \frac{4 \pm \sqrt{12}}{2} \\
 & = \frac{4 \pm 2\sqrt{3}}{2} \\
 & = 2 \pm \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & x^2 - 4x + 2 = 0 \\
 \text{has } & a = 1, \quad b = -4, \quad c = 2 \\
 \therefore & x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\
 & = \frac{4 \pm \sqrt{8}}{2} \\
 & = \frac{4 \pm 2\sqrt{2}}{2} \\
 & = 2 \pm \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & x^2 - 2\sqrt{2}x + 2 = 0 \\
 \text{has } & a = 1, \quad b = -2\sqrt{2}, \quad c = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & x^2 + 6x + 7 = 0 \\
 \text{has } & a = 1, \quad b = 6, \quad c = 7 \\
 \therefore & x = \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} \\
 & = \frac{-6 \pm \sqrt{8}}{2} \\
 & = \frac{-6 \pm 2\sqrt{2}}{2} \\
 & = -3 \pm \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & x^2 + 4x = 1 \\
 \therefore & x^2 + 4x - 1 = 0 \\
 \text{which has } & a = 1, \quad b = 4, \quad c = -1 \\
 \therefore & x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)} \\
 & = \frac{-4 \pm \sqrt{20}}{2} \\
 & = \frac{-4 \pm 2\sqrt{5}}{2} \\
 & = -2 \pm \sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 2x^2 - 2x - 3 = 0 \\
 \text{has } & a = 2, \quad b = -2, \quad c = -3 \\
 \therefore & x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)} \\
 & = \frac{2 \pm \sqrt{28}}{4} \\
 & = \frac{2 \pm 2\sqrt{7}}{4} \\
 & = \frac{1}{2} \pm \frac{\sqrt{7}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore & x = \frac{-(-2\sqrt{2}) \pm \sqrt{(-2\sqrt{2})^2 - 4(1)(2)}}{2(1)} \\
 & = \frac{2\sqrt{2} \pm \sqrt{8-8}}{2} \\
 & = \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & (3x+1)^2 = -2x \\
 \therefore & 9x^2 + 6x + 1 = -2x \\
 \therefore & 9x^2 + 8x + 1 = 0 \\
 & \text{which has } a = 9, \quad b = 8, \quad c = 1 \\
 \therefore & x = \frac{-8 \pm \sqrt{8^2 - 4(9)(1)}}{2(9)} \\
 & = \frac{-8 \pm \sqrt{28}}{18} \\
 & = \frac{-8 \pm 2\sqrt{7}}{18} \quad \text{or} \quad -\frac{4}{9} \pm \frac{\sqrt{7}}{9}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & (x+2)(x-1) = 2-3x \\
 \therefore & x^2 - x + 2x - 2 = 2-3x \\
 \therefore & x^2 + 4x - 4 = 0 \\
 & \text{which has } a = 1, \quad b = 4, \quad c = -4 \\
 \therefore & x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-4)}}{2(1)} \\
 & = \frac{-4 \pm \sqrt{32}}{2} \\
 & = \frac{-4 \pm 4\sqrt{2}}{2} \\
 & = -2 \pm 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (x-2)^2 = 1+x \\
 \therefore & x^2 - 4x + 4 = 1+x \\
 \therefore & x^2 - 5x + 3 = 0 \\
 & \text{which has } a = 1, \quad b = -5, \quad c = 3 \\
 \therefore & x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} \\
 & = \frac{5 \pm \sqrt{25-12}}{2} \\
 & = \frac{5}{2} \pm \frac{\sqrt{13}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & x - \frac{1}{x} = 1 \\
 \therefore & x^2 - 1 = x \\
 \therefore & x^2 - x - 1 = 0 \\
 & \text{which has } a = 1, \quad b = -1, \quad c = -1 \\
 \therefore & x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\
 & = \frac{1 \pm \sqrt{1+4}}{2} \\
 & = \frac{1}{2} \pm \frac{\sqrt{5}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & (x+3)(2x+1) = 9 \\
 \therefore & 2x^2 + x + 6x + 3 = 9 \\
 \therefore & 2x^2 + 7x - 6 = 0 \\
 & \text{which has } a = 2, \quad b = 7, \quad c = -6 \\
 \therefore & x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-6)}}{2(2)} \\
 & = \frac{-7 \pm \sqrt{49+48}}{4} \\
 & = -\frac{7}{4} \pm \frac{\sqrt{97}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (2x+1)^2 = 3-x \\
 \therefore & 4x^2 + 4x + 1 = 3-x \\
 \therefore & 4x^2 + 5x - 2 = 0 \\
 & \text{which has } a = 4, \quad b = 5, \quad c = -2 \\
 \therefore & x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)} \\
 & = \frac{-5 \pm \sqrt{25+32}}{8} \\
 & = -\frac{5}{8} \pm \frac{\sqrt{57}}{8}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \frac{x-1}{2-x} = 2x+1 \\
 \therefore & x-1 = (2x+1)(2-x) \\
 \therefore & x-1 = 4x-2x^2+2-x \\
 \therefore & 2x^2-2x-3 = 0 \\
 & \text{which has } a = 2, \quad b = -2, \quad c = -3 \\
 \therefore & x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)} \\
 & = \frac{2 \pm \sqrt{28}}{4} \\
 & = \frac{2 \pm 2\sqrt{7}}{4} \quad \text{or} \quad \frac{1}{2} \pm \frac{\sqrt{7}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 2x - \frac{1}{x} = 3 \\
 \therefore & 2x^2 - 1 = 3x \\
 \therefore & 2x^2 - 3x - 1 = 0 \\
 & \text{which has } a = 2, \quad b = -3, \quad c = -1 \\
 \therefore & x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)} \\
 & = \frac{3 \pm \sqrt{9+8}}{4} \\
 & = \frac{3}{4} \pm \frac{\sqrt{17}}{4}
 \end{aligned}$$

EXERCISE 8F

- 1 **a** $x = -3.414$ or -0.586 **b** $x = 0.317$ or -6.317 **c** $x = 2.77$ or -1.27
 d $x = -1.08$ or 3.41 **e** $x = -0.892$ or 3.64 **f** $x = 1.34$ or -2.54
- 2 **a** $x = -4.83$ or 0.828 **b** $x = -1.57$ or 0.319 **c** $x = 0.697$ or 4.30
 d $x = -0.823$ or 1.82 **e** $x = -0.618$ or 1.62 **f** $x = -0.281$ or 1.78

EXERCISE 8G

- 1 Let the smaller of the integers be x .
 Since they differ by 12,
 the other integer is $(x + 12)$.
 $\therefore$ the sum of their squares is

$$x^2 + (x + 12)^2 = 74$$

$$\therefore x^2 + x^2 + 24x + 144 = 74$$

$$\therefore 2x^2 + 24x + 70 = 0$$

$$\therefore x^2 + 12x + 35 = 0$$

$$\therefore (x + 7)(x + 5) = 0$$

$$\therefore x = -7 \text{ or } -5$$
 $\therefore$ the larger integer is 5 or 7
 i.e., the integers are -7 and 5 , or -5 and 7
- 2 Let the number be x , so its reciprocal is $\frac{1}{x}$.
 They have sum $x + \frac{1}{x} = 5\frac{1}{5}$

$$\therefore x^2 + 1 = \frac{26}{5}x$$

$$\therefore x^2 - \frac{26}{5}x + 1 = 0$$

$$\therefore 5x^2 - 26x + 5 = 0$$

$$\therefore (5x - 1)(x - 5) = 0$$

$$\therefore x = \frac{1}{5} \text{ or } 5$$
 i.e., the number is either $\frac{1}{5}$ or 5
- 3 Let the number be x so its square is x^2 .
 $\therefore$ the sum is $x + x^2 = 210$

$$\therefore x^2 + x - 210 = 0$$

$$\therefore (x + 15)(x - 14) = 0$$

$$\therefore x = -15 \text{ or } 14$$
 But x is a natural number, so $x > 0$,
 $\therefore$ the number is 14.
- 4 Suppose the numbers are x and $(x + 2)$.
 Then $x(x + 2) = 360$

$$\therefore x^2 + 2x - 360 = 0$$

$$\therefore (x + 20)(x - 18) = 0$$

$$\therefore x = -20 \text{ or } 18$$
 $\therefore$ the numbers are -20 and -18 ,
 or 18 and 20 .
- 5 Suppose the numbers are x and $(x + 2)$.
 Then $x(x + 2) = 255$

$$\therefore x^2 + 2x - 255 = 0$$

$$\therefore (x + 17)(x - 15) = 0$$

$$\therefore x = -17 \text{ or } 15$$
 $\therefore$ the numbers are -17 and -15 ,
 or 15 and 17 .
- 6 If the polygon has n sides, then

$$\frac{n}{2}(n - 3) = 90$$

$$\therefore \frac{1}{2}n^2 - \frac{3}{2}n = 90$$

$$\therefore n^2 - 3n - 180 = 0$$

$$\therefore (n - 15)(n + 12) = 0$$

$$\therefore n = -12 \text{ or } 15$$
 $\therefore$ the polygon has 15 sides. {as $n > 0$ }
- 7 **a** The base has sides of length x cm, so
 the areas of the top and bottom surfaces
 are both x^2 cm².
 The box has height $(x + 1)$ cm,
 so the area of each of the side faces
 is $x(x + 1)$ cm.
 $\therefore$ the total surface area is

$$A = 2x^2 + 4x(x + 1)$$

$$= 2x^2 + 4x^2 + 4x$$

$$= 6x^2 + 4x \text{ cm}^2$$
- b** $6x^2 + 4x = 240$

$$\therefore 3x^2 + 2x - 120 = 0$$

$$\therefore (3x + 20)(x - 6) = 0$$

$$\therefore x = -\frac{20}{3} \text{ or } 6$$
 but $x > 0$, so $x = 6$ cm
 $\therefore$ the box is $6 \text{ cm} \times 6 \text{ cm} \times 7 \text{ cm}$

- 8** If the width of the rectangle is w cm, then its length is $(w + 4)$ cm.

$$\therefore \text{the area is } w(w + 4) = 26$$

$$\therefore w^2 + 4w - 26 = 0$$

$$\text{which has } a = 1, \quad b = 4, \quad c = -26$$

$$\therefore w = \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{120}}{2}$$

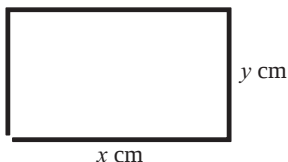
$$= \frac{-4 \pm 2\sqrt{30}}{2}$$

$$= -2 \pm \sqrt{30}$$

$$\text{But } w > 0, \text{ so } w = -2 + \sqrt{30} \\ \div 3.477 \text{ cm}$$

i.e., the width is approximately 3.48 cm.

10



Suppose one side of the rectangle has length x cm and the other has length y cm.

The perimeter is $(2x + 2y)$ cm,

$$\text{so } 2x + 2y = 20$$

$$\therefore 2y = 20 - 2x$$

$$\therefore y = 10 - x$$

The area of the rectangle is therefore

$$x(10 - x) \text{ cm}^2.$$

$\therefore$ if the area is 30 cm^2 , then

$$x(10 - x) = 30$$

$$\therefore 10x - x^2 = 30$$

$$\therefore x^2 - 10x + 30 = 0$$

$$\text{which has } a = 1, \quad b = -10, \quad c = 30$$

$$\therefore x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(30)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100 - 120}}{2}$$

$$= \frac{10 \pm \sqrt{-20}}{2}$$

$\therefore x$ has no real solutions, so it is not possible.

- 9** Suppose the tin plate was $x \text{ cm} \times x \text{ cm}$.

When $3 \text{ cm} \times 3 \text{ cm}$ squares are cut from the corners, the base of the open box formed is $(x - 6) \text{ cm} \times (x - 6) \text{ cm}$.

The open box has height 3 cm, so its volume

$$\text{is } 3 \times (x - 6) \times (x - 6) = 80$$

$$\therefore 3(x^2 - 12x + 36) = 80$$

$$\therefore 3x^2 - 36x + 108 = 80$$

$$\therefore 3x^2 - 36x + 28 = 0$$

$$\text{which has } a = 3, \quad b = -36, \quad c = 28$$

$$\therefore x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)}$$

$$= \frac{36 \pm \sqrt{960}}{6} \quad \text{and since } x > 0,$$

$$x = 6 + \frac{\sqrt{960}}{6} \div 11.16 \text{ cm}$$

$\therefore$ the original piece of tinplate was about 11.2 cm square.

- 11** The smaller rectangle is similar to the original rectangle.

$$\therefore \frac{AB}{AD} = \frac{BC}{BY}$$

$$\text{But } AD = BC \quad \text{and} \quad BY = AB - AY \\ = AB - AD$$

$$\therefore \frac{AB}{AD} = \frac{AD}{AB - AD}$$

$$\text{Suppose } AB = x \text{ units, and } AD = BC \\ = 1 \text{ unit}$$

$$\therefore \frac{x}{1} = \frac{1}{x - 1}$$

$$\therefore x(x - 1) = 1$$

$$\therefore x^2 - x - 1 = 0$$

$$\text{which has } a = 1, \quad b = -1, \quad c = -1$$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

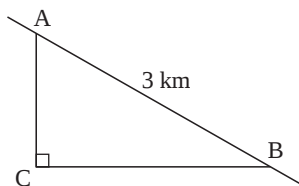
$$= \frac{1 \pm \sqrt{1 + 4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{1 + \sqrt{5}}{2}, \quad \text{since } x > 0$$

$$\text{But } \frac{AB}{AD} = x, \quad \text{which is the golden ratio}$$

$$\therefore \text{the golden ratio is } \frac{1 + \sqrt{5}}{2}$$

12

Suppose AC is x hundred metres,
so BC is $(x + 4)$ hundred metres.

Now $AC^2 + BC^2 = AB^2$ {Pythagoras}

$$\therefore x^2 + (x + 4)^2 = 30^2$$

$$\therefore x^2 + x^2 + 8x + 16 = 900$$

$$\therefore 2x^2 + 8x - 884 = 0$$

$$\therefore x^2 + 4x - 442 = 0$$

which has $a = 1$, $b = 4$, $c = -442$

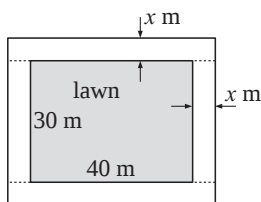
$$\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-442)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{1784}}{2}$$

$$\therefore \text{since } x > 0, x = \frac{-4 \pm \sqrt{1784}}{2} \div 19.12$$

$\therefore AC \div 19.12$ hundred metres and $BC \div 23.12$ hundred metres

$\therefore$ since the paddock is triangular, its area is $\frac{1}{2} \times 19.12 \times 23.12 \div 221$ hectares.

13

Suppose the concrete has width x m around the lawn. We divide the concrete up into four regions as shown.

The smaller regions have area $30x$ m², whilst the larger regions have area $x(40 + 2x)$ m².

Now the total area of concrete is one quarter the area of the lawn.

$$\therefore 2 \times 30x + 2 \times x(40 + 2x) = \frac{1}{4} \times 30 \times 40$$

$$\therefore 60x + 80x + 4x^2 = 300$$

$$\therefore 4x^2 + 140x = 300$$

$$\therefore x^2 + 35x - 75 = 0$$

which has $a = 1$, $b = 35$, $c = -75$

$$\therefore x = \frac{-35 \pm \sqrt{35^2 - 4(1)(-75)}}{2(1)}$$

$$= \frac{-35 \pm \sqrt{1525}}{2}$$

But $x > 0$,

$$\text{so } x = \frac{-35 + \sqrt{1525}}{2} \div 2.026 \text{ m}$$

$\therefore$ the path is about 2.03 m wide.

14 Suppose Hassan's speed is h kmph.

We know that $\text{speed} = \frac{\text{distance}}{\text{time}}$,

so $\text{time} = \frac{\text{distance}}{\text{speed}}$

$$\therefore \text{if it takes Hassan } t \text{ hours, } t = \frac{40}{h} \dots (1)$$

Now Chuong says he will drive home at speed $(h + 40)$ kmph and arrive in time $(t - \frac{1}{3})$ hrs.

$$\therefore t - \frac{1}{3} = \frac{40}{h + 40}$$

$$\text{i.e., } t = \frac{40}{h + 40} + \frac{1}{3} \dots (2)$$

$$\therefore \text{using (1) and (2), } \frac{40}{h} = \frac{40}{h + 40} + \frac{1}{3}$$

$$\therefore 40(h + 40) = 40h + \frac{1}{3}h(h + 40)$$

$$\therefore 40h + 1600 = 40h + \frac{1}{3}h^2 + \frac{40}{3}h$$

$$\therefore h^2 + 40h - 4800 = 0$$

which has $a = 1$, $b = 40$, $c = -4800$

$$\therefore h = \frac{-40 \pm \sqrt{40^2 - 4(1)(-4800)}}{2(1)}$$

$$= \frac{-40 \pm \sqrt{1600 + 19200}}{2}$$

$$= \frac{-40 \pm \sqrt{20800}}{2}$$

$$\text{But } h > 0, \text{ so } h = \frac{-40 + \sqrt{20800}}{2} \div 52.1 \text{ kmph}$$

i.e., Hassan's speed is approximately 52.1 kmph.

- 15** Suppose the speed of the plane is x kmph.

We know $\text{speed} = \frac{\text{distance}}{\text{time}}$, so $\text{time} = \frac{\text{distance}}{\text{speed}}$

Using the information given, $\frac{1000}{x} = \frac{1000}{x-120} - \frac{1}{2}$

$$\therefore 1000(x-120) = 1000x - \frac{1}{2}x(x-120)$$

$$\therefore 1000x - 120\,000 = 1000x - \frac{1}{2}x^2 + 60x$$

$$\therefore x^2 - 120x - 240\,000 = 0$$

which has $a = 1$, $b = -120$, $c = -240\,000$

$$\therefore x = \frac{-(-120) \pm \sqrt{(-120)^2 - 4(1)(-240\,000)}}{2(1)}$$

$$= \frac{120 \pm \sqrt{974\,400}}{2}$$

But $x > 0$, so $x = \frac{120 + \sqrt{974\,400}}{2} \div 553.6$ kmph,

$\therefore$ the plane has speed approximately 554 kmph.

- 16** Suppose the express train travels at x kmph (on average).

We know $\text{speed} = \frac{\text{distance}}{\text{time}}$, so $\text{time} = \frac{\text{distance}}{\text{speed}}$

$\therefore$ it takes the express train $\frac{105}{x}$ hours and the normal train $\frac{105}{x-10}$ hours.

$$\therefore \frac{105}{x} + \frac{1}{2} = \frac{105}{x-10}$$

$$\therefore 105(x-10) + \frac{1}{2}x(x-10) = 105x$$

$$\therefore 105x - 1050 + \frac{1}{2}x^2 - 5x = 105x$$

$$\therefore x^2 - 10x - 2100 = 0$$

which has $a = 1$, $b = -10$, $c = -2100$

$$\therefore x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-2100)}}{2(1)} = \frac{10 \pm \sqrt{8500}}{2}$$

But $x > 0$, so $x = \frac{10 + \sqrt{8500}}{2} \div 51.1$ kmph

$\therefore$ the express train travels on average at about 51.1 kmph.

- 17** Suppose n elderly citizens ended up going on the trip, so the cost per person was $\$ \frac{160}{n}$.

If the original number of elderly citizens had gone, there would have been $(n+8)$,

and the cost per person would have been $\$ \frac{160}{n+8}$.

$$\text{Hence } \frac{160}{n} = \frac{160}{n+8} + 1$$

$$\therefore 160(n+8) = 160n + n(n+8)$$

$$\therefore 160n + 1280 = 160n + n^2 + 8n$$

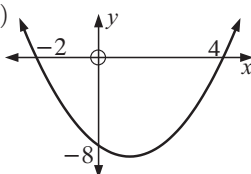
$$\therefore n^2 + 8n - 1280 = 0$$

$$\therefore (n-32)(n+40) = 0$$

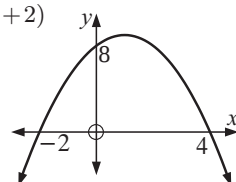
$\therefore$ since $n > 0$, $n = 32$, i.e., 32 elderly citizens went on the trip.

EXERCISE 8H

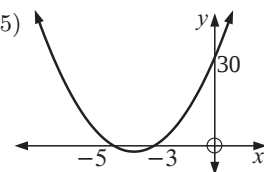
1 a $y = (x - 4)(x + 2)$

has x -intercepts
-2 and 4and y -intercept
-8

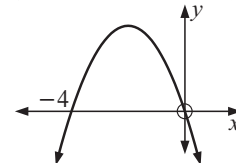
b $y = -(x - 4)(x + 2)$

has x -intercepts
-2 and 4and y -intercept 8

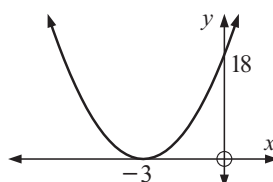
c $y = 2(x + 3)(x + 5)$

has x -intercepts
-5 and -3and y -intercept
30

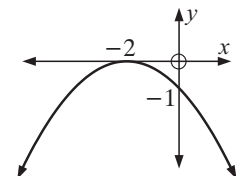
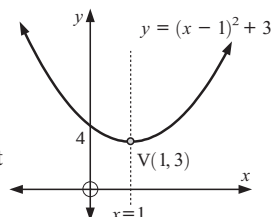
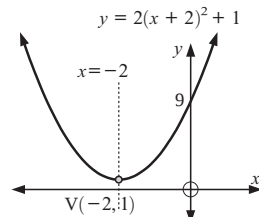
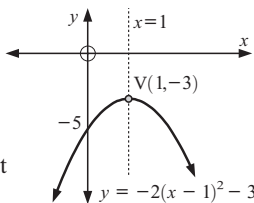
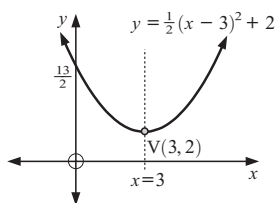
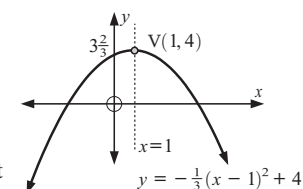
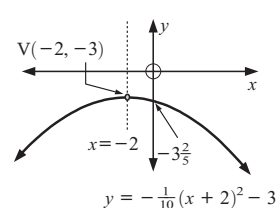
d $y = -3x(x + 4)$

has x -intercepts
0 and -4and y -intercept 0

e $y = 2(x + 3)^2$

has x -intercept
-3and y -intercept
18

f $y = -\frac{1}{4}(x + 2)^2$

has x -intercept
-2and y -intercept
-12 a The average of the x -intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.b The average of the x -intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.c The average of the x -intercepts is -4 $\therefore$ the axis of symmetry is $x = -4$.d The average of the x -intercepts is -2 $\therefore$ the axis of symmetry is $x = -2$.e The only x -intercept is -3, so the axis of symmetry is $x = -3$.f The only x -intercept is -2, so the axis of symmetry is $x = -2$.3 a The vertex is
(1, 3).The axis of
symmetry is
 $x = 1$.The y -intercept
is 4.b The vertex is
(-2, 1).The axis of
symmetry is
 $x = -2$.The y -intercept
is 9.c The vertex is
(1, -3).The axis of
symmetry is
 $x = 1$.The y -intercept
is -5.d The vertex is
(3, 2).The axis of
symmetry is
 $x = 3$.The y -intercept
is $\frac{13}{2}$.e The vertex is
(1, 4).The axis of
symmetry is
 $x = 1$.The y -intercept
is $3\frac{2}{3}$.f The vertex is
(-2, -3).The axis of
symmetry is
 $x = -2$.The y -intercept
is $-3\frac{2}{5}$.

4 a $y = x^2 - 4x + 2$
 has $a = 1$, $b = -4$, $c = 2$
 $\therefore -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$
 $\therefore$ the axis of symmetry is $x = 2$
 When $x = 2$,
 $y = 2^2 - 4 \times 2 + 2 = -2$
 $\therefore$ the vertex is at $(2, -2)$.

c $y = 2x^2 + 4$
 has $a = 2$, $b = 0$, $c = 4$
 $\therefore -\frac{b}{2a} = -\frac{0}{2(2)} = 0$
 $\therefore$ the axis of symmetry is $x = 0$
 When $x = 0$, $y = 4$
 $\therefore$ the vertex is at $(0, 4)$.

e $y = 2x^2 + 8x - 7$
 has $a = 2$, $b = 8$, $c = -7$
 $\therefore -\frac{b}{2a} = -\frac{8}{2(2)} = -2$
 $\therefore$ the axis of symmetry is $x = -2$
 When $x = -2$,
 $y = 2(-2)^2 + 8(-2) - 7 = -15$
 $\therefore$ the vertex is at $(-2, -15)$.

g $y = 2x^2 + 6x - 1$
 has $a = 2$, $b = 6$, $c = -1$
 $\therefore -\frac{b}{2a} = -\frac{6}{2(2)} = -\frac{3}{2}$
 $\therefore$ the axis of symmetry is $x = -\frac{3}{2}$
 When $x = -\frac{3}{2}$,
 $y = 2(-\frac{3}{2})^2 + 6(-\frac{3}{2}) - 1$
 $= \frac{9}{2} - 9 - 1$
 $= -\frac{11}{2}$
 $\therefore$ the vertex is at $(-\frac{3}{2}, -\frac{11}{2})$.

i $y = -\frac{1}{2}x^2 + x - 5$
 has $a = -\frac{1}{2}$, $b = 1$, $c = -5$
 $\therefore -\frac{b}{2a} = -\frac{1}{2(-\frac{1}{2})} = 1$
 $\therefore$ the axis of symmetry is $x = 1$
 When $x = 1$,
 $y = -\frac{1}{2}(1)^2 + 1 - 5 = -\frac{9}{2}$
 $\therefore$ the vertex is at $(1, -\frac{9}{2})$.

b $y = x^2 + 2x - 3$
 has $a = 1$, $b = 2$, $c = -3$
 $\therefore -\frac{b}{2a} = -\frac{2}{2(1)} = -1$
 $\therefore$ the axis of symmetry is $x = -1$
 When $x = -1$,
 $y = (-1)^2 + 2(-1) - 3 = -4$
 $\therefore$ the vertex is at $(-1, -4)$.

d $y = -3x^2 + 1$
 has $a = -3$, $b = 0$, $c = 1$
 $\therefore -\frac{b}{2a} = -\frac{0}{2(-3)} = 0$
 $\therefore$ the axis of symmetry is $x = 0$
 When $x = 0$, $y = 1$
 $\therefore$ the vertex is at $(0, 1)$.

f $y = -x^2 - 4x - 9$
 has $a = -1$, $b = -4$, $c = -9$
 $\therefore -\frac{b}{2a} = -\frac{(-4)}{2(-1)} = -2$
 $\therefore$ the axis of symmetry is $x = -2$
 When $x = -2$, $y = -(-2)^2 - 4(-2) - 9$
 $= -4 + 8 - 9$
 $= -5$
 $\therefore$ the vertex is at $(-2, -5)$.

h $y = 2x^2 - 10x + 3$
 has $a = 2$, $b = -10$, $c = 3$
 $\therefore -\frac{b}{2a} = -\frac{(-10)}{2(2)} = \frac{5}{2}$
 $\therefore$ the axis of symmetry is $x = \frac{5}{2}$
 When $x = \frac{5}{2}$, $y = 2(\frac{5}{2})^2 - 10(\frac{5}{2}) + 3$
 $= \frac{25}{2} - \frac{50}{2} + 3$
 $= -\frac{19}{2}$
 $\therefore$ the vertex is at $(\frac{5}{2}, -\frac{19}{2})$.

j $y = -2x^2 + 8x - 2$
 has $a = -2$, $b = 8$, $c = -2$
 $\therefore -\frac{b}{2a} = -\frac{8}{2(-2)} = 2$
 $\therefore$ the axis of symmetry is $x = 2$
 When $x = 2$, $y = -2(2)^2 + 8(2) - 2$
 $= -8 + 16 - 2$
 $= 6$
 $\therefore$ the vertex is at $(2, 6)$.

5 a When $y = 0$, $x^2 - 9 = 0$
 $\therefore (x+3)(x-3) = 0$
 $\therefore x = \pm 3$
 $\therefore$ the x -intercepts are ± 3

c When $y = 0$, $x^2 + 7x + 10 = 0$
 $\therefore (x+5)(x+2) = 0$
 $\therefore x = -5$ or -2
 $\therefore$ the x -intercepts are -5 and -2

e When $y = 0$, $4x - x^2 = 0$
 $\therefore x(4-x) = 0$
 $\therefore x = 0$ or 4
 $\therefore$ the x -intercepts are 0 and 4

g When $y = 0$, $-2x^2 - 4x - 2 = 0$
 $\therefore x^2 + 2x + 1 = 0$
 $\therefore (x+1)^2 = 0$
 $\therefore x = -1$
 $\therefore$ the x -intercept is -1 (touching)

i When $y = 0$, $x^2 - 4x + 1 = 0$
 $a = 1$, $b = -4$ and $c = 1$
 $\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$
 $= \frac{4 \pm \sqrt{12}}{2}$
 $= \frac{4 \pm 2\sqrt{3}}{2}$
 $= 2 \pm \sqrt{3}$
 $\therefore$ the x -intercepts are $2 \pm \sqrt{3}$

k When $y = 0$, $x^2 - 6x - 2 = 0$
 $a = 1$, $b = -6$ and $c = -2$
 $\therefore x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-2)}}{2(1)}$
 $= \frac{6 \pm \sqrt{44}}{2}$
 $= \frac{6 \pm 2\sqrt{11}}{2}$
 $= 3 \pm \sqrt{11}$
 $\therefore$ the x -intercepts are $3 \pm \sqrt{11}$

b When $y = 0$, $2x^2 - 6 = 0$
 $\therefore x^2 - 3 = 0$
 $\therefore (x+\sqrt{3})(x-\sqrt{3}) = 0$
 $\therefore x = \pm\sqrt{3}$
 $\therefore$ the x -intercepts are $\pm\sqrt{3}$

d When $y = 0$, $x^2 + x - 12 = 0$
 $\therefore (x+4)(x-3) = 0$
 $\therefore x = -4$ or 3
 $\therefore$ the x -intercepts are -4 and 3

f When $y = 0$, $-x^2 - 6x - 8 = 0$
 $\therefore x^2 + 6x + 8 = 0$
 $\therefore (x+4)(x+2) = 0$
 $\therefore x = -4$ or -2
 $\therefore$ the x -intercepts are -4 and -2

h When $y = 0$, $4x^2 - 24x + 36 = 0$
 $\therefore x^2 - 6x + 9 = 0$
 $\therefore (x-3)^2 = 0$
 $\therefore x = 3$
 $\therefore$ the x -intercept is 3 (touching)

j When $y = 0$, $x^2 + 4x - 3 = 0$
 $a = 1$, $b = 4$ and $c = -3$
 $\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$
 $= \frac{-4 \pm \sqrt{28}}{2}$
 $= \frac{-4 \pm 2\sqrt{7}}{2}$
 $= -2 \pm \sqrt{7}$
 $\therefore$ the x -intercepts are $-2 \pm \sqrt{7}$

l When $y = 0$, $x^2 + 8x + 11 = 0$
 $a = 1$, $b = 8$ and $c = 11$
 $\therefore x = \frac{-8 \pm \sqrt{8^2 - 4(1)(11)}}{2(1)}$
 $= \frac{-8 \pm \sqrt{20}}{2}$
 $= \frac{-8 \pm 2\sqrt{5}}{2}$
 $= -4 \pm \sqrt{5}$
 $\therefore$ the x -intercepts are $-4 \pm \sqrt{5}$

- 6 a i** $y = x^2 - 2x + 5$
has $a = 1$, $b = -2$, $c = 5$
 $\therefore -\frac{b}{2a} = -\frac{(-2)}{2(1)} = 1$
 $\therefore$ the axis of symmetry is $x = 1$

iii When $x = 0$, $y = 5$,
so the y -intercept is 5
When $y = 0$, $x^2 - 2x + 5 = 0$
 $\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$
 $= \frac{2 \pm \sqrt{4 - 20}}{2}$

This has no real solutions,
so there are no x -intercepts.

- b i** $y = x^2 + 4x - 1$
has $a = 1$, $b = 4$, $c = -1$
 $\therefore -\frac{b}{2a} = -\frac{4}{2(1)} = -2$
 $\therefore$ the axis of symmetry is $x = -2$

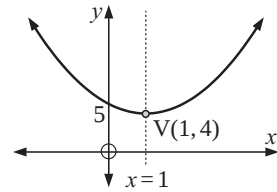
iii When $x = 0$, $y = -1$,
so the y -intercept is -1 .
When $y = 0$, $x^2 + 4x - 1 = 0$
 $\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)}$
 $= \frac{-4 \pm \sqrt{20}}{2}$
 $= \frac{-4 \pm 2\sqrt{5}}{2}$
 $= -2 \pm \sqrt{5}$
 $\therefore$ the x -intercepts are $-2 \pm \sqrt{5}$

- c i** $y = 2x^2 - 5x + 2$
has $a = 2$, $b = -5$, $c = 2$
 $\therefore -\frac{b}{2a} = -\frac{(-5)}{2(2)} = \frac{5}{4}$
 $\therefore$ the axis of symmetry is $x = \frac{5}{4}$

iii When $x = 0$, $y = 2$,
so the y -intercept is 2.
When $y = 0$, $2x^2 - 5x + 2 = 0$
 $\therefore (2x - 1)(x - 2) = 0$
 $\therefore x = \frac{1}{2}$ or 2
 $\therefore$ the x -intercepts are $\frac{1}{2}$ and 2

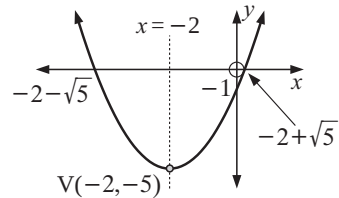
- ii** When $x = 1$,
 $y = 1^2 - 2(1) + 5$
 $= 1 - 2 + 5$
 $= 4$
 $\therefore$ the vertex is at $(1, 4)$

iv



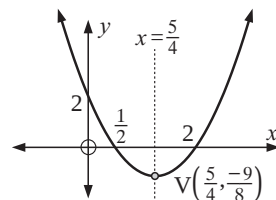
- ii** When $x = -2$,
 $y = (-2)^2 + 4(-2) - 1$
 $= 4 - 8 - 1$
 $= -5$
 $\therefore$ the vertex is at $(-2, -5)$

iv



- ii** When $x = \frac{5}{4}$,
 $y = 2(\frac{5}{4})^2 - 5(\frac{5}{4}) + 2$
 $= \frac{50}{16} - \frac{25}{4} + 2$
 $= -\frac{9}{8}$
 $\therefore$ the vertex is at $(\frac{5}{4}, -\frac{9}{8})$

iv



d i $y = -x^2 + 3x - 2$
 has $a = -1$, $b = 3$, $c = -2$
 $\therefore -\frac{b}{2a} = -\frac{3}{2(-1)} = \frac{3}{2}$
 $\therefore$ the axis of symmetry is $x = \frac{3}{2}$

iii When $x = 0$, $y = -2$,
 so the y -intercept is -2 .
 When $y = 0$, $-x^2 + 3x - 2 = 0$
 $\therefore x^2 - 3x + 2 = 0$
 $\therefore (x-1)(x-2) = 0$
 $\therefore x = 1$ or 2
 $\therefore$ the x -intercepts are 1 and 2

e i $y = -3x^2 + 4x - 1$
 has $a = -3$, $b = 4$, $c = -1$
 $\therefore -\frac{b}{2a} = -\frac{4}{2(-3)} = \frac{2}{3}$
 $\therefore$ the axis of symmetry is $x = \frac{2}{3}$

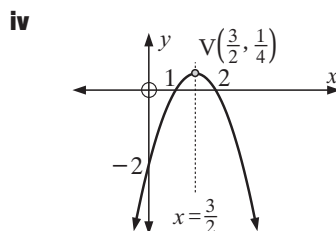
iii When $x = 0$, $y = -1$,
 so the y -intercept is -1 .
 When $y = 0$, $-3x^2 + 4x - 1 = 0$
 $\therefore 3x^2 - 4x + 1 = 0$
 $\therefore (3x-1)(x-1) = 0$
 $\therefore x = \frac{1}{3}$ or 1
 $\therefore$ the x -intercepts are $\frac{1}{3}$ and 1

f i $y = -2x^2 + x + 1$
 has $a = -2$, $b = 1$, $c = 1$
 $\therefore -\frac{b}{2a} = -\frac{1}{2(-2)} = \frac{1}{4}$
 $\therefore$ the axis of symmetry is $x = \frac{1}{4}$

iii When $x = 0$, $y = 1$,
 so the y -intercept is 1 .
 When $y = 0$, $-2x^2 + x + 1 = 0$
 $\therefore 2x^2 - x - 1 = 0$
 $\therefore (2x+1)(x-1) = 0$
 $\therefore x = -\frac{1}{2}$ or 1
 $\therefore$ the x -intercepts are $-\frac{1}{2}$ and 1

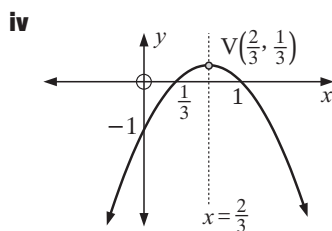
ii When $x = \frac{3}{2}$,
 $y = -(\frac{3}{2})^2 + 3(\frac{3}{2}) - 2$
 $= -\frac{9}{4} + \frac{9}{2} - 2$
 $= \frac{1}{4}$

$\therefore$ the vertex is at $(\frac{3}{2}, \frac{1}{4})$



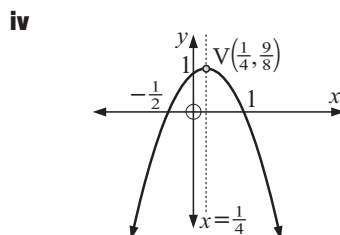
ii When $x = \frac{2}{3}$,
 $y = -3(\frac{2}{3})^2 + 4(\frac{2}{3}) - 1$
 $= -\frac{4}{3} + \frac{8}{3} - 1$
 $= \frac{1}{3}$

$\therefore$ the vertex is at $(\frac{2}{3}, \frac{1}{3})$



ii When $x = \frac{1}{4}$,
 $y = -2(\frac{1}{4})^2 + \frac{1}{4} + 1$
 $= -\frac{1}{8} + \frac{1}{4} + 1$
 $= \frac{9}{8}$

$\therefore$ the vertex is at $(\frac{1}{4}, \frac{9}{8})$



g i $y = 6x - x^2$
has $a = -1$, $b = 6$, $c = 0$
 $\therefore -\frac{b}{2a} = -\frac{6}{2(-1)} = 3$

$\therefore$ the axis of symmetry is $x = 3$

iii When $x = 0$, $y = 0$,
so the y -intercept is 0.
When $y = 0$, $6x - x^2 = 0$
 $\therefore x(6 - x) = 0$
 $\therefore x = 0$ or 6

$\therefore$ the x -intercepts are 0 and 6

h i $y = -x^2 - 6x - 8$
has $a = -1$, $b = -6$, $c = -8$
 $\therefore -\frac{b}{2a} = -\frac{(-6)}{2(-1)} = -3$

$\therefore$ the axis of symmetry is $x = -3$

iii When $x = 0$, $y = -8$,
so the y -intercept is -8 .
When $y = 0$, $-x^2 - 6x - 8 = 0$
 $\therefore x^2 + 6x + 8 = 0$
 $\therefore (x + 4)(x + 2) = 0$
 $\therefore x = -4$ or -2

$\therefore$ the x -intercepts are -4 and -2

i i $y = -\frac{1}{4}x^2 + 2x + 1$
has $a = -\frac{1}{4}$, $b = 2$, $c = 1$
 $\therefore -\frac{b}{2a} = -\frac{2}{2(-\frac{1}{4})} = 4$

$\therefore$ the axis of symmetry is $x = 4$

iii When $x = 0$, $y = 1$,
so the y -intercept is 1.
When $y = 0$, $-\frac{1}{4}x^2 + 2x + 1 = 0$
 $\therefore x^2 - 8x - 4 = 0$
which has $a = 1$, $b = -8$, $c = -4$

$$\therefore x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{80}}{2}$$

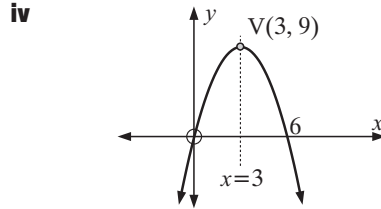
$$= \frac{8 \pm 4\sqrt{5}}{2}$$

$$= 4 \pm 2\sqrt{5}$$

$\therefore$ the x -intercepts are $4 \pm 2\sqrt{5}$

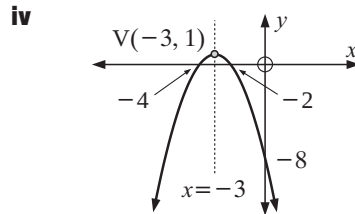
ii When $x = 3$,
 $y = 6 \times 3 - 3^2$
 $= 9$

$\therefore$ the vertex is at $(3, 9)$



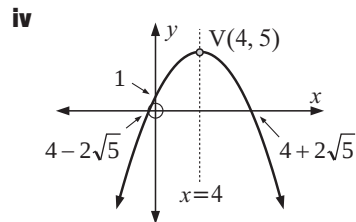
ii When $x = -3$,
 $y = -(-3)^2 - 6(-3) - 8$
 $= -9 + 18 - 8$
 $= 1$

$\therefore$ the vertex is at $(-3, 1)$



ii When $x = 4$,
 $y = -\frac{1}{4}(4)^2 + 2(4) + 1$
 $= -4 + 8 + 1$
 $= 5$

$\therefore$ the vertex is at $(4, 5)$



EXERCISE 8I.1

1 a $x^2 + 7x - 2 = 0$

has $a = 1, b = 7, c = -2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 7^2 - 4(1)(-2) \\ &= 57\end{aligned}$$

Since $\Delta > 0$, there are 2 distinct real solutions.

c $2x^2 + 3x - 1 = 0$

has $a = 2, b = 3, c = -1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 3^2 - 4(2)(-1) \\ &= 17\end{aligned}$$

Since $\Delta > 0$, there are 2 distinct real solutions.

e $x^2 + x + 6 = 0$

has $a = 1, b = 1, c = 6$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 1^2 - 4(1)(6) \\ &= -23\end{aligned}$$

Since $\Delta < 0$, there are no real roots.

2 a $2x^2 + 7x - 4 = 0$

has $a = 2, b = 7, c = -4$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 7^2 - 4(2)(-4) \\ &= 81\end{aligned}$$

$\therefore \sqrt{\Delta} = 9$, so the equation has rational roots.

c $2x^2 + 6x + 1 = 0$

has $a = 2, b = 6, c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 6^2 - 4(2)(1) \\ &= 28\end{aligned}$$

$\therefore \sqrt{\Delta} = \sqrt{28} = 2\sqrt{7}$, so the equation does not have rational roots.

e $4x^2 - 3x + 3 = 0$

has $a = 4, b = -3, c = 3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(4)(3) \\ &= -39\end{aligned}$$

Since $\Delta < 0$, the equation does not have rational roots.

b $x^2 + 4\sqrt{2}x + 8 = 0$

has $a = 1, b = 4\sqrt{2}, c = 8$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (4\sqrt{2})^2 - 4(1)(8) \\ &= 32 - 32 \\ &= 0\end{aligned}$$

$\therefore$ there is one repeated real root.

d $6x^2 + 5x - 4 = 0$

has $a = 6, b = 5, c = -4$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 5^2 - 4(6)(-4) \\ &= 121\end{aligned}$$

Since $\Delta > 0$, there are 2 distinct real solutions.

f $9x^2 + 6x + 1 = 0$

has $a = 9, b = 6, c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 6^2 - 4(9)(1) \\ &= 0\end{aligned}$$

$\therefore$ there is one repeated real root.

b $3x^2 - 7x - 6 = 0$

has $a = 3, b = -7, c = -6$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(3)(-6) \\ &= 121\end{aligned}$$

$\therefore \sqrt{\Delta} = 11$, so the equation has rational roots.

d $6x^2 + 19x + 10 = 0$

has $a = 6, b = 19, c = 10$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 19^2 - 4(6)(10) \\ &= 121\end{aligned}$$

$\therefore \sqrt{\Delta} = 11$, so the equation has rational roots.

f $8x^2 - 10x - 3 = 0$

has $a = 8, b = -10, c = -3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-10)^2 - 4(8)(-3) \\ &= 196\end{aligned}$$

$\therefore \sqrt{\Delta} = 14$, so the equation has rational roots.

3 a $x^2 + 3x + m = 0$ has $a = 1$, $b = 3$, $c = m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = 3^2 - 4(1)m \\ &= 9 - 4m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 9 - 4m = 0$$

$$\therefore 4m = 9$$

$$\therefore m = \frac{9}{4}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 9 - 4m > 0$$

$$\therefore 4m < 9$$

$$\therefore m < \frac{9}{4}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 9 - 4m < 0$$

$$\therefore 4m > 9$$

$$\therefore m > \frac{9}{4}$$

b $x^2 - 5x + m = 0$ has $a = 1$, $b = -5$, $c = m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = (-5)^2 - 4(1)m \\ &= 25 - 4m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 25 - 4m = 0$$

$$\therefore 4m = 25$$

$$\therefore m = \frac{25}{4}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 25 - 4m > 0$$

$$\therefore 4m < 25$$

$$\therefore m < \frac{25}{4}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 25 - 4m < 0$$

$$\therefore 4m > 25$$

$$\therefore m > \frac{25}{4}$$

c $mx^2 - x + 1 = 0$ has $a = m$, $b = -1$, $c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = (-1)^2 - 4m(1) \\ &= 1 - 4m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 1 - 4m = 0$$

$$\therefore 4m = 1$$

$$\therefore m = \frac{1}{4}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 1 - 4m > 0$$

$$\therefore 4m < 1$$

$$\therefore m < \frac{1}{4}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 1 - 4m < 0$$

$$\therefore 4m > 1$$

$$\therefore m > \frac{1}{4}$$

d $mx^2 + 2x + 3 = 0$ has $a = m$, $b = 2$, $c = 3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = 2^2 - 4m(3) \\ &= 4 - 12m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 4 - 12m = 0$$

$$\therefore 12m = 4$$

$$\therefore m = \frac{1}{3}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 4 - 12m > 0$$

$$\therefore 12m < 4$$

$$\therefore m < \frac{1}{3}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 4 - 12m < 0$$

$$\therefore 12m > 4$$

$$\therefore m > \frac{1}{3}$$

e $2x^2 + 7x + m = 0$ has $a = 2$, $b = 7$, $c = m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = 7^2 - 4(2)m \\ &= 49 - 8m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 49 - 8m = 0$$

$$\therefore 8m = 49$$

$$\therefore m = \frac{49}{8}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 49 - 8m > 0$$

$$\therefore 8m < 49$$

$$\therefore m < \frac{49}{8}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 49 - 8m < 0$$

$$\therefore 8m > 49$$

$$\therefore m > \frac{49}{8}$$

f $mx^2 - 5x + 4 = 0$ has $a = m$, $b = -5$, $c = 4$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = (-5)^2 - 4m(4) \\ &= 25 - 16m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 25 - 16m = 0$$

$$\therefore 16m = 25$$

$$\therefore m = \frac{25}{16}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 25 - 16m > 0$$

$$\therefore 16m < 25$$

$$\therefore m < \frac{25}{16}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 25 - 16m < 0$$

$$\therefore 16m > 25$$

$$\therefore m > \frac{25}{16}$$

EXERCISE 8I.2

1 a $y = x^2 + 7x - 2$

has $a = 1$, $b = 7$, $c = -2$

$$\therefore \Delta = b^2 - 4ac$$

$$= 7^2 - 4(1)(-2)$$

$$= 57$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

c $y = -2x^2 + 3x + 1$

has $a = -2$, $b = 3$, $c = 1$

$$\therefore \Delta = b^2 - 4ac$$

$$= 3^2 - 4(-2)(1)$$

$$= 17$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

e $y = -x^2 + x + 6$

has $a = -1$, $b = 1$, $c = 6$

$$\therefore \Delta = b^2 - 4ac$$

$$= 1^2 - 4(-1)(6)$$

$$= 25$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

2 a $x^2 - 3x + 6$

has $a = 1$, $b = -3$, $c = 6$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-3)^2 - 4(1)(6)$$

$$= -15$$

 $\therefore$ since $a > 0$ and $\Delta < 0$,
 $x^2 - 3x + 6 > 0$ for all x .

c $2x^2 - 4x + 7$

has $a = 2$, $b = -4$, $c = 7$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-4)^2 - 4(2)(7)$$

$$= -40$$

 $\therefore$ since $a > 0$ and $\Delta < 0$,
 $2x^2 - 4x + 7 > 0$ for all x .
 $\therefore$ it is positive definite.

b $y = x^2 + 4\sqrt{2}x + 8$

has $a = 1$, $b = 4\sqrt{2}$, $c = 8$

$$\therefore \Delta = b^2 - 4ac$$

$$= (4\sqrt{2})^2 - 4(1)(8)$$

$$= 0$$

 $\therefore$ the graph touches the x -axis.

d $y = 6x^2 + 5x - 4$

has $a = 6$, $b = 5$, $c = -4$

$$\therefore \Delta = b^2 - 4ac$$

$$= 5^2 - 4(6)(-4)$$

$$= 121$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

f $y = 9x^2 + 6x + 1$

has $a = 9$, $b = 6$, $c = 1$

$$\therefore \Delta = b^2 - 4ac$$

$$= 6^2 - 4(9)(1)$$

$$= 0$$

 $\therefore$ the graph touches the x -axis.

b $4x - x^2 - 6$

has $a = -1$, $b = 4$, $c = -6$

$$\therefore \Delta = b^2 - 4ac$$

$$= 4^2 - 4(-1)(-6)$$

$$= -8$$

 $\therefore$ since $a < 0$ and $\Delta < 0$,
 $4x - x^2 - 6 < 0$ for all x .

d $-2x^2 + 3x - 4$

has $a = -2$, $b = 3$, $c = -4$

$$\therefore \Delta = b^2 - 4ac$$

$$= 3^2 - 4(-2)(-4)$$

$$= -23$$

 $\therefore$ since $a < 0$ and $\Delta < 0$,
 $-2x^2 + 3x - 4 < 0$ for all x .
 $\therefore$ it is negative definite.

$$3 \quad 3x^2 + kx - 1$$

$$\text{has } a = 3, \quad b = k, \quad c = -1$$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= k^2 - 4(3)(-1) \\ &= k^2 + 12\end{aligned}$$

$$\text{Now } k^2 \geq 0 \text{ for all } k$$

$$\therefore k^2 + 12 > 0 \text{ for all } k$$

$$\therefore \Delta > 0 \text{ for all } k$$

$$\therefore 3x^2 + kx - 1 \text{ has two real distinct roots for all } k.$$

$$\therefore \text{it can never be positive definite.}$$

$$4 \quad 2x^2 + kx + 2$$

$$\text{has } a = 2, \quad b = k, \quad c = 2$$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= k^2 - 4(2)(2) \\ &= k^2 - 16\end{aligned}$$

$$\text{Now } 2x^2 + kx + 2 \text{ has } a > 0.$$

$$\therefore \text{it is positive definite provided } k^2 - 16 < 0$$

$$\therefore k^2 < 16$$

$$\therefore -4 < k < 4$$

EXERCISE 8J

$$1 \quad a \quad \text{The } x\text{-intercepts are 1 and 2.}$$

$$\therefore y = a(x-1)(x-2)$$

$$\text{for some } a \neq 0.$$

$$\text{But the } y\text{-intercept is 4.}$$

$$\therefore a(-1)(-2) = 4$$

$$\therefore 2a = 4$$

$$\therefore a = 2$$

$$\therefore y = 2(x-1)(x-2)$$

$$c \quad \text{The } x\text{-intercepts are 1 and 3.}$$

$$\therefore y = a(x-1)(x-3)$$

$$\text{for some } a \neq 0.$$

$$\text{But the } y\text{-intercept is 3.}$$

$$\therefore a(-1)(-3) = 3$$

$$\therefore 3a = 3$$

$$\therefore a = 1$$

$$\therefore y = (x-1)(x-3)$$

$$e \quad \text{The graph touches the } x\text{-axis when } x = 1.$$

$$\therefore y = a(x-1)^2 \text{ for some } a \neq 0.$$

$$\text{But the } y\text{-intercept is } -3.$$

$$\therefore a(-1)^2 = -3$$

$$\therefore a = -3$$

$$\therefore y = -3(x-1)^2$$

$$b \quad \text{The graph touches the } x\text{-axis when } x = 2.$$

$$\therefore y = a(x-2)^2 \text{ for some } a \neq 0.$$

$$\text{But the } y\text{-intercept is 8.}$$

$$\therefore a(-2)^2 = 8$$

$$\therefore 4a = 8$$

$$\therefore a = 2$$

$$\therefore y = 2(x-2)^2$$

$$d \quad \text{The } x\text{-intercepts are } -1 \text{ and } 3.$$

$$\therefore y = a(x+1)(x-3)$$

$$\text{for some } a \neq 0.$$

$$\text{But the } y\text{-intercept is 3.}$$

$$\therefore a(1)(-3) = 3$$

$$\therefore a = -1$$

$$\therefore y = -(x+1)(x-3)$$

$$f \quad \text{The } x\text{-intercepts are } -2 \text{ and } 3.$$

$$\therefore y = a(x+2)(x-3)$$

$$\text{for some } a \neq 0.$$

$$\text{But the } y\text{-intercept is 12.}$$

$$\therefore a(2)(-3) = 12$$

$$\therefore -6a = 12$$

$$\therefore a = -2$$

$$\therefore y = -2(x+2)(x-3)$$

$$2 \quad a \quad y = 2(x-1)(x-4)$$

$$\text{has } x\text{-intercepts 1 and 4, and } y\text{-intercept 8}$$

$$\therefore \text{its graph is C}$$

$$d \quad y = (x+1)(x-4)$$

$$\text{has } x\text{-intercepts } -1 \text{ and } 4, \text{ and } y\text{-intercept } -4$$

$$\therefore \text{its graph is F}$$

$$b \quad y = -(x+1)(x-4)$$

$$\text{has } x\text{-intercepts } -1 \text{ and } 4, \text{ and } y\text{-intercept 4}$$

$$\therefore \text{its graph is E}$$

$$e \quad y = 2(x+4)(x-1)$$

$$\text{has } x\text{-intercepts } -4 \text{ and } 1, \text{ and } y\text{-intercept } -8$$

$$\therefore \text{its graph is G}$$

$$c \quad y = (x-1)(x-4)$$

$$\text{has } x\text{-intercepts 1 and 4, and } y\text{-intercept 4}$$

$$\therefore \text{its graph is B}$$

$$f \quad y = -3(x+4)(x-1)$$

$$\text{has } x\text{-intercepts } -4 \text{ and } 1, \text{ and } y\text{-intercept 12}$$

$$\therefore \text{its graph is H}$$

g $y = -(x-1)(x-4)$

has x -intercepts 1 and 4,

and y -intercept -4

$\therefore$ its graph is **A**

h $y = -3(x-1)(x-4)$

has x -intercepts 1 and 4,

and y -intercept -12

$\therefore$ its graph is **D**

- 3 a** As the axis of symmetry is $x = 3$,
the other x -intercept is 4.

$$\therefore y = a(x-2)(x-4)$$

for some $a \neq 0$.

But the y -intercept $= 12$

$$\therefore a(-2)(-4) = 12$$

$$\therefore 8a = 12$$

$$\therefore a = \frac{12}{8} = \frac{3}{2}$$

$$\therefore y = \frac{3}{2}(x-2)(x-4)$$

- b** As the axis of symmetry is $x = -1$,
the other x -intercept is 2.

$$\therefore y = a(x+4)(x-2)$$

for some $a \neq 0$.

But the y -intercept $= 4$

$$\therefore a(4)(-2) = 4$$

$$\therefore -8a = 4$$

$$\therefore a = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}(x+4)(x-2)$$

- c** The graph touches the x -axis at $x = -3$,

$$\therefore y = a(x+3)^2 \text{ for some } a \neq 0.$$

But the y -intercept $= -12$

$$\therefore a(3)^2 = -12$$

$$\therefore 9a = -12$$

$$\therefore a = -\frac{12}{9} = -\frac{4}{3} \quad \therefore y = -\frac{4}{3}(x+3)^2$$

- 4 a** Since the x -intercepts are 5 and 1, the
equation is $y = a(x-5)(x-1)$
for some $a \neq 0$.

But when $x = 2$, $y = -9$

$$\therefore -9 = a(2-5)(2-1)$$

$$\therefore -9 = a(-3)(1)$$

$$\therefore -3a = -9$$

$$\therefore a = 3$$

$\therefore$ the equation is

$$y = 3(x-5)(x-1)$$

$$\text{i.e., } y = 3(x^2 - 6x + 5)$$

$$\text{i.e., } y = 3x^2 - 18x + 15$$

- b** Since the x -intercepts are 2 and $-\frac{1}{2}$, the
equation is $y = a(x-2)(x+\frac{1}{2})$
for some $a \neq 0$.

But when $x = 3$, $y = -14$

$$\therefore -14 = a(3-2)(3+\frac{1}{2})$$

$$\therefore -14 = a(1)(\frac{7}{2})$$

$$\therefore \frac{7}{2}a = -14$$

$$\therefore a = -4$$

$\therefore$ the equation is

$$y = -4(x-2)(x+\frac{1}{2})$$

$$\text{i.e., } y = -4(x^2 - \frac{3}{2}x - 1)$$

$$\text{i.e., } y = -4x^2 + 6x + 4$$

- c** Since the graph touches the x -axis at 3,
its equation is $y = a(x-3)^2$,
for some $a \neq 0$.

But when $x = -2$, $y = -25$

$$\therefore -25 = a(-2-3)^2$$

$$\therefore -25 = 25a$$

$$\therefore a = -1$$

$\therefore$ the equation is

$$y = -(x-3)^2$$

$$\text{i.e., } y = -(x^2 - 6x + 9)$$

$$\text{i.e., } y = -x^2 + 6x - 9$$

- d** Since the graph touches the x -axis at -2 ,
its equation is $y = a(x+2)^2$,
for some $a \neq 0$.

But when $x = -1$, $y = 4$

$$\therefore 4 = a(-1+2)^2$$

$$\therefore 4 = a$$

$\therefore$ the equation is

$$y = 4(x+2)^2$$

$$\text{i.e., } y = 4(x^2 + 4x + 4)$$

$$\text{i.e., } y = 4x^2 + 16x + 16$$

- e** Since the graph cuts the x -axis at 3 and has axis of symmetry $x = 2$, it must also cut the x -axis at 1.
 $\therefore$ the x -intercepts are 3 and 1, and the equation is $y = a(x - 3)(x - 1)$ for some $a \neq 0$.

But when $x = 5$, $y = 12$

$$\therefore 12 = a(5 - 3)(5 - 1)$$

$$\therefore 12 = a(2)(4)$$

$$\therefore 8a = 12$$

$$\therefore a = \frac{3}{2}$$

$\therefore$ the equation is

$$y = \frac{3}{2}(x - 3)(x - 1)$$

$$\text{i.e., } y = \frac{3}{2}(x^2 - 4x + 3)$$

$$\text{i.e., } y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$$

- 5 a** The vertex is (2, 4),
 so the quadratic has equation
 $y = a(x - 2)^2 + 4$ for some $a \neq 0$.
 But the graph passes through the origin
 $\therefore 0 = a(0 - 2)^2 + 4$
 $\therefore 4a + 4 = 0$
 $\therefore a = -1$
 $\therefore$ the equation is $y = -(x - 2)^2 + 4$

- c** The vertex is (3, 8),
 so the quadratic has equation
 $y = a(x - 3)^2 + 8$ for some $a \neq 0$.
 But the graph passes through (1, 0)
 $\therefore 0 = a(1 - 3)^2 + 8$
 $\therefore 0 = 4a + 8$
 $\therefore a = -2$
 $\therefore$ the equation is $y = -2(x - 3)^2 + 8$

- e** The vertex is (2, 3),
 so the quadratic has equation
 $y = a(x - 2)^2 + 3$ for some $a \neq 0$.
 But the graph passes through (3, 1)
 $\therefore 1 = a(3 - 2)^2 + 3$
 $\therefore 1 = a + 3$
 $\therefore a = -2$
 $\therefore$ the equation is $y = -2(x - 2)^2 + 3$

- f** Since the graph cuts the x -axis at 5 and has axis of symmetry $x = 1$, it must also cut the x -axis at -3 .
 $\therefore$ the x -intercepts are 5 and -3 , and the equation is $y = a(x - 5)(x + 3)$ for some $a \neq 0$.

But when $x = 2$, $y = 5$

$$\therefore 5 = a(2 - 5)(2 + 3)$$

$$\therefore 5 = a(-3)(5)$$

$$\therefore -3a = 1$$

$$\therefore a = -\frac{1}{3}$$

$\therefore$ the equation is

$$y = -\frac{1}{3}(x - 5)(x + 3)$$

$$\text{i.e., } y = -\frac{1}{3}(x^2 - 2x - 15)$$

$$\text{i.e., } y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$$

- b** The vertex is (2, -1),
 so the quadratic has equation
 $y = a(x - 2)^2 - 1$ for some $a \neq 0$.
 But the graph passes through (0, 7)
 $\therefore 7 = a(0 - 2)^2 - 1$
 $\therefore 7 = 4a - 1$
 $\therefore 4a = 8$
 $\therefore a = 2$
 $\therefore$ the equation is $y = 2(x - 2)^2 - 1$

- d** The vertex is (4, -6),
 so the quadratic has equation
 $y = a(x - 4)^2 - 6$ for some $a \neq 0$.
 But the graph passes through (7, 0)
 $\therefore 0 = a(7 - 4)^2 - 6$
 $\therefore 9a - 6 = 0$
 $\therefore a = \frac{2}{3}$
 $\therefore$ the equation is $y = \frac{2}{3}(x - 4)^2 - 6$

- f** The vertex is $(\frac{1}{2}, -\frac{3}{2})$,
 so the quadratic has equation
 $y = a(x - \frac{1}{2})^2 - \frac{3}{2}$ for some $a \neq 0$.
 But the graph passes through $(\frac{3}{2}, \frac{1}{2})$
 $\therefore \frac{1}{2} = a(\frac{3}{2} - \frac{1}{2})^2 - \frac{3}{2}$
 $\therefore \frac{1}{2} = a - \frac{3}{2}$
 $\therefore a = 2$
 $\therefore$ the equation is $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 8K

- 1 a** $y = x^2 - 2x + 8$ meets $y = x + 6$
 when $x^2 - 2x + 8 = x + 6$
 $\therefore x^2 - 3x + 2 = 0$
 $\therefore (x - 1)(x - 2) = 0$
 $\therefore x = 1$ or 2
 Substituting into $y = x + 6$,
 when $x = 1$, $y = 7$
 and when $x = 2$, $y = 8$
 $\therefore$ the graphs meet at $(1, 7)$ and $(2, 8)$
- b** $y = -x^2 + 3x + 9$ meets $y = 2x - 3$
 when $-x^2 + 3x + 9 = 2x - 3$
 $\therefore x^2 - x - 12 = 0$
 $\therefore (x - 4)(x + 3) = 0$
 $\therefore x = 4$ or -3
 Substituting into $y = 2x - 3$,
 when $x = -3$, $y = 2(-3) - 3 = -9$
 and when $x = 4$, $y = 2(4) - 3 = 5$
 $\therefore$ the graphs meet at $(-3, -9)$ and $(4, 5)$
- c** $y = x^2 - 4x + 3$ meets $y = 2x - 6$
 when $x^2 - 4x + 3 = 2x - 6$
 $\therefore x^2 - 6x + 9 = 0$
 $\therefore (x - 3)^2 = 0$
 $\therefore x = 3$
 Substituting into $y = 2x - 6$,
 when $x = 3$, $y = 0$
 $\therefore$ the graphs touch at $(3, 0)$
- d** $y = -x^2 + 4x - 7$ meets $y = 5x - 4$
 when $-x^2 + 4x - 7 = 5x - 4$
 $\therefore x^2 + x + 3 = 0$
 which has $a = 1$, $b = 1$, $c = 3$
 $\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2}$
 $= \frac{-1 \pm \sqrt{-11}}{2}$
 $\therefore$ there are no real solutions
 $\therefore$ the graphs do not meet.
- 2 a** $(0.59, 5.59)$ and $(3.41, 8.41)$ **b** $(3, -4)$ touching **c** graphs do not meet
d $(-2.56, -18.81)$ and $(1.56, 1.81)$
- 3 a** $y = x^2$ meets $y = x + 2$
 when $x^2 = x + 2$
 $\therefore x^2 - x - 2 = 0$
 $\therefore (x + 1)(x - 2) = 0$
 $\therefore x = -1$ or 2
 Substituting into $y = x + 2$,
 when $x = -1$, $y = 1$
 and when $x = 2$, $y = 4$
 $\therefore$ the graphs meet at $(-1, 1)$ and $(2, 4)$.
- b** $y = x^2 + 2x - 3$ meets $y = x - 1$
 when $x^2 + 2x - 3 = x - 1$
 $\therefore x^2 + x - 2 = 0$
 $\therefore (x - 1)(x + 2) = 0$
 $\therefore x = 1$ or -2
 Substituting into $y = x - 1$,
 when $x = 1$, $y = 0$
 and when $x = -2$, $y = -3$
 $\therefore$ the graphs meet at $(1, 0)$ and $(-2, -3)$.
- c** $y = 2x^2 - x + 3$ meets $y = 2 + x + x^2$
 when $2x^2 - x + 3 = 2 + x + x^2$
 $\therefore x^2 - 2x + 1 = 0$
 $\therefore (x - 1)^2 = 0$
 $\therefore x = 1$
 Substituting into $y = 2 + x + x^2$,
 when $x = 1$, $y = 2 + 1 + 1 = 4$
 $\therefore$ the graphs meet at $(1, 4)$
- d** Now if $xy = 4$, then $y = \frac{4}{x}$
 $xy = 4$ meets $y = x + 3$
 when $\frac{4}{x} = x + 3$
 $\therefore 4 = x^2 + 3x$
 $\therefore x^2 + 3x - 4 = 0$
 $\therefore (x + 4)(x - 1) = 0$
 $\therefore x = -4$ or 1
 Substituting into $y = x + 3$,
 when $x = -4$, $y = -1$
 and when $x = 1$, $y = 4$
 $\therefore$ the graphs meet at $(-4, -1)$ and $(1, 4)$

EXERCISE 8L

- 1 a** $H(t) = 36t - 2t^2$
has $a = -2$ and $b = 36$
Since $a < 0$,
the graph is 

The maximum height reached occurs

$$\text{when } t = -\frac{b}{2a} = -\frac{36}{2(-2)} = 9$$

i.e., the maximum height is reached after 9 seconds.

- 2 a** $C(x) = x^2 - 24x + 244$
has $a = 1$, $b = -24$ and $c = 244$
Since $a > 0$,
the graph is 

The minimum cost occurs

$$\text{when } x = -\frac{b}{2a} = -\frac{(-24)}{2(1)} = 12$$

i.e., the minimum cost is when twelve skateboards are produced.

b $H(9) = 36 \times 9 - 2 \times 9^2$
 $= 324 - 162$
 $= 162$

$\therefore$ the maximum height reached is 162 m.

c The ball hits the ground when $H(t) = 0$
 $\therefore 36t - 2t^2 = 0$
 $\therefore 2t(18 - t) = 0$
 $\therefore t = 0 \text{ or } 18$

$\therefore$ the ball hits the ground after 18 seconds.

b $C(12) = 12^2 - 24(12) + 244$
 $= 144 - 288 + 244$
 $= 100$

$\therefore$ the minimum cost is \$100.

c $C(0) = 0 - 24(0) + 244$
 $= 244$

$\therefore$ if no skateboards are made, there is a fixed cost of \$244.

- 3 a** The driver applies the brakes when $t = 0$.
Now $v(t) = -\frac{1}{2}t^2 + \frac{1}{2}t + 15 \quad \therefore v(0) = 15$
 $\therefore$ the car was travelling at 15 m/s when the driver applied the brakes.

Note: In the reprint of the text, the function $v(t)$ in the question was altered to be more physically realistic.

- b** $v(t) = -\frac{1}{2}t^2 + \frac{1}{2}t + 15$ has $a = -\frac{1}{2}$, $b = \frac{1}{2}$ and $c = 15$
Since $a < 0$, the graph is


The maximum velocity was when $t = -\frac{b}{2a} = -\frac{\frac{1}{2}}{2(-\frac{1}{2})} = \frac{1}{2}$ sec

i.e., the maximum velocity occurred half a second after the brake was applied.

Since the car was travelling downhill, it was accelerating. Therefore, when the brake was applied, it took a half a second before the deceleration, due to the brake, cancelled the acceleration due to the hill. Therefore the speed of the vehicle increased for a short time after the brake was applied.

c $v(\frac{1}{2}) = -\frac{1}{2}(\frac{1}{2})^2 + \frac{1}{2}(\frac{1}{2}) + 15$
 $= -\frac{1}{8} + \frac{1}{4} + 15$
 $= 15\frac{1}{8}$ m/s

$\therefore$ the maximum velocity was $15\frac{1}{8}$ m/s.

d The car is stopped when $v(t) = 0$.
 $\therefore -\frac{1}{2}t^2 + \frac{1}{2}t + 15 = 0$
 $\therefore t^2 - t - 30 = 0$
 $\therefore (t - 6)(t + 5) = 0$
 $\therefore t = 6 \text{ or } -5$

but $t > 0$, so $t = 6$

$\therefore$ the car is stopped after 6 seconds.

- 4 a** $P(n) = 84n - 45 - 2n^2$ has $a = -2$, $b = 84$ and $c = -45$

Since $a < 0$, the graph is

$$\therefore \text{ the maximum value of } P \text{ is when } n = -\frac{b}{2a} = -\frac{84}{2(-2)} = 21$$

$\therefore$ the maximum profit occurs for a fleet of 21 taxis.

b $P(21) = 84 \times 21 - 45 - 2 \times 21^2$
 $= 837$

$\therefore$ the maximum profit is \$837 per hour.

c $P(0) = -45$, so if no taxis are on the road, \$45 is lost per hour.

- 5 a** Dusk corresponds to $t = 0$.

Now $T(t) = \frac{1}{4}t^2 - 5t + 30$

$\therefore T(0) = 30^\circ\text{C}$

$\therefore$ the temperature at dusk was 30°C .

b $T(t) = \frac{1}{4}t^2 - 5t + 30$

has $a = \frac{1}{4}$, $b = -5$ and $c = 30$

Since $a > 0$,

the graph is

$\therefore$ the temperature was a minimum

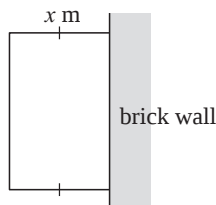
when $t = -\frac{b}{2a} = -\frac{(-5)}{2(\frac{1}{4})} = 10$

Now 10 hours after 7 pm is 5 am, so the temperature was a minimum at 5 am.

c $T(10) = \frac{1}{4} \times 10^2 - 5 \times 10 + 30$
 $= 5^\circ\text{C}$

$\therefore$ the minimum temperature was 5°C .

- 6 a**



If the sides shown are x m long, and the total length of fencing is 40 m, then the other side must have length $(40 - 2x)$ m.

$\therefore$ the area is $A = x(40 - 2x) \text{ m}^2$
 i.e., $A = -2x^2 + 40x \text{ m}^2$

b $A = -2x^2 + 40x$

has $a = -2$ and $b = 40$

Since $a < 0$,

the graph of A has shape

$$\therefore \text{ the area } A \text{ is maximised when } x = -\frac{b}{2a} = -\frac{40}{2(-2)} = 10$$

$\therefore$ the area is maximised when $x = 10$.

c $A(10) = -2 \times 10^2 + 40 \times 10$
 $= -200 + 400$
 $= 200 \text{ m}^2$

$\therefore$ the maximum area is 200 m^2 .

- 7 a** With the axes as described, the parabola has vertex $(0, 70)$.

$\therefore$ its equation is $y = ax^2 + 70$ for some $a \neq 0$.

The end of the bridge is 80 m from A,

so the arch meets the vertical end supports at the point $(80, 6)$.

Since $(80, 6)$ must lie on the curve, $6 = a(80)^2 + 70$

$\therefore 6400a = -64$

$\therefore a = -\frac{1}{100}$

$\therefore$ the arch has equation $y = -\frac{1}{100}x^2 + 70$

- b** The supports occur every 10 m.

$$\text{When } x = 10, \quad y = -\frac{1}{100} \times 10^2 + 70 = 69 \text{ m}$$

$$\text{When } x = 20, \quad y = -\frac{1}{100} \times 20^2 + 70 = 66 \text{ m}$$

$$\text{When } x = 30, \quad y = -\frac{1}{100} \times 30^2 + 70 = 61 \text{ m}$$

$$\text{When } x = 40, \quad y = -\frac{1}{100} \times 40^2 + 70 = 54 \text{ m}$$

$$\text{When } x = 50, \quad y = -\frac{1}{100} \times 50^2 + 70 = 45 \text{ m}$$

$$\text{When } x = 60, \quad y = -\frac{1}{100} \times 60^2 + 70 = 34 \text{ m}$$

$$\text{When } x = 70, \quad y = -\frac{1}{100} \times 70^2 + 70 = 21 \text{ m}$$

$\therefore$ the other supports have lengths 21 m, 34 m, 45 m, 54 m, 61 m, 66 m and 69 m.

- 8 a** The maximum sag occurs in the middle of the cable.

This is halfway between the towers, which are at $x = 0$ and at $x = 60$ m

$\therefore$ the maximum sag is when $x = 30$ m.

The towers are 50 m and the sag is 20 m, $\therefore$ at the vertex $y = 50 - 20 = 30$ m

$\therefore$ the vertex of the parabola is at (30, 30).

- b** Using **a**, the parabola has equation

$$y = a(x - 30)^2 + 30 \quad \text{for some } a \neq 0.$$

$$\text{When } x = 0, \quad y = 50$$

$$\therefore 50 = a(0 - 30)^2 + 30$$

$$\therefore 900a = 20$$

$$\therefore a = \frac{20}{900} = \frac{1}{45}$$

$$\therefore \text{the equation is } y = \frac{1}{45}(x - 30)^2 + 30$$

- c** Point X lies at $x = 50$ m

$$\therefore y = \frac{1}{45}(50 - 30)^2 + 30$$

$$= \frac{1}{45} \times 400 + 30$$

$$= \frac{350}{9}$$

$$\div 38.89$$

$\therefore$ the cable is about 38.9 m above X.

REVIEW SET 8A

- 1 a** The x -intercepts are -2 and 1 .

- b** The axis of symmetry lies midway between the x -intercepts, so its equation is $x = -\frac{1}{2}$.

- c** When $x = -\frac{1}{2}$, $y = -2(-\frac{1}{2} + 2)(-\frac{1}{2} - 1)$ **e**

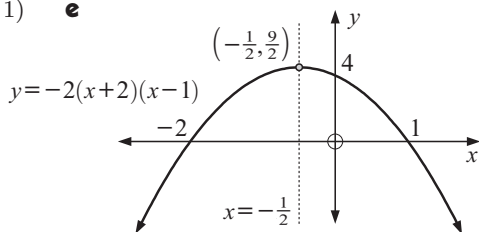
$$= -2(\frac{3}{2})(-\frac{3}{2})$$

$$= \frac{9}{2}$$

$\therefore$ the vertex is $(-\frac{1}{2}, \frac{9}{2})$

- d** When $x = 0$, $y = -2(2)(-1) = 4$

$\therefore$ the y -intercept is 4



- 2 a** The axis of symmetry is $x = 2$.

- b** When $x = 2$, $y = \frac{1}{2}(2 - 2)^2 - 4$

$$= -4$$

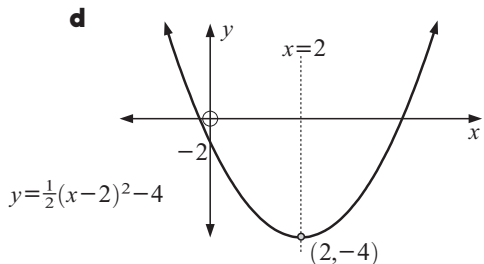
$\therefore$ the vertex is (2, -4)

- c** When $x = 0$, $y = \frac{1}{2}(-2)^2 - 4$

$$= 2 - 4$$

$$= -2$$

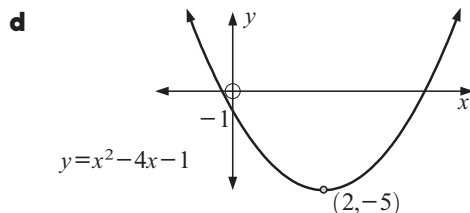
$\therefore$ the y -intercept is -2



$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad y &= x^2 - 4x - 1 \\ &= x^2 - 4x + 4 - 4 - 1 \\ &= (x - 2)^2 - 5 \end{aligned}$$

b The vertex is $(2, -5)$

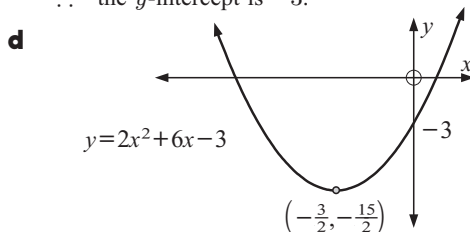
c When $x = 0$, $y = (-2)^2 - 5 = -1$
 $\therefore$ the y -intercept is -1



$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad y &= 2x^2 + 6x - 3 \\ &= 2\left[x^2 + 3x - \frac{3}{2}\right] \\ &= 2\left[x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - \frac{3}{2}\right] \\ &= 2\left[x^2 + 3x + \frac{9}{4} - \frac{9}{4} - \frac{3}{2}\right] \\ &= 2\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{2} - 3 \\ &= 2\left(x + \frac{3}{2}\right)^2 - \frac{15}{2} \end{aligned}$$

b The vertex is $\left(-\frac{3}{2}, -\frac{15}{2}\right)$.

c When $x = 0$, $y = -3$
 $\therefore$ the y -intercept is -3 .



$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad x^2 - 11x &= 60 \\ \therefore x^2 - 11x - 60 &= 0 \\ \therefore (x + 4)(x - 15) &= 0 \\ \therefore x &= -4 \text{ or } 15 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 3x^2 - x - 10 &= 0 \\ \therefore (3x + 5)(x - 2) &= 0 \\ \therefore x &= -\frac{5}{3} \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 3x^2 - 12x &= 0 \\ \therefore 3x(x - 4) &= 0 \\ \therefore x &= 0 \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad x^2 + 10 &= 7x \\ \therefore x^2 - 7x + 10 &= 0 \\ \therefore (x - 2)(x - 5) &= 0 \\ \therefore x &= 2 \text{ or } 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad x + \frac{12}{x} &= 7 \\ \therefore x^2 + 12 &= 7x \\ \therefore x^2 - 7x + 12 &= 0 \\ \therefore (x - 3)(x - 4) &= 0 \\ \therefore x &= 3 \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2x^2 - 7x + 3 &= 0 \\ \therefore (2x - 1)(x - 3) &= 0 \\ \therefore x &= \frac{1}{2} \text{ or } 3 \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad x^2 + 7x - 4 &= 0 \\ \therefore x^2 + 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 - 4 &= 0 \\ \therefore \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} - 4 &= 0 \\ \therefore \left(x + \frac{7}{2}\right)^2 &= \frac{65}{4} \\ \therefore x + \frac{7}{2} &= \pm \frac{\sqrt{65}}{2} \\ \therefore x &= -\frac{7}{2} \pm \frac{\sqrt{65}}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{8} \quad x^2 + 4x + 1 &= 0 \\ \therefore x^2 + 4x + 4 - 4 + 1 &= 0 \\ \therefore (x + 2)^2 &= 3 \\ \therefore x + 2 &= \pm \sqrt{3} \\ \therefore x &= -2 \pm \sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad x^2 - 7x + 3 &= 0 \\ \text{has } a &= 1, \quad b = -7 \quad \text{and} \quad c = 3 \\ \therefore x &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{7 \pm \sqrt{49 - 12}}{2} \\ &= \frac{7 \pm \sqrt{37}}{2} \\ \text{i.e., } x &= \frac{7}{2} \pm \frac{\sqrt{37}}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2x^2 - 5x + 4 &= 0 \\ \text{has } a &= 2, \quad b = -5 \quad \text{and} \quad c = 4 \\ \therefore x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(4)}}{2(2)} \\ &= \frac{5 \pm \sqrt{25 - 32}}{4} \\ \text{i.e., } x &= \frac{5 \pm \sqrt{-7}}{4} \end{aligned}$$

$\therefore x$ has no real solutions.

REVIEW SET 8B

1 $y = -x^2 + 2x = x(2 - x)$

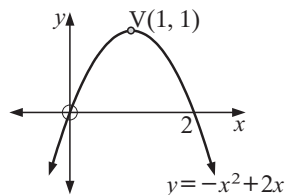
$\therefore$ the graph has x -intercepts 0 and 2, and y -intercept 0

Its axis of symmetry is midway between the x -intercepts,

i.e., at $x = 1$

and when $x = 1$, $y = -1^2 + 2 = 1$

$\therefore$ the vertex is (1, 1)



2 $y = -3x^2 + 8x + 7$ has $a = -3$, $b = 8$ and $c = 7$

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{8}{2(-3)}$

i.e., $x = \frac{4}{3}$

When $x = \frac{4}{3}$, $y = -3(\frac{4}{3})^2 + 8(\frac{4}{3}) + 7$

$$= -\frac{16}{3} + \frac{32}{3} + 7$$

$$= \frac{37}{3}$$

$\therefore$ the axis of symmetry is $x = \frac{4}{3}$ and the vertex is $(\frac{4}{3}, \frac{37}{3})$.

3 *Method 1*

$$y = 2x^2 + 4x - 3$$

$$= 2[x^2 + 2x - \frac{3}{2}]$$

$$= 2[x^2 + 2x + 1 - 1 - \frac{3}{2}]$$

$$= 2[(x+1)^2 - 1 - \frac{3}{2}]$$

$$= 2(x+1)^2 - 2 - 3$$

$$= 2(x+1)^2 - 5$$

$\therefore$ the axis of symmetry is $x = -1$

and the vertex is $(-1, -5)$

Method 2

$$y = 2x^2 + 4x - 3$$

has $a = 2$, $b = 4$, $c = -3$

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{4}{2(2)}$
i.e., $x = -1$

and when $x = -1$,

$$y = 2(-1)^2 + 4(-1) - 3$$

$$= 2 - 4 - 3$$

$$= -5$$

$\therefore$ the axis of symmetry is $x = -1$

and the vertex is $(-1, -5)$.

4 a $3x^2 - 5x + 7 = 0$

has $a = 3$, $b = -5$ and $c = 7$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-5)^2 - 4(3)(7)$$

$$= -59$$

Since $\Delta < 0$, there are no real solutions.

b $-2x^2 - 4x + 3 = 0$

has $a = -2$, $b = -4$ and $c = 3$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-4)^2 - 4(-2)(3)$$

$$= 40$$

Since $\Delta > 0$, there are two real solutions.

5 $5 + 7x + 3x^2$ has $a = 3$, $b = 7$ and $c = 5$

$$\therefore \Delta = b^2 - 4ac$$

$$= 7^2 - 4(3)(5)$$

$$= -11$$

Since $\Delta < 0$, the graph of $y = 5 + 7x + 3x^2$ never cuts or touches the x -axis.

$\therefore$ since $a > 0$, the graph lies completely above the x -axis,

and $5 + 7x + 3x^2$ is positive definite.

- 6** $y = -2x^2 + 4x - 3$ has $a = -2$, $b = 4$ and $c = 3$

Since $a < 0$, the graph is  and will have a maximum.

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{4}{2(-2)}$
i.e., $x = 1$

When $x = 1$, $y = -2(1)^2 + 4(1) + 3$
 $= 5$

$\therefore$ the maximum is 5, and this occurs when $x = 1$.

- 7** $y = x^2 - 3x$ meets $y = 3x^2 - 5x - 24$

when $x^2 - 3x = 3x^2 - 5x - 24$

$$\therefore 2x^2 - 2x - 24 = 0$$

$$\therefore x^2 - x - 12 = 0$$

$$\therefore (x - 4)(x + 3) = 0$$

$$\therefore x = 4 \text{ or } -3$$

Substituting into $y = x^2 - 3x$,

when $x = 4$, $y = 4^2 - 3 \times 4 = 4$

and when $x = -3$, $y = (-3)^2 - 3(-3)$
 $= 9 + 9 = 18$

$\therefore$ the graphs meet at $(4, 4)$ and $(-3, 18)$.

- 8** $y = -2x^2 + 5x + k$

has $a = -2$, $b = 5$ and $c = k$.

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 5^2 - 4(-2)k \\ &= 25 + 8k\end{aligned}$$

The graph does not cut the x -axis

if $\Delta < 0$

$$\therefore 25 + 8k < 0$$

$$\therefore 8k < -25$$

$$\therefore k < -\frac{25}{8}$$

$$\text{i.e., } k < -3\frac{1}{8}$$

- 9 a** The total length of wire for the fence is 60 m.

$$\therefore AB + BC + CD = 60$$

Since the enclosure is rectangular,

$$CD = AB$$

$$\therefore 2AB + x = 60$$

$$\therefore 2AB = 60 - x$$

$$\therefore AB = 30 - \frac{1}{2}x$$

$\therefore$ the area of the rectangle is

$$\begin{aligned}A &= x \left(30 - \frac{1}{2}x \right) \\ &= \left(30x - \frac{1}{2}x^2 \right) \text{ m}^2\end{aligned}$$

- b** $A = 30x - \frac{1}{2}x^2$

has $a = -\frac{1}{2}$ and $b = 30$.

Since $a < 0$, A has a maximum at the axis of symmetry, and this is at

$$x = -\frac{b}{2a} = -\frac{30}{2(-\frac{1}{2})} = 30$$

When $x = 30$, $AB = 30 - \frac{1}{2} \times 30$
 $= 15 \text{ m}$

$\therefore$ the enclosure is 15 m by 30 m.

REVIEW SET 8C

- 1 a** $x^2 + 5x + 3 = 0$

has $a = 1$, $b = 5$, $c = 3$

$$\therefore x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{-5 \pm \sqrt{25 - 12}}{2}$$

$$= -\frac{5}{2} \pm \frac{\sqrt{13}}{2}$$

- b** $3x^2 + 11x - 2 = 0$

has $a = 3$, $b = 11$, $c = -2$

$$\therefore x = \frac{-11 \pm \sqrt{11^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{-11 \pm \sqrt{121 + 24}}{6}$$

$$= -\frac{11}{6} \pm \frac{\sqrt{145}}{6}$$

2 a $x^2 - 5x - 3 = 0$
has $a = 1$, $b = -5$, $c = -3$

$$\begin{aligned}\therefore x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)} \\ &= \frac{5 \pm \sqrt{25 + 12}}{2} \\ &= \frac{5}{2} \pm \frac{\sqrt{37}}{2}\end{aligned}$$

b $2x^2 - 7x - 3 = 0$
has $a = 2$, $b = -7$, $c = -3$

$$\begin{aligned}\therefore x &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{7 \pm \sqrt{49 + 24}}{4} \\ &= \frac{7}{4} \pm \frac{\sqrt{73}}{4}\end{aligned}$$

3 a $x = -5.828$ or -0.1716

b $x = -1.135$ or 1.468

4 a $x = 0.5858$ or 3.414

b $x = -0.1861$ or 2.686

5 a $2x^2 - 5x - 7 = 0$
has $a = 2$, $b = -5$, $c = -7$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(2)(-7) \\ &= 25 + 56 \\ &= 81\end{aligned}$$

$$\therefore \Delta > 0 \text{ and } \sqrt{\Delta} = 9$$

$\therefore$ there are two distinct real rational roots

b $3x^2 - 24x + 48 = 0$
has $a = 3$, $b = -24$, $c = 48$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-24)^2 - 4(3)(48) \\ &= 576 - 576 \\ &= 0\end{aligned}$$

$\therefore$ there is a repeated real root

6 $2x^2 - 3x + m = 0$
has $a = 2$, $b = -3$ and $c = m$
 $\therefore \Delta = b^2 - 4ac$
 $= (-3)^2 - 4(2)m$
 $= 9 - 8m$

b There are two distinct real roots if $\Delta > 0$
i.e., $9 - 8m > 0$
 $\therefore 8m < 9$
 $\therefore m < \frac{9}{8}$

a There is a repeated root if $\Delta = 0$
i.e., $9 - 8m = 0$
 $\therefore m = \frac{9}{8}$

c There are no real roots if $\Delta < 0$
i.e., $9 - 8m < 0$
 $\therefore 8m > 9$
 $\therefore m > \frac{9}{8}$

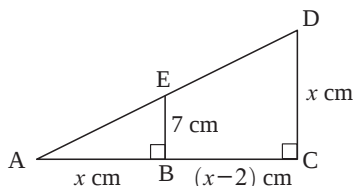
7 $3x^2 + 4x + t = 0$
has $a = 3$, $b = 4$ and $c = t$
 $\therefore \Delta = b^2 - 4ac$
 $= 4^2 - 4(3)t$
 $= 16 - 12t$

b There are two distinct real roots if $\Delta > 0$
i.e., $16 - 12t > 0$
 $\therefore 12t < 16$
 $\therefore t < \frac{4}{3}$

a There is a repeated root if $\Delta = 0$
i.e., $16 - 12t = 0$
 $\therefore 12t = 16$
 $\therefore t = \frac{4}{3}$

c There are no real roots if $\Delta < 0$
i.e., $16 - 12t < 0$
 $\therefore 12t > 16$
 $\therefore t > \frac{4}{3}$

- 8** Suppose AB is x cm in length.
Then, using the information
given, we can label the diagram:



Now by similar triangles, $\frac{BE}{AB} = \frac{CD}{AC}$ $\therefore \frac{7}{x} = \frac{x}{x + (x - 2)}$

$$\therefore \frac{7}{x} = \frac{x}{2x - 2}$$

$$\therefore 7(2x - 2) = x^2$$

$$\therefore 14x - 14 = x^2$$

$$\therefore x^2 - 14x + 14 = 0$$

which has $a = 1$, $b = -14$ and $c = 14$

$$\therefore x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(14)}}{2(1)} = \frac{14 \pm \sqrt{140}}{2}$$

Now $x > 0$, so $x = \frac{14 + \sqrt{140}}{2} \div 12.92$ cm

$\therefore$ AB is approximately 12.9 cm long.

9 Let the hypotenuse have length x cm.

$\therefore$ the longer of the remaining sides has length $(x - 2)$ cm,

and the third side has length $(x - 9)$ cm.

By Pythagoras' theorem,

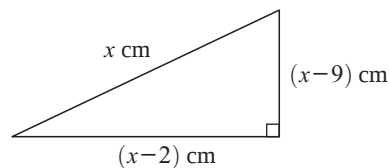
$$(x - 2)^2 + (x - 9)^2 = x^2$$

$$\therefore x^2 - 4x + 4 + x^2 - 18x + 81 = x^2$$

$$\therefore x^2 - 22x + 85 = 0$$

$$\therefore (x - 5)(x - 17) = 0$$

$$\therefore x = 5 \text{ or } 17$$

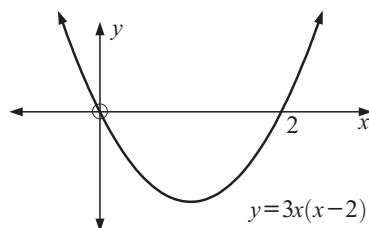


But if x was 5, the shortest side would have negative length.

$\therefore$ the only solution is that the hypotenuse has length 17 cm.

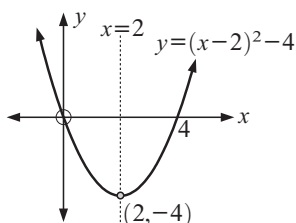
REVIEW SET 8D

- 1** $y = 3x(x - 2)$ has x -intercepts 0 and 2
and y -intercept 0



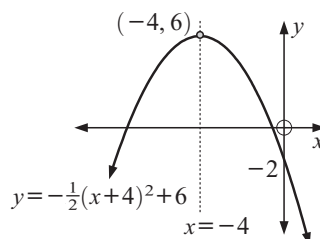
- 2 a** $y = (x - 2)^2 - 4$ has vertex $(2, -4)$
and axis of symmetry $x = 2$.

When $x = 0$, $y = (-2)^2 - 4 = 0$
so the y -intercept is 0.



- b** $y = -\frac{1}{2}(x + 4)^2 + 6$ has vertex $(-4, 6)$
and axis of symmetry $x = -4$.

When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$
so the y -intercept is -2 .



3 a $y = 2x^2 + 4x - 1$

has $a = 2$, $b = 4$ and $c = -1$

The axis of symmetry is $x = -\frac{b}{2a}$
i.e., $x = -\frac{4}{2 \times 2}$

i.e., $x = -1$

c When $x = 0$, $y = -1$,

so the y -intercept is -1

When $y = 0$, $2x^2 + 4x - 1 = 0$

$$\begin{aligned}\therefore x &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{24}}{4} \\ &= \frac{-4 \pm 2\sqrt{6}}{4} = -1 \pm \frac{1}{2}\sqrt{6}\end{aligned}$$

$\therefore$ the x -intercepts are $-1 \pm \frac{1}{2}\sqrt{6}$

4 a $y = 2x^2 + 3x - 7$

has $a = 2$, $b = 3$ and $c = -7$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 3^2 - 4(2)(-7) \\ &= 65\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a > 0$, the graph is



5 a $y = -2x^2 + 3x + 2$

has $a = -2$, $b = 3$ and $c = 2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(2) \\ &= 25\end{aligned}$$

Since $\Delta > 0$, the function is neither positive definite nor negative definite.

6 a The graph has x -intercepts ± 3 , so its equation is
 $y = a(x+3)(x-3)$ for some $a \neq 0$.

Its y -intercept is -27 , so

$$a(3)(-3) = -27$$

$$\therefore -9a = -27$$

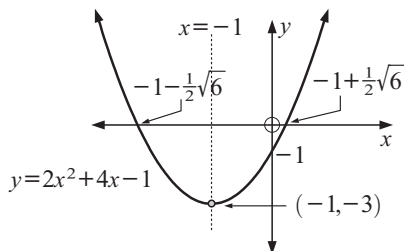
$$\therefore a = 3$$

$\therefore$ the equation is $y = 3(x+3)(x-3)$

b When $x = -1$, $y = 2(-1)^2 + 4(-1) - 1$
 $= 2 - 4 - 1$
 $= -3$

$\therefore$ the vertex is $(-1, -3)$

d



b $y = -3x^2 - 7x + 4$

has $a = -3$, $b = -7$ and $c = 4$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(-3)(4) \\ &= 97\end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a < 0$, the graph is



b $y = 3x^2 + x + 11$

has $a = 3$, $b = 1$ and $c = 11$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 1^2 - 4(3)(11) \\ &= -131\end{aligned}$$

$\therefore \Delta < 0$, and since $a > 0$, the function is positive definite.

b The quadratic has vertex $(2, 25)$
 $\therefore$ its equation is $y = a(x-2)^2 + 25$
The y -intercept is 1, so

$$a(-2)^2 + 25 = 1$$

$$\therefore 4a = -24$$

$$\therefore a = -6$$

$\therefore$ the equation is $y = -6(x-2)^2 + 25$

7 Let the number be x .

$\therefore$ its reciprocal is $\frac{1}{x}$.

$$\therefore x + \frac{1}{x} = 2\frac{1}{30} = \frac{61}{30}$$

$$\therefore x^2 + 1 = \frac{61}{30}x$$

$$\therefore 30x^2 + 30 = 61x$$

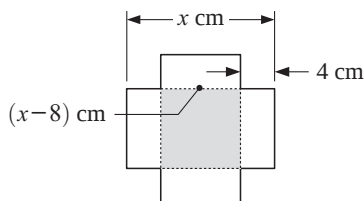
$$\therefore 30x^2 - 61x + 30 = 0$$

$$\therefore (6x - 5)(5x - 6) = 0$$

$$\therefore x = \frac{5}{6} \text{ or } \frac{6}{5}$$

$\therefore$ the number is $\frac{5}{6}$ or $\frac{6}{5}$

8



Since the container has a square base, the original tinplate must have been square.

Suppose its side was x cm long, so the base of the container is $(x - 8)$ cm by $(x - 8)$ cm.

The height of the container is 4 cm, so its capacity is $4(x - 8)(x - 8) \text{ cm}^3$.

$$\therefore 4(x - 8)^2 = 120$$

$$\therefore (x - 8)^2 = 30$$

$$\therefore x - 8 = \pm\sqrt{30}$$

$$\therefore x = 8 \pm \sqrt{30}$$

Clearly, $x > 8$, so $x = 8 + \sqrt{30} \div 13.48$

$\therefore$ the tinplate was about 13.5 cm by 13.5 cm

9 $y = -x^2 - 5x + 3$ meets $y = x^2 + 3x + 11$

when $-x^2 - 5x + 3 = x^2 + 3x + 11$

$$\therefore 2x^2 + 8x + 8 = 0$$

$$\therefore x^2 + 4x + 4 = 0$$

$$\therefore (x + 2)^2 = 0$$

$$\therefore x = -2$$

$\therefore$ the graphs touch at $(-2, 9)$.

Substituting into $y = x^2 + 3x + 11$,

$$\begin{aligned} \text{when } x = -2, \quad y &= (-2)^2 + 3(-2) + 11 \\ &= 4 - 6 + 11 \\ &= 9 \end{aligned}$$

REVIEW SET 8E

1 a The graph has x -intercepts -3 and 1 , so its equation is

$$y = a(x + 3)(x - 1) \text{ for some } a \neq 0.$$

Its y -intercept is 18 , so

$$a(3)(-1) = 18$$

$$\therefore a = -6$$

So the equation is

$$y = -6(x + 3)(x - 1).$$

2 a Since one x -intercept is 7 and the axis of symmetry is $x = 4$, the other x -intercept is $x = 1$.

$\therefore$ the graph has equation

$$y = a(x - 7)(x - 1) \text{ for some } a \neq 0.$$

The y -intercept is -2

$$\therefore a(-7)(-1) = -2$$

$$\therefore a = -\frac{2}{7}$$

$\therefore$ the equation is $y = -\frac{2}{7}(x - 7)(x - 1)$.

b The graph has vertex $(2, -20)$, so its equation is

$$y = a(x - 2)^2 - 20 \text{ for some } a \neq 0.$$

Now an x -intercept is 5

$$\therefore a(5 - 2)^2 - 20 = 0$$

$$\therefore 9a = 20 \text{ and so } a = \frac{20}{9}$$

So the equation is $y = \frac{20}{9}(x - 2)^2 - 20$.

b The graph has vertex $(-3, 0)$, so its equation is

$$y = a(x + 3)^2 \text{ for some } a \neq 0.$$

The y -intercept is 2

$$\therefore a(3)^2 = 2$$

$$\therefore 9a = 2 \text{ and so } a = \frac{2}{9}$$

So the equation is $y = \frac{2}{9}(x + 3)^2$.

- 3** The x -intercepts are 3 and -2 , so the equation is $y = a(x - 3)(x + 2)$ for some $a \neq 0$.

But the y -intercept is 24 $\therefore a(-3)(2) = 24$

$$\therefore -6a = 24$$

$$\therefore a = -4$$

$\therefore$ the equation is $y = -4(x - 3)(x + 2)$ i.e., $y = -4(x^2 - x - 6)$

$$\text{i.e., } y = -4x^2 + 4x + 24$$

- 4** The graph touches the x -axis at 4, so its vertex is $(4, 0)$.

$\therefore$ its equation is $y = a(x - 4)^2$ for some $a \neq 0$.

The graph also passes through $(2, 12)$ $\therefore a(2 - 4)^2 = 12$

$$\therefore 4a = 12$$

$$\therefore a = 3$$

$\therefore$ the equation is $y = 3(x - 4)^2$ i.e., $y = 3(x^2 - 8x + 16)$

$$\text{i.e., } y = 3x^2 - 24x + 48$$

- 5** The quadratic has vertex $(-4, 1)$, so its equation is $y = a(x + 4)^2 + 1$ for some $a \neq 0$.

The graph also passes through $(1, 11)$

$$\therefore 11 = a(1 + 4)^2 + 1$$

$$\therefore 25a = 10$$

$$\therefore a = \frac{2}{5}$$

$\therefore$ the equation is $y = \frac{2}{5}(x + 4)^2 + 1$ i.e., $y = \frac{2}{5}(x^2 + 8x + 16) + 1$

$$\text{i.e., } y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$$

- 6 a** $y = 3x^2 + 4x + 7$
has $a = 3$, $b = 4$ and $c = 7$
Since $a > 0$,
the graph is



and so has a minimum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

$$\text{i.e., } x = -\frac{4}{2(3)} = -\frac{2}{3}$$

When $x = -\frac{2}{3}$,

$$y = 3\left(-\frac{2}{3}\right)^2 + 4\left(-\frac{2}{3}\right) + 7$$

$$= \frac{4}{3} - \frac{8}{3} + 7$$

$$= \frac{17}{3}$$

$\therefore$ the minimum is $\frac{17}{3}$ when $x = -\frac{2}{3}$

- b** $y = -2x^2 - 5x + 2$
has $a = -2$, $b = -5$ and $c = 2$
Since $a < 0$,
the graph is



and so has a maximum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

$$\text{i.e., } x = -\frac{(-5)}{2(-2)} = -\frac{5}{4}$$

When $x = -\frac{5}{4}$,

$$y = -2\left(-\frac{5}{4}\right)^2 - 5\left(-\frac{5}{4}\right) + 2$$

$$= -\frac{50}{16} + \frac{25}{4} + 2$$

$$= \frac{-25 + 50 + 16}{8}$$

$$= \frac{41}{8}$$

$\therefore$ the maximum is $\frac{41}{8}$ when $x = -\frac{5}{4}$

7 $y = x^2 - 2x + k$ has $a = 1$, $b = -2$ and $c = k$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)k \\ &= 4 - 4k\end{aligned}$$

The graph cuts the x -axis twice if $\Delta > 0$

$$\therefore 4 - 4k > 0$$

$$\therefore 4k < 4$$

$$\therefore k < 1$$

8 a The total length of fencing is

$$(8x + 9y) \text{ m}$$

$$\therefore 8x + 9y = 600$$

$$\therefore 9y = 600 - 8x$$

$$\therefore y = \frac{600 - 8x}{9}$$

c $A = x \left(\frac{600 - 8x}{9} \right)$

$$= \frac{600}{9}x - \frac{8}{9}x^2$$

which has $a = -\frac{8}{9}$, $b = \frac{600}{9}$

Since $a < 0$, A is maximised at the axis

of symmetry, which is $x = -\frac{b}{2a}$

$$\text{i.e., } x = -\frac{\frac{600}{9}}{2(-\frac{8}{9})}$$

$$\text{i.e., } x = \frac{600}{16} = \frac{75}{2}$$

$$\text{When } x = \frac{75}{2}, y = \frac{600 - 8(\frac{75}{2})}{9} = \frac{600 - 300}{9} = \frac{300}{9} = \frac{100}{3}$$

$\therefore$ for maximum area, each pen should be

$$37\frac{1}{2} \text{ m} \times 33\frac{1}{3} \text{ m}$$

9 a $AB = CD = x \text{ cm}$

$$\therefore \text{ since } AB + BC + CD = 24 \text{ cm,}$$

$$BC = 24 - 2x \text{ cm}$$

$$\therefore \text{ the cross-sectional area is } A = x(24 - 2x) \text{ cm}^2$$

b $A = x(24 - 2x)$

$$= 24x - 2x^2 \text{ which has } a = -2 \text{ and } b = 24$$

Since $a < 0$, A is maximised at the axis of symmetry, which is $x = -\frac{b}{2a}$

$$\text{i.e., } x = -\frac{24}{2(-2)} = 6$$

$\therefore$ the bends should be 6 cm from the ends of the sheet.

b The area of each pen is

$$A = xy$$

$$= x \left(\frac{600 - 8x}{9} \right) \text{ m}^2$$

d The maximum area of each pen is

$$37\frac{1}{2} \times 33\frac{1}{3}$$

$$= \frac{75}{2} \times \frac{100}{3}$$

$$= 1250 \text{ m}^2$$

Chapter 9

THE BINOMIAL THEOREM

EXERCISE 9A

1 a $(x+1)^3$
 $= x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3$
 $= x^3 + 3x^2 + 3x + 1$

c $(x-4)^3$
 $= x^3 + 3x^2(-4)^1 + 3x(-4)^2 + (-4)^3$
 $= x^3 - 12x^2 + 48x - 64$

e $(2x-1)^3$
 $= (2x)^3 + 3(2x)^2(-1) + 3(2x)(-1)^2 + (-1)^3$
 $= 8x^3 - 12x^2 + 6x - 1$

f $(3x-1)^3$
 $= (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3$
 $= 27x^3 - 27x^2 + 9x - 1$

g $(2x+5)^3$
 $= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$
 $= 8x^3 + 60x^2 + 150x + 125$

b $(x+2)^3$
 $= x^3 + 3x^2(2)^1 + 3x(2)^2 + (2)^3$
 $= x^3 + 6x^2 + 12x + 8$

d $(2x+1)^3$
 $= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3$
 $= 8x^3 + 12x^2 + 6x + 1$

h $\left(2x + \frac{1}{x}\right)^3$
 $= (2x)^3 + 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3$
 $= 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$

2 a $(x+2)^4 = x^4 + 4x^3(2)^1 + 6x^2(2)^2 + 4x(2)^3 + 2^4$
 $= x^4 + 8x^3 + 24x^2 + 32x + 16$

b $(x-2)^4 = x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$
 $= x^4 - 8x^3 + 24x^2 - 32x + 16$

c $(2x+3)^4 = (2x)^4 + 4(2x)^3(3)^1 + 6(2x)^2(3)^2 + 4(2x)(3)^3 + (3)^4$
 $= 16x^4 + 12 \times 8x^3 + 54 \times 4x^2 + 108 \times 2x + 81$
 $= 16x^4 + 96x^3 + 216x^2 + 216x + 81$

d $(3x-1)^4 = (3x)^4 + 4(3x)^3(-1) + 6(3x)^2(-1)^2 + 4(3x)(-1)^3 + (-1)^4$
 $= 81x^4 - 4 \times 27x^3 + 6 \times 9x^2 - 4 \times 3x + 1$
 $= 81x^4 - 108x^3 + 54x^2 - 12x + 1$

e $\left(x + \frac{1}{x}\right)^4 = x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4$
 $= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

f $\left(2x - \frac{1}{x}\right)^4 = (2x)^4 + 4(2x)^3\left(-\frac{1}{x}\right) + 6(2x)^2\left(-\frac{1}{x}\right)^2 + 4(2x)\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4$
 $= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$

3 a $(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5$
 $= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

$$\begin{aligned}\mathbf{b} \quad (x-2)^5 &= x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (2x+1)^5 &= (2x)^5 + 5(2x)^4(1) + 10(2x)^3(1)^2 + 10(2x)^2(1)^3 + 5(2x)(1)^4 + (1)^5 \\ &= 32x^5 + 5 \times 16x^4 + 10 \times 8x^3 + 10 \times 4x^2 + 10x + 1 \\ &= 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad \left(2x - \frac{1}{x}\right)^5 &= (2x)^5 + 5(2x)^4\left(-\frac{1}{x}\right) + 10(2x)^3\left(-\frac{1}{x}\right)^2 + 10(2x)^2\left(-\frac{1}{x}\right)^3 + 5(2x)\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5 \\ &= 32x^5 - 80x^3 + 80x - \frac{40}{x} + \frac{10}{x^3} - \frac{1}{x^5}\end{aligned}$$

$$\begin{array}{llllllll}\mathbf{4} & \mathbf{a} & 1 & 5 & 10 & 10 & 5 & 1 & \longleftarrow \text{the 5th row} \\ & & 1 & 6 & 15 & 20 & 15 & 6 & 1 & \longleftarrow \text{the 6th row}\end{array}$$

$$\begin{aligned}\mathbf{b} \quad \mathbf{i} \quad (x+2)^6 &= x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6 \\ &= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64\end{aligned}$$

$$\begin{aligned}\mathbf{ii} \quad (2x-1)^6 &= (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4 \\ &\quad + 6(2x)(-1)^5 + (-1)^6 \\ &= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1 \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1\end{aligned}$$

$$\begin{aligned}\mathbf{iii} \quad \left(x + \frac{1}{x}\right)^6 &= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}\end{aligned}$$

$$\begin{aligned}\mathbf{5} \quad \mathbf{a} \quad (1+\sqrt{2})^3 &= (1)^3 + 3(1)^2(\sqrt{2}) + 3(1)(\sqrt{2})^2 + (\sqrt{2})^3 \\ &= 1 + 3\sqrt{2} + 3 \times 2 + 2 \times \sqrt{2} \\ &= 1 + 3\sqrt{2} + 6 + 2\sqrt{2} \\ &= 7 + 5\sqrt{2}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (1+\sqrt{5})^4 &= (1)^4 + 4(1)^3(\sqrt{5}) + 6(1)^2(\sqrt{5})^2 + 4(1)(\sqrt{5})^3 + (\sqrt{5})^4 \\ &= 1 + 4\sqrt{5} + 30 + 20\sqrt{5} + 25 \\ &= 56 + 24\sqrt{5}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (2-\sqrt{2})^5 &= (2)^5 + 5(2)^4(-\sqrt{2}) + 10(2)^3(-\sqrt{2})^2 + 10(2)^2(-\sqrt{2})^3 + 5(2)^1(-\sqrt{2})^4 + (-\sqrt{2})^5 \\ &= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2} \\ &= 232 - 164\sqrt{2}\end{aligned}$$

$$\begin{aligned}\mathbf{6} \quad \mathbf{a} \quad (2+x)^6 &= (2)^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6 \\ &= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6\end{aligned}$$

$$\mathbf{b} \quad (2.01)^6 \text{ is obtained by letting } x = 0.01$$

$$\begin{aligned}\therefore (2.01)^6 &= 64 + 192 \times (0.01) + 240 \times (0.01)^2 + 160 \times (0.01)^3 + 60 \times (0.01)^4 \\ &\quad + 12 \times (0.01)^5 + (0.01)^6 \\ &= 65.944\,160\,601\,201\end{aligned}$$

$$\begin{array}{r}64 \\ 1.92 \\ 0.024 \\ 0.000\,16 \\ 0.000\,000\,6 \\ 0.000\,000\,001\,2 \\ 0.000\,000\,000\,001 \\ \hline 65.944\,160\,601\,201\end{array}$$

$$\begin{aligned}7 \quad (2x+3)(x+1)^4 &= (2x+3)(x^4+4x^3+6x^2+4x+1) \\ &= 2x^5+8x^4+12x^3+8x^2+2x \\ &\quad + 3x^4+12x^3+18x^2+12x+3 \\ &= 2x^5+11x^4+24x^3+26x^2+14x+3\end{aligned}$$

$$8 \quad \mathbf{a} \quad (3a+b)^5 = (3a)^5 + 5(3a)^4b + 10(3a)^3b^2 + \dots$$

$$\therefore \text{the coefficient of } a^3b^2 \text{ is } 10 \times 3^3 = 270$$

$$\mathbf{b} \quad (2a+3b)^6 = (2a)^6 + 6(2a)^5(3b) + 15(2a)^4(3b)^2 + 20(2a)^3(3b)^3 + \dots$$

$$\therefore \text{the coefficient of } a^3b^3 \text{ is } 20 \times 2^3 \times 3^3 = 4320$$

EXERCISE 9B

$$\begin{aligned}1 \quad \mathbf{a} \quad (1+2x)^{11} &= 1^{11} + \binom{11}{1}1^{10}(2x)^1 + \binom{11}{2}1^9(2x)^2 + \dots + \binom{11}{10}1^1(2x)^{10} + \binom{11}{11}(2x)^{11} \\ &= 1 + \binom{11}{1}(2x) + \binom{11}{2}(2x)^2 + \dots + \binom{11}{10}(2x)^{10} + \binom{11}{11}(2x)^{11}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \left(3x + \frac{2}{x}\right)^{15} &= (3x)^{15} + \binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right) + \binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2 + \dots + \binom{15}{14}(3x)\left(\frac{2}{x}\right)^{14} + \binom{15}{15}\left(\frac{2}{x}\right)^{15}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \left(2x - \frac{3}{x}\right)^{20} &= (2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right) + \binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2 + \dots + \binom{20}{19}(2x)\left(-\frac{3}{x}\right)^{19} + \binom{20}{20}\left(-\frac{3}{x}\right)^{20}\end{aligned}$$

$$2 \quad \mathbf{a} \quad \text{For } (2x+5)^{15}, \quad a = (2x), \quad b = 5 \quad \text{and} \quad n = 15$$

$$\text{Now } T_{r+1} = \binom{n}{r}a^{n-r}b^r \quad \text{and letting } r = 5 \text{ gives } T_6 = \binom{15}{5}(2x)^{10}5^5.$$

$$\mathbf{b} \quad \text{For } \left(x^2 + \frac{5}{x}\right)^9, \quad a = (x^2), \quad b = \left(\frac{5}{x}\right) \quad \text{and} \quad n = 9$$

$$\text{Now } T_{r+1} = \binom{n}{r}a^{n-r}b^r \quad \text{and letting } r = 3 \text{ gives } T_4 = \binom{9}{3}(x^2)^6\left(\frac{5}{x}\right)^3.$$

$$\mathbf{c} \quad \text{For } \left(x - \frac{2}{x}\right)^{17}, \quad a = x, \quad b = \left(-\frac{2}{x}\right) \quad \text{and} \quad n = 17$$

$$\text{Now } T_{r+1} = \binom{n}{r}a^{n-r}b^r \quad \text{and letting } r = 9 \text{ gives } T_{10} = \binom{17}{9}x^8\left(-\frac{2}{x}\right)^9$$

$$\mathbf{d} \quad \text{For } \left(2x^2 - \frac{1}{x}\right)^{21}, \quad a = (2x^2), \quad b = \left(-\frac{1}{x}\right) \quad \text{and} \quad n = 21$$

$$\text{Now } T_{r+1} = \binom{n}{r}a^{n-r}b^r \quad \text{and letting } r = 8 \text{ gives } T_9 = \binom{21}{8}(2x^2)^{13}\left(-\frac{1}{x}\right)^8.$$

3 a In $(3 + 2x^2)^{10}$, $a = 3$, $b = (2x^2)$ and $n = 10$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{10}{r} 3^{10-r} (2x^2)^r \\ &= \binom{10}{r} 3^{10-r} 2^r x^{2r}\end{aligned}$$

$$\begin{aligned}\text{We now let } 2r &= 10 \\ \therefore r &= 5 \\ \text{So, } T_6 &= \underbrace{\binom{10}{5} 3^5 2^5}_{\text{coefficient}} x^{10} \\ \therefore \text{ the coefficient is } & \binom{10}{5} 3^5 2^5.\end{aligned}$$

b In $(2x^2 - \frac{3}{x})^6$, $a = (2x^2)$, $b = (-\frac{3}{x})$ and $n = 6$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (2x^2)^{6-r} \left(-\frac{3}{x}\right)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} \frac{(-3)^r}{x^r} \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-3r}\end{aligned}$$

$$\begin{aligned}\text{We now let } 12 - 3r &= 3 \\ \therefore 3r &= 9 \\ \therefore r &= 3 \\ \text{So, } T_4 &= \underbrace{\binom{6}{3} 2^3 (-3)^3}_{\text{coefficient}} x^3 \\ \therefore \text{ the coefficient is } & \binom{6}{3} 2^3 (-3)^3.\end{aligned}$$

c In $(2x^2 - \frac{1}{x})^{12}$, $a = (2x^2)$, $b = (-\frac{1}{x})$ and $n = 12$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{12}{r} (2x^2)^{12-r} \left(-\frac{1}{x}\right)^r \\ &= \binom{12}{r} 2^{12-r} x^{24-2r} \frac{(-1)^r}{x^r} \\ &= \binom{12}{r} 2^{12-r} (-1)^r x^{24-3r}\end{aligned}$$

$$\begin{aligned}\text{We now let } 24 - 3r &= 12 \\ \therefore 3r &= 12 \\ \therefore r &= 4 \\ \text{So, } T_5 &= \underbrace{\binom{12}{4} 2^8 (-1)^4}_{\text{coefficient}} x^{12} \\ \therefore \text{ the coefficient is } & \binom{12}{4} 2^8.\end{aligned}$$

4 a For $(x + \frac{2}{x^2})^{15}$, $a = x$, $b = \frac{2}{x^2}$ and $n = 15$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{15}{r} x^{15-r} \left(\frac{2}{x^2}\right)^r \\ &= \binom{15}{r} x^{15-r} \frac{2^r}{x^{2r}} \\ &= \binom{15}{r} 2^r x^{15-3r}\end{aligned}$$

$$\begin{aligned}\text{The constant term does not contain } x. \\ \therefore 15 - 3r &= 0 \\ \therefore r &= 5 \\ \text{and } T_6 &= \binom{15}{5} 2^5 x^0 \\ \therefore \text{ the constant term is } & \binom{15}{5} 2^5.\end{aligned}$$

b For $(x - \frac{3}{x^2})^9$, $a = x$, $b = (-\frac{3}{x^2})$ and $n = 9$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{9}{r} x^{9-r} \left(-\frac{3}{x^2}\right)^r \\ &= \binom{9}{r} x^{9-r} \frac{(-3)^r}{x^{2r}} \\ &= \binom{9}{r} (-3)^r x^{9-3r}\end{aligned}$$

$$\begin{aligned}\text{The constant term does not contain } x. \\ \therefore 9 - 3r &= 0 \\ \therefore r &= 3 \\ \text{and } T_4 &= \binom{9}{3} (-3)^3 x^0 \\ \therefore \text{ the constant term is } & \binom{9}{3} (-3)^3.\end{aligned}$$

5 a, b	Row 1	1 1	←	sum = 1 + 1 = 2	= 2 ¹
	Row 2	1 2 1	←	sum = 1 + 2 + 1 = 4	= 2 ²
	Row 3	1 3 3 1	←	sum = 1 + 3 + 3 + 1 = 8	= 2 ³
	Row 4	1 4 6 4 1	←	sum = 1 + 4 + 6 + 4 + 1 = 16	= 2 ⁴
	Row 5	1 5 10 10 5 1	←	sum = 1 + 5 + 10 + 10 + 5 + 1 = 32	= 2 ⁵

c It seems that the sum of the numbers in row n of Pascal's triangle is 2^n .

$$\begin{aligned} \mathbf{d} \quad (1+x)^n &= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}x + \binom{n}{2}1^{n-2}x^2 + \binom{n}{3}1^{n-3}x^3 + \dots + \binom{n}{n-1}1^1x^{n-1} + \binom{n}{n}x^n \\ &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \\ &\quad \{\text{as all powers of 1 are 1}\} \end{aligned}$$

$$\text{Now letting } x = 1 \text{ gives } \text{LHS} = (1+1)^n = 2^n$$

$$\text{and } \text{RHS} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

$$\therefore \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad &(x+2)(x^2+1)^8 \\ &= (x+2) \left[(x^2)^8 + \binom{8}{1}(x^2)^7 \cdot 1 + \binom{8}{2}(x^2)^6 \cdot 1^2 + \dots + \binom{8}{6}(x^2)^2 \cdot 1^6 + \binom{8}{7}(x^2)^1 \cdot 1^7 + \binom{8}{8}1^8 \right] \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad \text{only terms which when multiplied give an } x^5 \end{aligned}$$

$$\therefore \text{coefficient of } x^5 \text{ is } 1 \times \binom{8}{6} = \binom{8}{6} = 28.$$

$$\begin{aligned} \mathbf{b} \quad &(2-x)(3x+1)^9 \\ &= (2-x) \left[(3x)^9 + \binom{9}{1}(3x)^8 + \binom{9}{2}(3x)^7 + \binom{9}{3}(3x)^6 + \binom{9}{4}(3x)^5 + \dots \right] \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\therefore \text{coefficient of } x^6 \text{ is } 2 \times \binom{9}{3} \times 3^6 + (-1) \times \binom{9}{4} \times 3^5 = 2\binom{9}{3}3^6 - \binom{9}{4}3^5 \\ &\qquad \qquad \qquad = 91854 \end{aligned}$$

7 a Notice:

$$\binom{n}{1} = n \quad \{\text{the 2nd member in each row of Pascal's triangle}\}$$

$$\binom{3}{1} = 3 \quad \text{and} \quad \binom{3}{2} = \frac{3 \times 2}{2} = 3 \quad \checkmark$$

$$\binom{8}{1} = 8 \quad \text{and} \quad \binom{8}{2} = \frac{8 \times 7}{2} = 28 \quad \checkmark$$

$$\binom{20}{1} = 20 \quad \text{and} \quad \binom{20}{2} = \frac{20 \times 19}{2} = 190 \quad \checkmark$$

$$\begin{array}{ccccccccc} & & & & 1 & 1 & & & & \text{these are } \binom{n}{1} \text{ values} \\ & & & 1 & 2 & 1 & & & & \\ & & 1 & 3 & 3 & 1 & & & & \\ & 1 & 4 & 6 & 4 & 1 & & & & \\ 1 & 5 & 10 & 10 & 5 & 1 & & & & \end{array}$$

$$\mathbf{b} \quad (1+x)^n \text{ has } T_3 = \binom{n}{2}1^{n-2}x^2 \text{ and } n \geq 2$$

$$= \binom{n}{2}x^2$$

$$\text{But this term is } 36x^2 \quad \therefore \binom{n}{2} = 36$$

$$\therefore \frac{n(n-1)}{2} = 36$$

$$\therefore n(n-1) = 72$$

$$\therefore n^2 - n - 72 = 0$$

$$\therefore (n-9)(n+8) = 0$$

$$\therefore n = 9 \text{ or } -8$$

$$\text{But } n \geq 2 \quad \therefore n = 9$$

$$\begin{aligned} \text{and } T_4 &= \binom{n}{3}1^{n-3}x^3 \\ &= \binom{9}{3}x^3 \\ &= 84x^3 \end{aligned}$$

$$\begin{aligned} \text{c } (1+kx)^n &= 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + \dots \\ &= 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + \dots \end{aligned}$$

$$\therefore \binom{n}{1}k = -12 \quad \text{and} \quad \binom{n}{2}k^2 = 60$$

$$\therefore nk = -12 \quad \text{and} \quad \frac{n(n-1)}{2}k^2 = 60$$

$$\therefore n(n-1)k^2 = 120$$

$$\text{But } k = -\frac{12}{n} \quad \therefore n(n-1)\frac{144}{n^2} = 120$$

$$\therefore 144(n-1) = 120n \quad \{n \geq 2\}$$

$$\therefore 144n - 120n = 144$$

$$\therefore 24n = 144$$

$$\therefore n = 6$$

$$\text{and so, as } k = -\frac{12}{n}, \quad k = -2$$

REVIEW SET 9

1 The sixth row of Pascal's triangle is 1 6 15 20 15 6 1

$$\therefore (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$\begin{aligned} \text{a } (x-3)^6 &= x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6 \\ &= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729 \end{aligned}$$

$$\begin{aligned} \text{b } \left(1 + \frac{1}{x}\right)^6 &= (1)^6 + 6(1)^5\left(\frac{1}{x}\right) + 15(1)^4\left(\frac{1}{x}\right)^2 + 20(1)^3\left(\frac{1}{x}\right)^3 + 15(1)^2\left(\frac{1}{x}\right)^4 + 6(1)\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= 1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6} \end{aligned}$$

2 In the expansion of $(2x+5)^6$, $a = (2x)$, $b = 5$, $n = 6$

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r \quad \text{For the coefficient of } x^3 \text{ we let } 6-r = 3$$

$$= \binom{6}{r}(2x)^{6-r}5^r \quad \text{and} \quad T_4 = \binom{6}{3}2^35^3x^3 \quad \therefore r = 3$$

$$= \binom{6}{r}2^{6-r}x^{6-r}5^r$$

$$\therefore \text{the coefficient is } \binom{6}{3}2^35^3 = 20\,000.$$

$$\begin{aligned} \text{3 } (\sqrt{3}+2)^5 &= (\sqrt{3})^5 + 5(\sqrt{3})^4(2) + 10(\sqrt{3})^3(2)^2 + 10(\sqrt{3})^2(2)^3 + 5(\sqrt{3})(2)^4 + 2^5 \\ &= 9\sqrt{3} + 90 + 120\sqrt{3} + 240 + 80\sqrt{3} + 32 \\ &= 362 + 209\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{4 } (4+x)^3 &= 4^3 + 3(4)^2x^1 + 3(4)^1x^2 + x^3 \\ &= 64 + 48x + 12x^2 + x^3 \end{aligned}$$

$$\begin{aligned} \text{Letting } x = 0.02 \text{ gives } (4.02)^3 &= 64 + 48(0.02) + 12(0.02)^2 + (0.02)^3 \\ &= 64 + 0.96 + 0.0048 + 0.000\,008 \\ &= 64.964\,808 \end{aligned}$$

5 For $\left(3x^2 + \frac{1}{x}\right)^8$, $a = (3x^2)$, $b = \left(\frac{1}{x}\right)$, $n = 8$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$= \binom{8}{r} (3x^2)^{8-r} \left(\frac{1}{x}\right)^r$$

$$= \binom{8}{r} 3^{8-r} x^{16-2r-r}$$

$$= \binom{8}{r} 3^{8-r} x^{16-3r}$$

Now a constant term does not contain x

$$\therefore 16 - 3r = 0$$

$$\therefore 3r = 16$$

$$\therefore r = 5\frac{1}{3}$$

which is impossible as r is in $\mathbf{Z}$

$\therefore$ no constant term exists.

6 $\left(3a^2 - \frac{2}{b}\right)^5$

$$= (3a^2)^5 + 5(3a^2)^4 \left(-\frac{2}{b}\right) + 10(3a^2)^3 \left(-\frac{2}{b}\right)^2 + 10(3a^2)^2 \left(-\frac{2}{b}\right)^3 + 5(3a^2)^1 \left(-\frac{2}{b}\right)^4 + \left(-\frac{2}{b}\right)^5$$

$$= 243a^{10} - \frac{810a^8}{b} + \frac{1080a^6}{b^2} - \frac{720a^4}{b^3} + \frac{240a^2}{b^4} - \frac{32}{b^5}$$

7 In $\left(2x - \frac{3}{x^2}\right)^{12}$, $a = (2x)$, $b = \left(-\frac{3}{x^2}\right)$, $n = 12$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$= \binom{12}{r} (2x)^{12-r} \left(-\frac{3}{x^2}\right)^r$$

$$= \binom{12}{r} 2^{12-r} x^{12-r} \frac{(-3)^r}{x^{2r}}$$

$$= \binom{12}{r} 2^{12-r} (-3)^r x^{12-3r}$$

For the coefficient of x^{-6} we let $12 - 3r = -6$

$$\therefore 3r = 18$$

$$\therefore r = 6$$

$$\text{So, } T_7 = \binom{12}{6} 2^6 (-3)^6 x^{-6}$$

$$\therefore \text{ the coefficient is } \binom{12}{6} 2^6 (-3)^6.$$

8 For $\left(x^2 + \frac{5}{x}\right)^{15}$, $a = (x^2)$, $b = \left(\frac{5}{x}\right)$, $n = 15$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$= \binom{15}{r} (x^2)^{15-r} \left(\frac{5}{x}\right)^r$$

$$= \binom{15}{r} x^{30-2r} \frac{5^r}{x^r}$$

$$= \binom{15}{r} 5^r x^{30-3r}$$

For the coefficient of x^{15} we let $30 - 3r = 15$

$$\therefore 3r = 15$$

$$\therefore r = 5$$

$$\text{So, } T_6 = \binom{15}{5} 5^5 x^{15}$$

$$\therefore \text{ the coefficient is } \binom{15}{5} 5^5.$$

9 $(2x + 3)(x - 2)^6$

$$= (2x + 3) [x^6 + 6x^5(-2) + 15x^4(-2)^2 + \dots]$$

$$\uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow$$

$$\begin{aligned} \therefore \text{ coefficient of } x^5 \text{ is } & 2 \times 15 \times (-2)^2 + 3 \times 6 \times (-2) \\ & = 120 - 36 \\ & = 84 \end{aligned}$$

Chapter 10

PRACTICAL TRIGONOMETRY WITH RIGHT ANGLED TRIANGLES

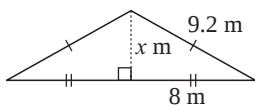
EXERCISE 10A

1 a $1^2 + x^2 = 1.2^2$ {Pythagoras}
 $\therefore x^2 = 1.2^2 - 1^2$
 $\therefore x = \sqrt{1.2^2 - 1^2}$
 $\therefore x \doteq 0.663$

c $x^2 = 1.8^2 + 1.32^2$ {Pythagoras}
 $\therefore x = \sqrt{1.8^2 + 1.32^2}$
 $\therefore x \doteq 2.23$

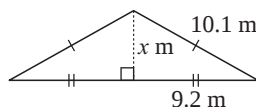
b $x^2 = 3.8^2 + 2.1^2$ {Pythagoras}
 $\therefore x = \sqrt{3.8^2 + 2.1^2}$
 $\therefore x \doteq 4.34$

2 a



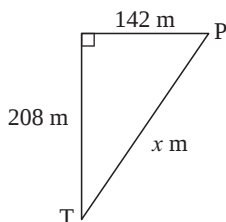
$x^2 + 8^2 = 9.2^2$ {Pythagoras}
 $\therefore x^2 = 9.2^2 - 8^2$
 $\therefore x = \sqrt{9.2^2 - 8^2}$
 $\therefore x \doteq 4.54$
 $\therefore$ the height is 4.54 m

b



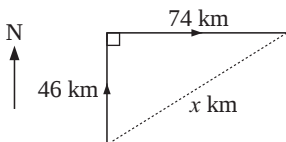
$x^2 + 9.2^2 = 10.1^2$ {Pythagoras}
 $\therefore x^2 = 10.1^2 - 9.2^2$
 $\therefore x = \sqrt{10.1^2 - 9.2^2}$
 $\therefore x \doteq 4.17$
 $\therefore$ the height is 4.17 m

3



$x^2 = 208^2 + 142^2$ {Pythagoras}
 $\therefore x = \sqrt{208^2 + 142^2}$
 $\therefore x \doteq 252$ m
 $\therefore$ must hit the ball (in the air)
 $(252 - 15)$ m
 $= 237$ m

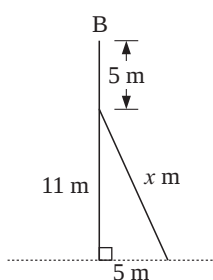
4 a



b $x^2 = 46^2 + 74^2$ {Pythagoras}
 $\therefore x = \sqrt{46^2 + 74^2}$
 $\therefore x \doteq 87.1$
 $\therefore$ is 87.1 km from the start.

EXERCISE 10B

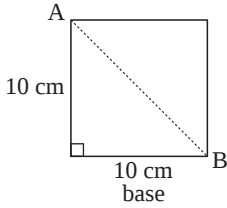
1



$x^2 = 5^2 + 11^2$ {Pythagoras}
 $\therefore x = \sqrt{5^2 + 11^2}$
 $\therefore x \doteq 12.083$

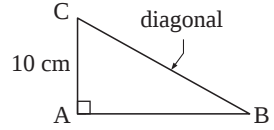
But $4x + 2$ m is needed
 i.e., $(4 \times 12.083 + 2)$ m
 $\doteq 50.3$ m

2

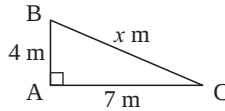
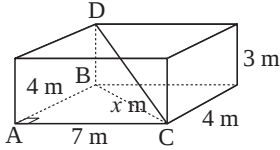


$$\begin{aligned} AB^2 &= 10^2 + 10^2 \quad \{\text{Pythagoras}\} \\ &= 100 + 100 \\ &= 200 \end{aligned}$$

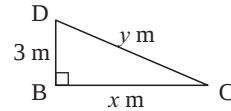
$$\begin{aligned} \text{But } BC^2 &= AB^2 + AC^2 \quad \{\text{Pythagoras, again}\} \\ \therefore BC^2 &= 200 + 10^2 \\ \therefore BC &= \sqrt{300} \div 17.3 \\ \therefore \text{the diagonal is 17.3 cm long.} \end{aligned}$$



3



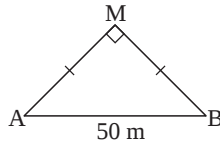
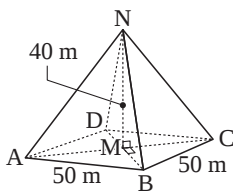
$$x^2 = 4^2 + 7^2 \quad \{\text{Pythagoras}\}$$



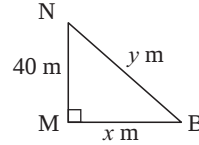
$$\begin{aligned} y^2 &= x^2 + 3^2 \quad \{\text{Pythagoras}\} \\ \therefore y^2 &= 4^2 + 7^2 + 3^2 \\ \therefore y &= \sqrt{4^2 + 7^2 + 3^2} \\ \therefore y &\div 8.60 \end{aligned}$$

So, the distance is about 8.60 m

4



$$\begin{aligned} x^2 + x^2 &= 50^2 \quad \{\text{Pythagoras}\} \\ \therefore 2x^2 &= 50^2 \\ \therefore x^2 &= \frac{50^2}{2} \end{aligned}$$



$$\begin{aligned} y^2 &= 40^2 + x^2 \quad \{\text{Pythagoras}\} \\ \therefore y^2 &= 40^2 + \frac{50^2}{2} \\ \therefore y &= \sqrt{40^2 + \frac{50^2}{2}} \\ \therefore y &\div 53.4 \end{aligned}$$

$\therefore$ each slant edge is 53.4 m long.

5

a Both are right angles.

b

In $\triangle PQR$,
 $PR^2 = a^2 + b^2$
 {Pythagoras}

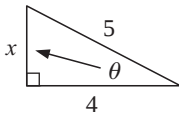
c

In $\triangle KPR$,
 $KR^2 = KP^2 + PR^2$ {Pythagoras}
 $\therefore KR^2 = c^2 + a^2 + b^2$
 $\therefore KR = \sqrt{a^2 + b^2 + c^2}$

EXERCISE 10C

1

a



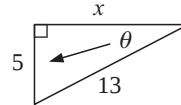
$$\begin{aligned} x^2 + 4^2 &= 5^2 \quad \{\text{Pythagoras}\} \\ \therefore x &= \sqrt{5^2 - 4^2} \\ \therefore x &= 3 \end{aligned}$$

$$\text{So, } \sin \theta = \frac{3}{5}$$

$$\cos \theta = \frac{4}{5}$$

$$\tan \theta = \frac{3}{4}$$

b

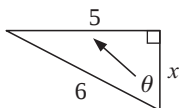


$$\begin{aligned} x^2 + 5^2 &= 13^2 \quad \{\text{Pythagoras}\} \\ \therefore x &= \sqrt{13^2 - 5^2} \\ \therefore x &= 12 \end{aligned}$$

$$\text{So, } \sin \theta = \frac{5}{13}$$

$$\cos \theta = \frac{12}{13}$$

$$\tan \theta = \frac{5}{12}$$

c


$$x^2 + 5^2 = 6^2 \quad \{\text{Pythagoras}\}$$

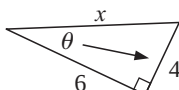
$$\therefore x = \sqrt{6^2 - 5^2}$$

$$\therefore x = \sqrt{11}$$

$$\text{So, } \sin \theta = \frac{5}{6}$$

$$\cos \theta = \frac{\sqrt{11}}{6}$$

$$\tan \theta = \frac{5}{\sqrt{11}}$$

e


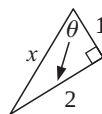
$$x^2 = 4^2 + 6^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{52}$$

$$\text{So, } \sin \theta = \frac{4}{\sqrt{52}}$$

$$\cos \theta = \frac{6}{\sqrt{52}}$$

$$\tan \theta = \frac{4}{6} = \frac{2}{3}$$

d


$$x^2 = 1^2 + 2^2 \quad \{\text{Pythagoras}\}$$

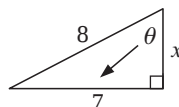
$$\therefore x = \sqrt{1+4}$$

$$\therefore x = \sqrt{5}$$

$$\text{So, } \sin \theta = \frac{2}{\sqrt{5}}$$

$$\cos \theta = \frac{1}{\sqrt{5}}$$

$$\tan \theta = \frac{2}{1} = 2$$

f


$$x^2 + 7^2 = 8^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x^2 = 8^2 - 7^2$$

$$\therefore x = \sqrt{64 - 49}$$

$$\therefore x = \sqrt{15}$$

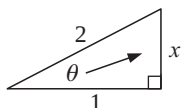
$$\text{So, } \sin \theta = \frac{7}{8}$$

$$\cos \theta = \frac{\sqrt{15}}{8}$$

$$\tan \theta = \frac{7}{\sqrt{15}}$$

2 a

$$\text{as } \cos \theta = \frac{1}{2}$$



$$x^2 + 1^2 = 2^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x^2 = 4 - 1$$

$$\therefore x = \sqrt{3}$$

$$\text{So, } \sin \theta = \frac{x}{2} = \frac{\sqrt{3}}{2}$$

$$\tan \theta = \frac{x}{1} = \sqrt{3}$$

b

$$\text{as } \sin \alpha = \frac{2}{3}$$

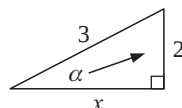
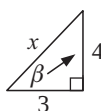
$$\therefore x^2 + 2^2 = 3^2$$

$$\therefore x^2 = 9 - 4$$

$$\therefore x = \sqrt{5}$$

$$\text{So, } \cos \alpha = \frac{x}{3} = \frac{\sqrt{5}}{3}$$

$$\tan \alpha = \frac{2}{x} = \frac{2}{\sqrt{5}}$$


c


$$\text{as } \tan \beta = \frac{4}{3}$$

$$x = 5 \quad \{3-4-5 \Delta\}$$

3 a

$$\sin \theta = \frac{b}{c}$$

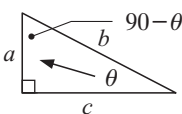
$$\cos \theta = \frac{a}{c}$$

$$\tan \theta = \frac{b}{a}$$

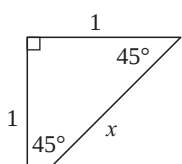
b

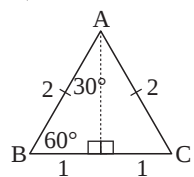
$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$$

$$\therefore \frac{\sin \theta}{\cos \theta} = \tan \theta$$

4 a  **i** $\sin \theta = \frac{a}{b}$ **ii** $\cos \theta = \frac{c}{b}$ **iii** $\sin(90 - \theta) = \frac{c}{b}$ **iv** $\cos(90 - \theta) = \frac{a}{b}$

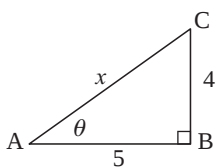
- b** **i** The sine of an angle is the cosine of its complement.
ii The cosine of an angle is the sine of its complement.

5  **a** $x^2 = 1^2 + 1^2 = 2$
 $\therefore x = \sqrt{2}$ {Pythagoras} **b** $\sin 45^\circ = \frac{1}{x} = \frac{1}{\sqrt{2}}$
 $\cos 45^\circ = \frac{1}{x} = \frac{1}{\sqrt{2}}$
 $\tan 45^\circ = \frac{1}{1} = 1$

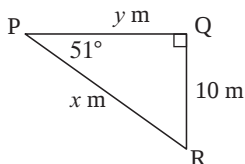
6  **a** $\angle ABN = 60^\circ$
 $\angle BAN = 30^\circ$ **b** $\triangle ABC$ is equilateral and AN is perpendicular to BC .
 $\therefore N$ is the midpoint of BC .
 $\therefore BN = 1$
 $\therefore AN^2 = 2^2 - 1^2$ {Pythagoras}
 $\therefore AN = \sqrt{3}$
c i $\sin 60^\circ = \frac{AN}{2} = \frac{\sqrt{3}}{2}$ **ii** $\sin 30^\circ = \frac{1}{2}$
 $\cos 60^\circ = \frac{1}{2}$ $\cos 30^\circ = \frac{AN}{2} = \frac{\sqrt{3}}{2}$
 $\tan 60^\circ = \frac{AN}{1} = \sqrt{3}$ $\tan 30^\circ = \frac{1}{AN} = \frac{1}{\sqrt{3}}$

EXERCISE 10D

1 a $\sin 35^\circ = \frac{x}{30}$ **b** $\cos 50^\circ = \frac{x}{400}$ **c** $\tan 60^\circ = \frac{x}{8.7}$
 $\therefore 30 \times \sin 35^\circ = x$ $\therefore 400 \times \cos 50^\circ = x$ $\therefore \tan 60^\circ \times 8.7 = x$
 $\therefore x \div 17.2$ $\therefore x \div 257$ $\therefore x \div 15.1$
d $\cos 65^\circ = \frac{3}{x}$ **e** $\tan 36.7^\circ = \frac{413}{x}$ **f** $\sin 53.9 = \frac{369}{x}$
 $\therefore x = \frac{3}{\cos 65^\circ}$ $\therefore x = \frac{413}{\tan 36.7}$ $\therefore x = \frac{369}{\sin 53.9}$
 $\therefore x \div 7.10$ $\therefore x \div 554$ $\therefore x \div 457$
2 a $\sin \theta = 0.9364$ **b** $\cos \theta = 0.2381$ **c** $\tan \theta = 1.7321$
 $\therefore \theta = \sin^{-1}(0.9364)$ $\therefore \theta = \cos^{-1}(0.2381)$ $\therefore \theta = \tan^{-1}(1.7321)$
 $\therefore \theta \div 69.5^\circ$ $\therefore \theta \div 76.2^\circ$ $\therefore \theta \div 60.0^\circ$
d $\cos \theta = \frac{2}{7}$ **e** $\sin \theta = \frac{1}{3}$ **f** $\tan \theta = \frac{14}{3}$
 $\therefore \theta = \cos^{-1}\left(\frac{2}{7}\right)$ $\therefore \theta = \sin^{-1}\left(\frac{1}{3}\right)$ $\therefore \theta = \tan^{-1}\left(\frac{14}{3}\right)$
 $\therefore \theta \div 73.4^\circ$ $\therefore \theta \div 19.5^\circ$ $\therefore \theta \div 77.9^\circ$
g $\sin \theta = \frac{\sqrt{3}}{11}$ **h** $\cos \theta = \frac{5}{\sqrt{37}}$
 $\therefore \theta = \sin^{-1}\left(\frac{\sqrt{3}}{11}\right)$ $\therefore \theta = \cos^{-1}\left(\frac{5}{\sqrt{37}}\right)$
 $\therefore \theta \div 9.06^\circ$ $\therefore \theta \div 34.7^\circ$
3 a $\sin \theta = \frac{5}{6}$ **b** $\tan \alpha = \frac{1}{12}$ **c** $\cos \beta = \frac{4}{6}$
 $\therefore \theta = \sin^{-1}\left(\frac{5}{6}\right)$ $\therefore \alpha = \tan^{-1}\left(\frac{1}{12}\right)$ $\therefore \beta = \cos^{-1}\left(\frac{2}{3}\right)$
 $\therefore \theta \div 56.4$ $\therefore \alpha \div 4.76$ $\therefore \beta \div 48.2$

4 a


$$\begin{aligned}
 x^2 &= 4^2 + 5^2 \quad \{\text{Pythagoras}\} & \tan \theta &= \frac{4}{5} \\
 \therefore x &= \sqrt{4^2 + 5^2} & \therefore \theta &= \tan^{-1}\left(\frac{4}{5}\right) \\
 \therefore x &= \sqrt{41} & \therefore \theta &\doteq 38.7 \\
 \therefore AC &= \sqrt{41} \text{ m} & \therefore \angle A &\doteq 38.7^\circ, \angle C \doteq 51.3^\circ \\
 &\text{or } \doteq 6.40 \text{ m}
 \end{aligned}$$

b


$$\begin{aligned}
 \angle R &= 90^\circ - 51^\circ = 39^\circ & \sin 51^\circ &= \frac{10}{x} & \tan 51^\circ &= \frac{10}{y} \\
 \therefore x &= \frac{10}{\sin 51^\circ} & \therefore y &= \frac{10}{\tan 51^\circ} \\
 \therefore x &\doteq 12.9 & \therefore y &\doteq 8.10 \\
 \text{So, PR} &\doteq 12.9 \text{ m} & \text{So, PQ} &\doteq 8.10 \text{ m}
 \end{aligned}$$

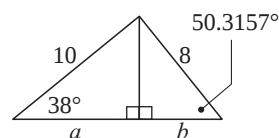
5 a

$$\begin{aligned}
 x^2 + 3^2 &= 4^2 \quad \{\text{Pythagoras}\} & \tan \theta &= \frac{2}{x} = \frac{2}{\sqrt{7}} \\
 \therefore x &= \sqrt{4^2 - 3^2} \\
 \therefore x &= \sqrt{7} \quad (\text{or } 2.65) & \therefore \theta &= \tan^{-1}\left(\frac{2}{\sqrt{7}}\right) \quad \text{and so } \theta \doteq 37.1
 \end{aligned}$$

b

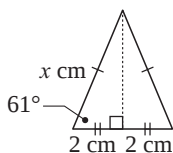
$$\begin{aligned}
 \sin 38^\circ &= \frac{x}{10} \\
 \therefore 10 \times \sin 38^\circ &= x \\
 \therefore x &\doteq 6.16
 \end{aligned}$$

$$\begin{aligned}
 \sin \theta &= \frac{x}{8} \\
 \therefore \sin \theta &\doteq \frac{6.1566}{8} \\
 \therefore \theta &\doteq \sin^{-1}\left(\frac{6.1566}{8}\right) \\
 \therefore \theta &\doteq 50.3
 \end{aligned}$$

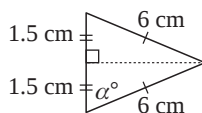


$$\cos 38^\circ = \frac{a}{10} \quad \text{and} \quad \cos 50.3157^\circ = \frac{b}{8}$$

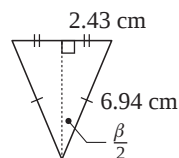
$$\begin{aligned}
 \text{Now } y &= a + b \\
 &= 10 \cos 38^\circ + 8 \cos 50.3157^\circ \\
 &\doteq 13.0
 \end{aligned}$$

6 a


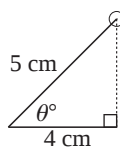
$$\begin{aligned}
 \cos 61^\circ &= \frac{2}{x} \\
 \therefore x &= \frac{2}{\cos 61^\circ} \\
 \therefore x &\doteq 4.13
 \end{aligned}$$

b


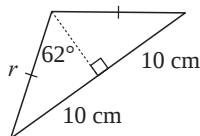
$$\begin{aligned}
 \cos \alpha &= \frac{1.5}{6} \\
 \therefore \alpha &= \cos^{-1}\left(\frac{1.5}{6}\right) \\
 \therefore \alpha &\doteq 75.5
 \end{aligned}$$

c


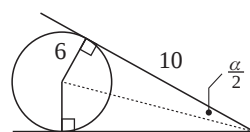
$$\begin{aligned}
 \sin\left(\frac{\beta}{2}\right) &= \frac{2.43}{6.94} \\
 \therefore \beta &= 2 \times \sin^{-1}\left(\frac{2.43}{6.94}\right) \\
 \therefore \beta &\doteq 41.0
 \end{aligned}$$

7 a


$$\begin{aligned}
 \cos \theta &= \frac{4}{5} \\
 \therefore \theta &= \cos^{-1}\left(\frac{4}{5}\right) \\
 \therefore \theta &\doteq 36.9
 \end{aligned}$$

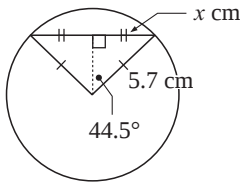
b


$$\begin{aligned}
 \sin 62^\circ &= \frac{10}{r} \\
 \therefore r &= \frac{10}{\sin 62^\circ} \\
 \therefore r &\doteq 11.3
 \end{aligned}$$

c


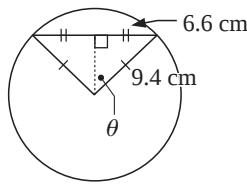
$$\begin{aligned}
 \tan\left(\frac{\alpha}{2}\right) &= \frac{6}{10} = 0.6 \\
 \therefore \alpha &= 2 \times \tan^{-1}(0.6) \\
 \therefore \alpha &\doteq 61.9
 \end{aligned}$$

8



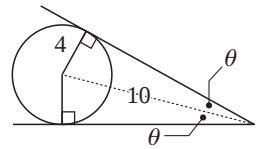
$$\begin{aligned}\sin 44.5^\circ &= \frac{x}{5.7} \\ \therefore 5.7 \times \sin 44.5^\circ &= x \\ \therefore x &\div 3.995 \\ \therefore 2x &\div 7.99 \\ \therefore \text{chord is } 7.99 \text{ cm long.}\end{aligned}$$

9



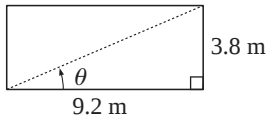
$$\begin{aligned}\sin \theta &= \frac{6.6}{9.4} \\ \therefore \theta &= \sin^{-1}\left(\frac{6.6}{9.4}\right) \\ \therefore \theta &\div 44.6 \\ \therefore 2\theta &\div 89.2 \\ \therefore \text{angle is } 89.2^\circ\end{aligned}$$

10



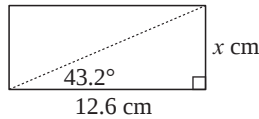
$$\begin{aligned}\sin \theta &= \frac{4}{10} = 0.4 \\ \therefore \theta &= \sin^{-1}(0.4) \\ \therefore 2\theta &= 2 \times \sin^{-1}(0.4) \\ &\div 47.2 \\ \therefore \text{angle is } 47.2^\circ\end{aligned}$$

11



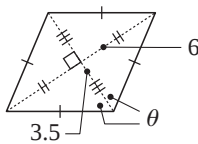
$$\begin{aligned}\tan \theta &= \frac{3.8}{9.2} \\ \therefore \theta &= \tan^{-1}\left(\frac{3.8}{9.2}\right) \\ \therefore \theta &\div 22.4 \\ \therefore \text{the angle is } 22.4^\circ\end{aligned}$$

12



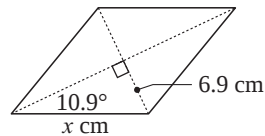
$$\begin{aligned}\tan 43.2^\circ &= \frac{x}{12.6} \\ \therefore 12.6 \times \tan 43.2^\circ &= x \\ \therefore x &\div 11.8 \\ \therefore \text{the shorter side is } 11.8 \text{ cm long}\end{aligned}$$

13



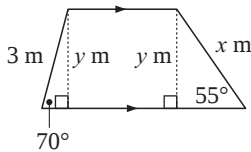
$$\begin{aligned}\tan \theta &= \frac{6}{3.5} \\ \therefore \theta &= \tan^{-1}\left(\frac{6}{3.5}\right) \\ \therefore 2\theta &= 2 \times \tan^{-1}\left(\frac{6}{3.5}\right) \div 119.49 \\ \therefore \text{So, the larger angle is } &\div 119^\circ\end{aligned}$$

14



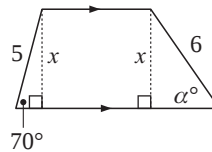
$$\begin{aligned}\sin 10.9^\circ &= \frac{6.9}{x} \\ \therefore x &= \frac{6.9}{\sin 10.9^\circ} \div 36.5 \\ \therefore \text{sides are } 36.5 \text{ cm long}\end{aligned}$$

15 a



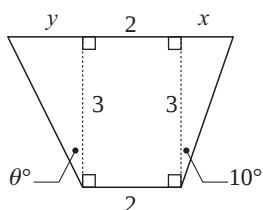
$$\begin{aligned}\sin 70^\circ &= \frac{y}{3} \\ \therefore y &= 3 \times \sin 70^\circ \\ \text{But } \sin 55^\circ &= \frac{y}{x} \\ \therefore \sin 55^\circ &= \frac{3 \times \sin 70^\circ}{x} \\ \therefore x &= \frac{3 \times \sin 70^\circ}{\sin 55^\circ} \div 3.44\end{aligned}$$

b



$$\begin{aligned}\sin 70^\circ &= \frac{x}{5} \\ \therefore 5 \times \sin 70^\circ &= x \\ \text{Now } \sin \alpha &= \frac{x}{6} \\ \therefore \sin \alpha &= \frac{5 \times \sin 70^\circ}{6} \\ \therefore \alpha &= \sin^{-1}\left(\frac{5 \times \sin 70^\circ}{6}\right) \div 51.5\end{aligned}$$

16



$$\tan 10^\circ = \frac{x}{3}$$

$$\text{Now } \tan \theta = \frac{y}{3}$$

$$\therefore 3 \times \tan 10^\circ = x$$

$$\therefore \tan \theta = 0.8237$$

$$\therefore x \doteq 0.5290$$

$$\therefore \theta \doteq 39.48^\circ$$

$$\text{But } y + 2 + x = 5$$

$$\text{So, } \beta \doteq 90 + 39.48^\circ$$

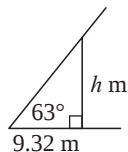
$$\therefore y \doteq 3 - 0.5290$$

$$\therefore \beta \doteq 129^\circ$$

$$\therefore y \doteq 2.471$$

EXERCISE 10E

1



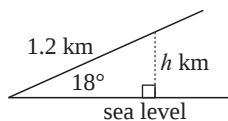
$$\tan 63^\circ = \frac{h}{9.32}$$

$$\therefore \tan 63^\circ \times 9.32 = h$$

$$\therefore h \doteq 18.3$$

$\therefore$ the height is 18.3 m

2 a



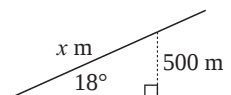
$$\sin 18^\circ = \frac{h}{1.2}$$

$$\therefore 1.2 \times \sin 18^\circ = h$$

$$\therefore h \doteq 0.371$$

$\therefore$ height is 371 m above sea level.

b



$$\sin 18^\circ = \frac{500}{x}$$

$$\therefore x = \frac{500}{\sin 18^\circ}$$

$$\therefore x \doteq 1618$$

$\therefore$ have walked 1.62 km up the hill.

3

Let $AB = x$ m

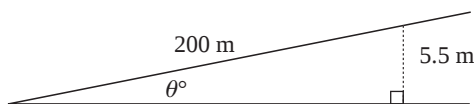
$$\text{Now } \tan 37^\circ = \frac{120}{x}$$

$$\therefore x = \frac{120}{\tan 37^\circ}$$

$$\therefore x \doteq 159$$

$\therefore$ the canal is 159 m wide

4



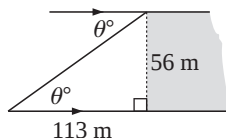
$$\sin \theta = \frac{5.5}{200}$$

$$\therefore \theta = \sin^{-1} \left(\frac{5.5}{200} \right)$$

$$\therefore \theta \doteq 1.58$$

i.e., an incline of 1.58°

5



$$\tan \theta = \frac{56}{113}$$

$$\therefore \theta \doteq \tan^{-1} \left(\frac{56}{113} \right)$$

$$\therefore \theta \doteq 26.4$$

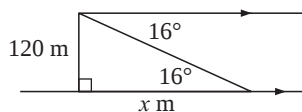
$\therefore$ angle of elevation is about 26.4°

The angle of depression is also θ° ,

i.e., about 26.4°

{equal alternate angles}

6



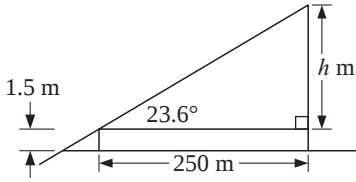
$$\tan 16^\circ = \frac{120}{x}$$

$$\therefore x = \frac{120}{\tan 16^\circ}$$

$$\therefore x \doteq 418$$

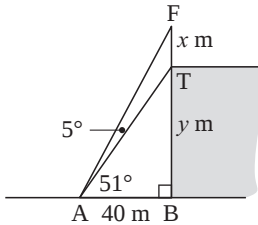
$\therefore$ boat is 418 m out from the base of the cliff.

7



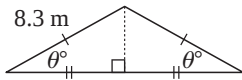
$$\begin{aligned}\tan 23.6^\circ &= \frac{h}{250} \\ \therefore 250 \times \tan 23.6^\circ &= h \\ \therefore h &= 109.2 \\ \therefore \text{tree height} &\div 109.2 + 1.5 \\ &\div 110.7 \text{ m} \\ &\div 111 \text{ m}\end{aligned}$$

9



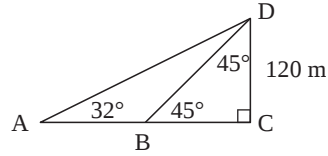
$$\begin{aligned}\tan 51^\circ &= \frac{y}{40} \\ \therefore 40 \times \tan 51^\circ &= y \\ \therefore y &\div 49.396 \\ \tan 56^\circ &= \frac{x+y}{40} \\ \therefore 40 \times \tan 56^\circ &= x+y \\ \therefore x+y &\div 59.302 \\ \therefore x &\div 59.302 - 49.396 \\ \therefore x &\div 9.91 \\ \text{i.e., it is } 9.91 \text{ m high}\end{aligned}$$

11



$$\begin{aligned}7.5 + 0.6 &= 8.1 \text{ m} \\ \cos \theta &= \frac{8.1}{8.3} \\ \therefore \theta &= \cos^{-1} \left(\frac{8.1}{8.3} \right) \\ \therefore \theta &\div 12.6\end{aligned}$$

8



$$\begin{aligned}\triangle BCD &\text{ is right-angled isosceles} \\ \therefore BC &= 120 \text{ m also} \\ \text{In } \triangle ACD, \tan 32^\circ &= \frac{120}{AC} \\ \therefore AC &= \frac{120}{\tan 32^\circ} \\ \therefore AC &\div 192 \\ \therefore \text{she walks } (192 - 120) \text{ m} \\ &\div 72 \text{ m}\end{aligned}$$

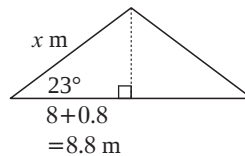
$$10 \text{ a } v = \sqrt{gr \tan \theta}$$

$$\begin{aligned}&= \sqrt{9.8 \times 100 \times \tan 15^\circ} \\ &\div 16.2 \text{ m/s}\end{aligned}$$

b

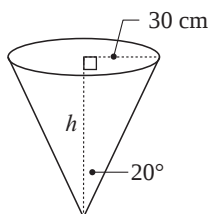
$$\begin{aligned}v &= \sqrt{gr \tan \theta} \\ \therefore 20 &= \sqrt{9.8 \times 200 \times \tan \theta} \\ \therefore 400 &= 9.8 \times 200 \times \tan \theta \\ \therefore \frac{2}{9.8} &= \tan \theta \\ \therefore \theta &= \tan^{-1} \left(\frac{2}{9.8} \right) \\ \therefore \theta &\div 11.5 \\ \text{i.e., a banked angle of } 11.5^\circ\end{aligned}$$

12



$$\begin{aligned}\cos 23^\circ &= \frac{8.8}{x} \\ \therefore x &= \frac{8.8}{\cos 23^\circ} \\ \therefore x &\div 9.56 \\ \therefore \text{a beam is } 9.56 \text{ m long}\end{aligned}$$

13



$$\tan 20^\circ = \frac{30}{h}$$

$$\therefore h = \frac{30}{\tan 20^\circ}$$

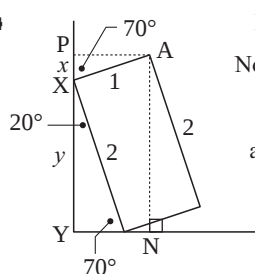
$$\therefore h \doteq 82.42 \text{ cm}$$

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \times 30^2 \times 82.42 \\ &\doteq 77\,683 \text{ cm}^3 \end{aligned}$$

$\therefore$ capacity is 77 683 mL

i.e., $\doteq 77.7 \text{ L}$

14



Height $AN = PY$ also

$$\text{Now } \cos 70^\circ = \frac{x}{1}$$

$$\therefore x = \cos 70^\circ$$

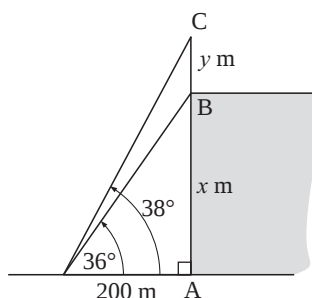
$$\text{and } \sin 70^\circ = \frac{y}{2}$$

$$\therefore y = 2 \sin 70^\circ$$

$$\begin{aligned} \text{Now } PY &= x + y \\ &= \cos 70^\circ + 2 \sin 70^\circ \\ &\doteq 2.22 \text{ m} \end{aligned}$$

$\therefore$ A is 2.22 m above the floor

15



$$\tan 36^\circ = \frac{x}{200}$$

$$\therefore 200 \times \tan 36^\circ = x$$

$$\therefore x \doteq 145.31 \quad \dots (1)$$

$$\tan 38^\circ = \frac{x+y}{200}$$

$$\therefore 200 \times \tan 38^\circ = x + y$$

$$\therefore x + y \doteq 156.26 \quad \dots (2)$$

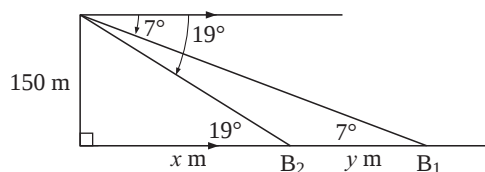
Using (1) and (2)

$$y \doteq 156.26 - 145.31$$

$$\therefore y \doteq 10.9$$

$\therefore$ the pole is about 10.9 m long.

16



$$\tan 19^\circ = \frac{150}{x}$$

$$\therefore x = \frac{150}{\tan 19^\circ} \doteq 435.63 \quad \dots (1)$$

$$\tan 7^\circ = \frac{150}{x+y}$$

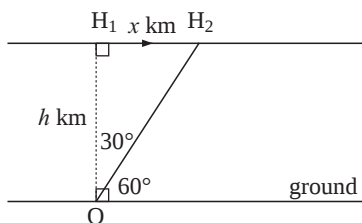
$$\therefore x + y = \frac{150}{\tan 7^\circ} \doteq 1221.65 \quad \dots (2)$$

From (1) and (2)

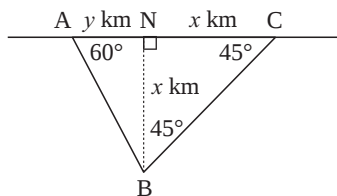
$$y \doteq 1221.65 - 435.63$$

$$\therefore y \doteq 786$$

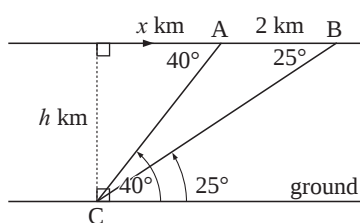
i.e., needs to be 786 m closer in.

17

$$\begin{aligned}\text{speed} &= \frac{\text{distance}}{\text{time}} \\ \therefore 100 &= \frac{x}{\frac{20}{3600} \text{ hours}} \\ \therefore x &= 100 \times \frac{20}{3600} \\ \therefore x &= \frac{20}{36} = \frac{5}{9} \\ \text{Now } \tan 30^\circ &= \frac{\frac{5}{9}}{h} \\ \therefore h &= \frac{\frac{5}{9}}{\tan 30^\circ} \\ \therefore h &\div 0.962 \\ \therefore &\text{ is about 962 m above ground level.}\end{aligned}$$

19

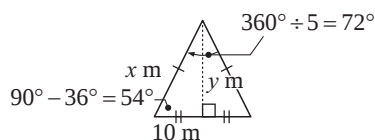
$$\begin{aligned}\triangle NBC \text{ is right angled isosceles} \\ \therefore BN &= x \text{ km} \\ \text{Now } \tan 60^\circ &= \frac{x}{y} \\ \therefore \sqrt{3} &= \frac{x}{5-x} \quad \{\text{as } x+y=5\} \\ \therefore 5\sqrt{3} - x\sqrt{3} &= x \\ \therefore 5\sqrt{3} &= x(1+\sqrt{3}) \\ \therefore \frac{5\sqrt{3}}{1+\sqrt{3}} &= x \\ \therefore x &\div 3.17 \\ \therefore &\text{ is 3.17 km from the shore}\end{aligned}$$

18

$$\begin{aligned}\text{Notice that} \\ \tan 40^\circ &= \frac{h}{x} \\ \text{and } \tan 25^\circ &= \frac{h}{x+2} \\ \therefore h &= x \tan 40^\circ \\ \text{and } h &= (x+2) \tan 25^\circ \\ \therefore x \tan 40^\circ &= (x+2) \tan 25^\circ \\ \therefore \frac{x+2}{x} &= \frac{\tan 40^\circ}{\tan 25^\circ} \\ \therefore \frac{x+2}{x} &\div 1.79945 \\ \therefore x+2 &\div 1.79945x \\ \therefore 2 &\div 0.79945x \\ \therefore x &\div 2.502 \\ \text{So } h &= 2.502 \times \tan 40^\circ \\ \therefore h &\div 2.10 \\ \text{i.e., } &2.10 \text{ km about ground level}\end{aligned}$$

20

Each triangle is:



$$\begin{aligned}\cos 54^\circ &= \frac{10}{x} \quad \therefore x = \frac{10}{\cos 54^\circ} \\ \therefore x &\div 17.01 \\ \tan 54^\circ &= \frac{y}{10} \quad \therefore y = 10 \tan 54^\circ \\ \therefore y &\div 13.76 \\ \text{But } d &= x + y \\ &\div 17.01 + 13.76 \\ &\div 30.8 \\ \text{i.e., width of land is } &30.8 \text{ m}\end{aligned}$$

EXERCISE 10F

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \tan \theta &= \frac{4-5}{-1-2} \\ \therefore \tan \theta &= \frac{-1}{-3} = \frac{1}{3} \\ \therefore \theta &= \tan^{-1}\left(\frac{1}{3}\right) \\ \therefore \theta &\doteq 18.4^\circ \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \tan \theta &= \frac{-1-4}{-2--7} \\ &= -\frac{5}{5} \\ &= -1 \\ \therefore \theta &= 135^\circ \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \tan \theta &= \frac{-4--2}{-1-3} \\ &= \frac{-2}{-4} \\ &= \frac{1}{2} \\ \therefore \theta &= \tan^{-1}\left(\frac{1}{2}\right) \\ \therefore \theta &\doteq 26.6^\circ \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad m &= \tan 60^\circ \\ &\doteq 1.73 \quad \text{or} \quad \sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \tan \theta &= \frac{-5-1}{1--2} \\ &= -\frac{6}{3} \\ &= -2 \\ \therefore \theta &= 180^\circ - \tan^{-1}(2) \\ \therefore \theta &\doteq 116.6^\circ \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\begin{array}{c} \text{Diagram: A line with a negative slope. A vertical line is perpendicular to it at a point. The angle between the vertical line and the line is } 75^\circ. \text{ The angle between the line and the horizontal axis is } 15^\circ. \text{ The angle between the line and the horizontal axis is } 165^\circ. \end{array} \\ m &= \tan 165^\circ \\ &\doteq -0.268 \end{aligned}$$

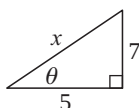
$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \theta &= 60^\circ \\ \therefore \tan \theta &= \sqrt{3} = m \\ \therefore y &= \sqrt{3}x + 2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \theta &= 120^\circ \\ \therefore m &= \tan 120^\circ \\ \therefore m &= -\sqrt{3} \\ \therefore y &= -\sqrt{3}x \end{aligned}$$

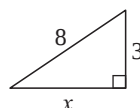
$$\begin{aligned} \mathbf{c} \quad \theta &= 30^\circ \\ \therefore \tan \theta &= \frac{1}{\sqrt{3}} \\ \text{So, } y &= \frac{1}{\sqrt{3}}x + c \\ \text{But } (2\sqrt{3}, 0) &\text{ lies on the line} \\ \therefore 0 &= \frac{1}{\sqrt{3}}(2\sqrt{3}) + c \\ \therefore 0 &= 2 + c \\ \therefore c &= -2 \\ \text{i.e., line is } y &= \frac{1}{\sqrt{3}}x - 2 \end{aligned}$$

REVIEW SET 10A

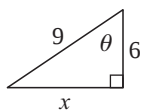
$$\mathbf{1} \quad \sin \theta = \frac{7}{11}$$

2


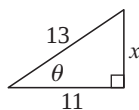
$$\begin{aligned} x^2 &= 5^2 + 7^2 \quad \{\text{Pythagoras}\} \\ \therefore x &= \sqrt{5^2 + 7^2} \\ \therefore x &= \sqrt{74} \\ \text{So, } \sin \theta &= \frac{7}{\sqrt{74}}, \quad \cos \theta = \frac{5}{\sqrt{74}} \end{aligned}$$

3


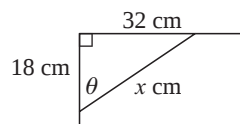
$$\begin{aligned} x^2 + 3^2 &= 8^2 \quad \{\text{Pythagoras}\} \\ \therefore x &= \sqrt{8^2 - 3^2} \\ \therefore x &= \sqrt{55} \\ \text{So, } \tan \theta &= \frac{3}{\sqrt{55}} \end{aligned}$$

4


$$\begin{aligned} x^2 + 6^2 &= 9^2 \quad \{\text{Pythagoras}\} \\ \therefore x &= \sqrt{9^2 - 6^2} \\ \therefore x &= \sqrt{45} \\ \text{So, } \sin \theta &= \frac{\sqrt{45}}{9} \\ \cos \theta &= \frac{6}{9} = \frac{2}{3} \\ \tan \theta &= \frac{\sqrt{45}}{6} \end{aligned}$$

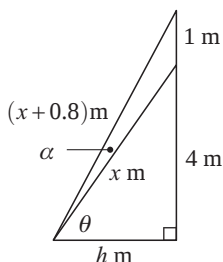
5


$$\begin{aligned} x^2 + 11^2 &= 13^2 \\ \therefore x &= \sqrt{13^2 - 11^2} \\ \therefore x &= \sqrt{48} \\ \text{So, } \sin \theta &= \frac{\sqrt{48}}{13} \\ \cos \theta &= \frac{11}{13} \\ \tan \theta &= \frac{\sqrt{48}}{11} \end{aligned}$$

6


$$\begin{aligned} \mathbf{a} \quad x^2 &= 18^2 + 32^2 \\ \therefore x &= \sqrt{18^2 + 32^2} \\ \therefore x &\doteq 36.715 \\ \text{So, the support is } &36.7 \text{ cm long} \\ \mathbf{b} \quad \tan \theta &= \frac{32}{18} \\ \therefore \theta &= \tan^{-1}\left(\frac{32}{18}\right) \doteq 60.6 \\ \therefore &\text{ makes } 60.6^\circ \text{ with the wall.} \end{aligned}$$

7



$$\mathbf{a} \quad x^2 = h^2 + 4^2 \dots (1)$$

$$\text{and } (x + 0.8)^2 = h^2 + 5^2$$

$$\therefore (x + 0.8)^2 - 25 = x^2 - 16$$

$$\therefore x^2 + 1.6x + 0.64 - 25 = x^2 - 16$$

$$\therefore 1.6x = 8.36$$

$$\therefore x \div 5.225$$

$$\text{and } x + 0.8 \div 6.025$$

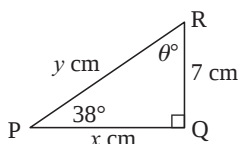
$\therefore$ the extended ladder is 6.025 m long

$$\mathbf{b} \quad \sin \theta = \frac{4}{5.225} \quad \sin(\theta + \alpha) = \frac{5}{6.025}$$

$$\therefore \theta \div 49.956 \quad \therefore \theta + \alpha \div 56.086$$

$$\therefore \alpha \div 56.086 - 49.956 \div 6.13 \quad \therefore \text{angle increases by } 6.13^\circ$$

9



$$\theta^\circ = 90^\circ - 38^\circ = 52^\circ$$

$$\therefore \angle PRQ = 52$$

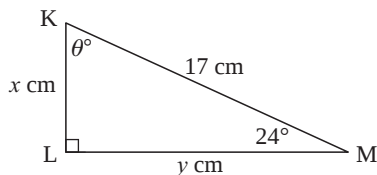
$$\tan 38^\circ = \frac{7}{x} \quad \text{and} \quad \sin 38^\circ = \frac{7}{y}$$

$$\therefore x = \frac{7}{\tan 38^\circ} \quad \therefore y = \frac{7}{\sin 38^\circ}$$

$$\therefore x \div 8.96 \quad \therefore y \div 11.4$$

$$\text{So } PQ \div 8.96 \text{ cm} \quad PR \div 11.4 \text{ cm}$$

8



$$\theta^\circ = 90^\circ - 24^\circ = 66^\circ$$

$$\cos 24^\circ = \frac{y}{17}, \quad \sin 24^\circ = \frac{x}{17}$$

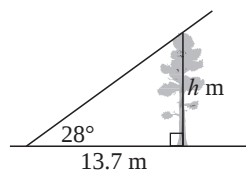
$$\therefore y = 17 \cos 24^\circ \quad \therefore x = 17 \sin 24^\circ$$

$$\therefore y \div 15.5 \quad \therefore x \div 6.91$$

$$\text{So, } KL \div 6.91 \text{ cm}$$

$$LM \div 15.5 \text{ cm}$$

10



$$\tan 28^\circ = \frac{h}{13.7}$$

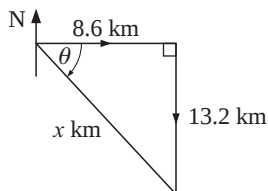
$$\therefore 13.7 \times \tan 28^\circ = h$$

$$\therefore h \div 7.28$$

$$\therefore \text{the height is } 7.28 \text{ m}$$

REVIEW SET 10B

1



$$x^2 = 8.6^2 + 13.2^2$$

$$\therefore x = \sqrt{8.6^2 + 13.2^2}$$

$$\therefore x \div 15.75$$

$$\text{So, distance is } 15.75 \text{ km}$$

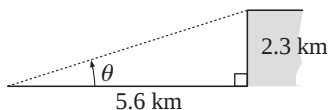
$$\tan \theta = \frac{13.2}{8.6}$$

$$\therefore \theta = \tan^{-1} \left(\frac{13.2}{8.6} \right)$$

$$\therefore \theta \div 56.9^\circ$$

$$\text{and } 90^\circ + 56.9^\circ \div 147^\circ$$

$$\therefore \text{bearing is } 147^\circ$$

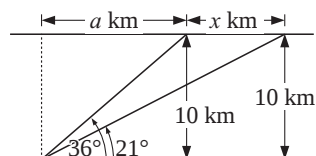
2


$$\tan \theta = \frac{2.3}{5.6} \quad \{\text{assuming a vertical mountain}\}$$

$$\therefore \theta = \tan^{-1} \left(\frac{2.3}{5.6} \right)$$

$$\therefore \theta \doteq 22.3^\circ$$

So, the angle of elevation is 22.3°

3


$$\tan 36^\circ = \frac{10}{a} \quad \text{and} \quad \tan 21^\circ = \frac{10}{a+x}$$

$$\therefore a = \frac{10}{\tan 36^\circ} \quad \text{and} \quad a+x = \frac{10}{\tan 21^\circ}$$

$$\therefore a \doteq 13.764 \quad \text{and} \quad a+x \doteq 26.051$$

$$\therefore x \doteq 26.051 - 13.764$$

$$\therefore x \doteq 12.287$$

$$\text{speed} = \frac{12.287 \text{ km}}{\frac{2}{60} \text{ hours}} \doteq 369 \text{ km/h}$$

4

a $\sin \theta = 0.8147$

$$\therefore \theta = \sin^{-1}(0.8147)$$

$$\therefore \theta \doteq 54.6^\circ$$

b

$$\cos \theta = 0.0917$$

$$\therefore \theta = \cos^{-1}(0.0917)$$

$$\therefore \theta \doteq 84.7^\circ$$

c

$$\tan \theta = 5.23$$

$$\therefore \theta = \tan^{-1}(5.23)$$

$$\therefore \theta \doteq 79.2^\circ$$

5

a $\sin \theta = \frac{\sqrt{11}}{5}$

$$\therefore \theta = \sin^{-1} \left(\frac{\sqrt{11}}{5} \right)$$

$$\therefore \theta \doteq 41.6^\circ$$

b

$$\cos \theta = \frac{5}{7}$$

$$\therefore \theta = \cos^{-1} \left(\frac{5}{7} \right)$$

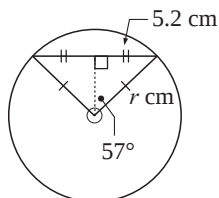
$$\therefore \theta \doteq 44.4^\circ$$

c

$$\tan \theta = 0.7452$$

$$\therefore \theta = \tan^{-1}(0.7452)$$

$$\therefore \theta \doteq 36.7^\circ$$

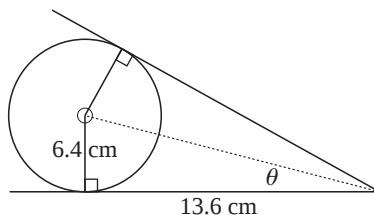
6


$$\sin 57^\circ = \frac{5.2}{r}$$

$$\therefore r = \frac{5.2}{\sin 57^\circ}$$

$$\therefore r \doteq 6.20$$

$\therefore$ the radius is 6.20 cm

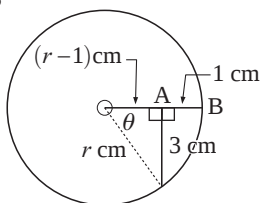
7


$$\tan \theta = \frac{6.4}{13.6}$$

$$\therefore \theta = \tan^{-1} \left(\frac{6.4}{13.6} \right)$$

$$\therefore \theta \doteq 25.2$$

So, the angle measures 25.2°

8


a If the radius OC is r cm

then $OA = (r-1)$ cm

$$\therefore (r-1)^2 + 3^2 = r^2 \quad \{\text{Pythagoras}\}$$

$$\therefore r^2 - 2r + 1 + 9 = r^2$$

$$\therefore -2r + 10 = 0$$

$$\therefore r = 5$$

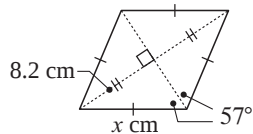
So, the radius is 5 cm

b $\sin \theta = \frac{3}{r} = \frac{3}{5}$

$$\therefore \theta = \sin^{-1}(0.6)$$

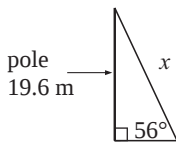
$$\therefore \theta \doteq 36.9$$

So, BC subtends 36.9° at O.

9

$$\sin 57^\circ = \frac{8.2}{x}$$

$$\therefore x = \frac{8.2}{\sin 57^\circ} \div 9.78 \quad \text{i.e., sides are 9.78 cm long}$$

10

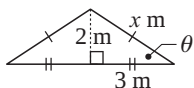
$$\sin 56^\circ = \frac{19.6}{x}$$

$$\therefore x = \frac{19.6}{\sin 56^\circ}$$

$$\therefore x \div 23.64$$

$$\text{and } 3x \div 70.9$$

So, the total length is 70.9 m

REVIEW SET 10C**1**

$$\mathbf{a} \quad x^2 = 2^2 + 3^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{2^2 + 3^2}$$

$$\therefore x \div 3.61$$

$\therefore$ beam AB is 3.61 m long.

$$\mathbf{b} \quad \tan \theta = \frac{2}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \theta \div 33.7$$

$\therefore$ beam is inclined at 33.7° to the horizontal.

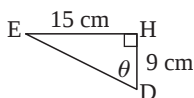
$$\mathbf{2} \quad \tan \theta = \frac{6 - (-4)}{1 - 7} = \frac{10}{-6} = -\frac{5}{3}$$

$$\therefore \theta = 180^\circ - \tan^{-1}\left(\frac{5}{3}\right)$$

$$\therefore \theta \div 121^\circ$$

$$\mathbf{3} \quad m = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore \text{equation is } y = \frac{1}{\sqrt{3}}x - 3$$

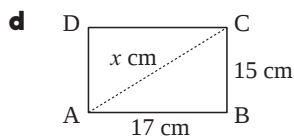
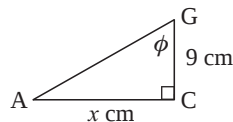
4 a

$$\mathbf{b} \quad \tan \theta = \frac{15}{9} = \frac{5}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{5}{3}\right)$$

$$\therefore \theta \div 59.0$$

So, $\angle HDE$ is 59.0°

c

$$\text{By Pythagoras } x^2 = 15^2 + 17^2 \quad \text{and } \tan \phi = \frac{x}{9}$$

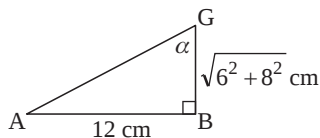
$$\therefore x = \sqrt{15^2 + 17^2}$$

$$\therefore x \div 22.67$$

$$\therefore \phi = \tan^{-1}\left(\frac{22.67}{9}\right)$$

$$\therefore \phi \div 68.3$$

i.e., $\angle AGC$ is 68.3°

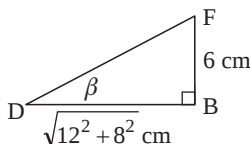
5 a

$$\tan \alpha = \frac{12}{\sqrt{6^2 + 8^2}}$$

$$\therefore \tan \alpha = \frac{12}{10}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{6}{5}\right) \div 50.2$$

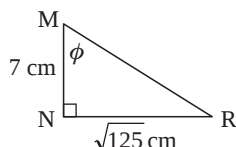
So, the angle measures 50.2°

b

$$\tan \beta = \frac{6}{\sqrt{12^2 + 8^2}} = \frac{6}{\sqrt{208}}$$

$$\therefore \beta = \tan^{-1}\left(\frac{6}{\sqrt{208}}\right) \div 22.6$$

So, the angle measures 22.6°

6 a

$$\mathbf{b} \quad RN = \sqrt{5^2 + 10^2}$$

$$= \sqrt{125}$$

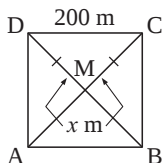
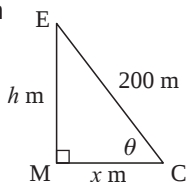
$$\div 11.180 \dots$$

$$\div 11.2 \text{ cm}$$

$$\mathbf{c} \quad \tan \phi = \frac{\sqrt{125}}{7}$$

$$\therefore \phi = \tan^{-1}\left(\frac{\sqrt{125}}{7}\right) \div 57.9$$

i.e., $\angle RMN$ is 57.9°

7 a


$$x^2 + x^2 = 200^2 \quad \{\text{Pythagoras}\}$$

$$\therefore 2x^2 = 40\,000$$

$$\therefore x^2 = 20\,000$$

Also
$$h^2 + x^2 = 200^2$$

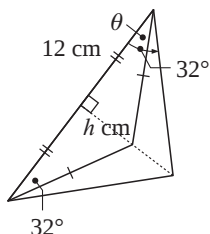
$$\therefore h^2 = 200^2 - 20\,000$$

$$\therefore h = \sqrt{200^2 - 20\,000} \div 141$$

i.e., 141 m high

b
$$\sin \theta = \frac{h}{200} \div \frac{141.42}{200}$$

$$\therefore \theta = \sin^{-1} \left(\frac{141.42}{200} \right) \div 45^\circ$$

8


$$\tan 32^\circ = \frac{h}{12}$$

$$\therefore 12 \times \tan 32^\circ = h$$

$$\therefore h \div 7.4984$$

$$\text{Area} = \frac{1}{2} \text{ base} \times \text{height}$$

$$= \frac{1}{2} \times 24 \times 7.4984$$

$$\div 89.981$$

$$\therefore \text{new area} = 2 \times \text{old area}$$

$$\div 179.96 \text{ cm}^2$$

 Now if the new height is H cm

$$\frac{1}{2} \times 24 \times H \div 179.96$$

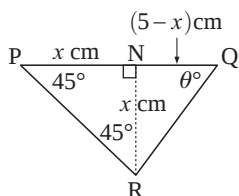
$$\therefore 12H \div 179.96$$

$$\therefore H \div 14.997$$

$$\therefore \tan \theta = \frac{14.997}{12}$$

$$\therefore \theta = \tan^{-1} \left(\frac{14.997}{12} \right)$$

$$\div 51.3$$

 So, the new base angles are 51.3°
9

 Let $NR = x$ cm

 Now $\triangle PNR$ is right angled isosceles

$$\therefore PN = x \text{ cm} \text{ also}$$

and so $QN = (5 - x) \text{ cm}$

Now $\tan \theta = \frac{x}{5 - x} \quad \{\text{in } \triangle RQN\}$

$$\therefore (5 - x) \tan \theta = x$$

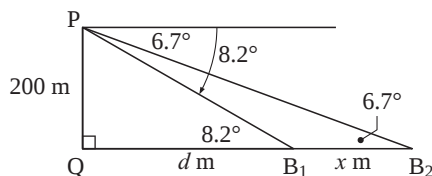
$$\therefore 5 \tan \theta - x \tan \theta = x$$

$$\therefore 5 \tan \theta = x + x \tan \theta$$

$$\therefore 5 \tan \theta = x(1 + \tan \theta)$$

$$\therefore \frac{5 \tan \theta}{1 + \tan \theta} = x$$

i.e., $RN = \frac{5 \tan \theta}{1 + \tan \theta} \text{ cm}$

10

 Let $QB_1 = d$ m and $B_1B_2 = x$ m

$$\therefore \tan 8.2^\circ = \frac{200}{d} \text{ and } \tan 6.7^\circ = \frac{200}{d + x}$$

$$\therefore d = \frac{200}{\tan 8.2} \div 1387.9$$

and $d + x = \frac{200}{\tan 6.7} \div 1702.5$

$$\text{So } x = 1702.5 - 1387.9 \div 315$$

i.e., the boats are 315 m apart.

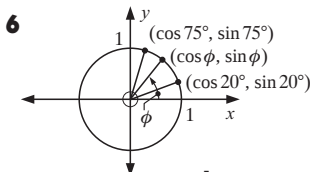
Chapter 11

THE UNIT CIRCLE

EXERCISE 11A

- 1** **a** The y -coordinate at 0° is 0, $\therefore \sin 0^\circ = 0$
b The y -coordinate at 15° is $\doteq 0.26$, $\therefore \sin 15^\circ = 0.26$
c The y -coordinate at 25° is $\doteq 0.42$, $\therefore \sin 25^\circ = 0.42$
d The y -coordinate at 30° is $\doteq 0.5$, $\therefore \sin 30^\circ = 0.50$
e The y -coordinate at 45° is $\doteq 0.71$, $\therefore \sin 45^\circ = 0.71$
f The y -coordinate at 60° is $\doteq 0.87$, $\therefore \sin 60^\circ = 0.87$
g The y -coordinate at 75° is $\doteq 0.97$, $\therefore \sin 75^\circ = 0.97$
h The y -coordinate at 90° is 1, $\therefore \sin 90^\circ = 1$
- 3** **a** The x -coordinate at 0° is 1, $\therefore \cos 0^\circ = 1$
b The x -coordinate at 15° is $\doteq 0.97$, $\therefore \cos 15^\circ \doteq 0.97$
c The x -coordinate at 25° is $\doteq 0.91$, $\therefore \cos 25^\circ \doteq 0.91$
d The x -coordinate at 30° is $\doteq 0.87$, $\therefore \cos 30^\circ \doteq 0.87$
e The x -coordinate at 45° is $\doteq 0.71$, $\therefore \cos 45^\circ \doteq 0.71$
f The x -coordinate at 60° is $\doteq 0.5$, $\therefore \cos 60^\circ \doteq 0.5$
g The x -coordinate at 75° is $\doteq 0.26$, $\therefore \cos 75^\circ \doteq 0.26$
h The x -coordinate at 90° is 0, $\therefore \cos 90^\circ = 0$

- 5** The coordinates
 $(\cos 55^\circ, \sin 55^\circ)$
 i.e., $(0.57, 0.82)$

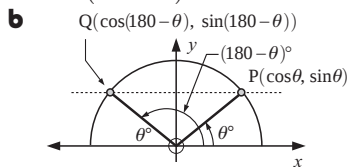


- 7** **a** As $\cos^2 \theta + \sin^2 \theta = 1$,
 then $\sin \theta = \sqrt{1 - \cos^2 \theta}$ as $\sin \theta > 0$
 $= \sqrt{1 - (0.8)^2}$
 $= 0.6$
- b** As $\cos^2 \theta + \sin^2 \theta = 1$,
 then $\cos \theta = \sqrt{1 - \sin^2 \theta}$ as $\cos \theta > 0$
 $= \sqrt{1 - (0.7)^2}$
 $\doteq 0.714$

EXERCISE 11B

- 1** **a** 0.98 **b** 0.98 **c** 0.87 **d** 0.87 **e** 0.5 **f** 0.5 **g** 0 **h** 0

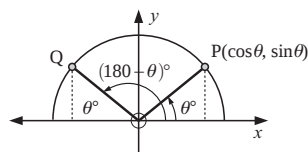
- 3** **a** It appears that
 $\sin(180 - \theta)^\circ = \sin \theta^\circ$



P and Q have the same y -coordinate.
 $\therefore \sin(180 - \theta)^\circ = \sin \theta^\circ$

- 4** **a** -0.34 **b** 0.34 **c** -0.64 **d** 0.64 **e** -0.77 **f** 0.77 **g** -1 **h** 1

- 6** **a** $\cos(180 - \theta)^\circ = -\cos \theta^\circ$



Q is
 $(\cos(180 - \theta)^\circ, \sin(180 - \theta)^\circ)$
 The x -coordinate of Q is the negative of the x -coordinate of P.
 $\therefore \cos(180 - \theta)^\circ = -\cos \theta^\circ$

- 7** **a** $\sin 45^\circ = \sin(180 - 45)^\circ = \sin 135^\circ$
 i.e., $\theta = 135^\circ$ **b** $\sin 51^\circ = \sin(180 - 51)^\circ = \sin 129^\circ$
 i.e., $\theta = 129^\circ$ **c** $\sin 74^\circ = \sin(180 - 74)^\circ = \sin 106^\circ$
 i.e., $\theta = 106^\circ$ **d** $\sin 82^\circ = \sin(180 - 82)^\circ = \sin 98^\circ$
 i.e., $\theta = 98^\circ$

$$\begin{array}{llll}
 \mathbf{8} \quad \mathbf{a} & \sin 130^\circ & \mathbf{b} & \sin 146^\circ & \mathbf{c} & \sin 162^\circ & \mathbf{d} & \sin 171^\circ \\
 & = \sin(180 - 130)^\circ & & = \sin(180 - 146)^\circ & & = \sin(180 - 162)^\circ & & = \sin(180 - 171)^\circ \\
 & = \sin 50^\circ & & = \sin 34^\circ & & = \sin 18^\circ & & = \sin 9^\circ \\
 & \text{i.e., } \theta = 50^\circ & & \text{i.e., } \theta = 34^\circ & & \text{i.e., } \theta = 18^\circ & & \text{i.e., } \theta = 9^\circ
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{9} \quad \mathbf{a} & \sin 137^\circ & \mathbf{b} & \sin 59^\circ & \mathbf{c} & \cos 143^\circ \\
 & = \sin(180 - 137)^\circ & & = \sin(180 - 59)^\circ & & = -\cos(180 - 143)^\circ \\
 & = \sin 43^\circ & & = \sin 121^\circ & & = -\cos 37^\circ \\
 & \div 0.6820 & & \div 0.8572 & & \div -0.7986 \\
 \mathbf{d} & \cos 24^\circ & \mathbf{e} & \sin 115^\circ & \mathbf{f} & \cos 132^\circ \\
 & = -\cos(180 - 24)^\circ & & = \sin(180 - 115)^\circ & & = -\cos(180 - 132)^\circ \\
 & = -\cos 156^\circ & & = \sin 65^\circ & & = -\cos 48^\circ \\
 & = 0.9135 & & \div 0.9063 & & = -0.6691
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{10} \quad \mathbf{a} & \angle \text{AOQ} + \angle \text{BOQ} = 180^\circ \\
 & \therefore \theta^\circ + \angle \text{BOQ} = 180^\circ \\
 & \angle \text{BOQ} = (180 - \theta)^\circ \\
 \mathbf{b} & \text{OQ is a reflection of OP in the } y\text{-axis} \\
 & \text{and so Q has coordinates } (-\cos \theta, \sin \theta) \\
 \mathbf{c} & \cos(180 - \theta)^\circ = -\cos \theta^\circ, \quad \sin(180 - \theta)^\circ = \sin \theta^\circ
 \end{array}$$

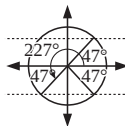
EXERCISE 11C

$$\begin{array}{ll}
 \mathbf{1} \quad \mathbf{a} & \text{Unit circle diagram showing angles } 0^\circ, 90^\circ, 180^\circ, 270^\circ \text{ and their corresponding coordinates.} \\
 \mathbf{b} & \text{Unit circle diagram showing angles } 0^\circ, 90^\circ, 180^\circ, 270^\circ \text{ and their corresponding coordinates.} \\
 \mathbf{c} & \text{Unit circle diagram showing angles } 0^\circ, 90^\circ, 180^\circ, 270^\circ \text{ and their corresponding coordinates.} \\
 \mathbf{2} \quad \mathbf{a} \quad \mathbf{i} & A(\cos 26^\circ, \sin 26^\circ) \quad B(\cos 146^\circ, \sin 146^\circ) \\
 & C(\cos 199^\circ, \sin 199^\circ) \\
 & \mathbf{ii} \quad A(0.899, 0.438) \quad B(-0.829, 0.559) \\
 & C(-0.946, -0.326) \\
 \mathbf{b} \quad \mathbf{i} & A(\cos 123^\circ, \sin 123^\circ) \quad B(\cos 251^\circ, \sin 251^\circ) \\
 & C(\cos(-35^\circ), \sin(-35^\circ)) \\
 & \mathbf{ii} \quad A(-0.545, 0.839) \quad B(-0.326, -0.946) \\
 & C(0.819, -0.574) \\
 \mathbf{3} & \text{Unit circle diagram showing angles } 0^\circ, 90^\circ, 180^\circ, 270^\circ \text{ and their corresponding coordinates.} \\
 \mathbf{a} & \cos 0^\circ = 1, \quad \sin 0^\circ = 0 \\
 \mathbf{c} & \cos 180^\circ = -1, \quad \sin 180^\circ = 0 \\
 \mathbf{e} & \cos(-90^\circ) = 0, \quad \sin(-90^\circ) = -1 \\
 \mathbf{b} & \cos 90^\circ = 0, \quad \sin 90^\circ = 1 \\
 \mathbf{d} & \cos 270^\circ = 0, \quad \sin 270^\circ = -1 \\
 \mathbf{f} & \cos 450^\circ = 0, \quad \sin 450^\circ = 1
 \end{array}$$

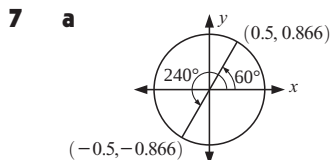
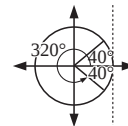
REVIEW SET 11

$$\begin{array}{ll}
 \mathbf{1} \quad \mathbf{a} & \sin 70^\circ \div 0.94 \quad \mathbf{b} \quad \cos 35^\circ \div 0.82 \\
 \mathbf{2} & M(\cos 73^\circ, \sin 73^\circ) \div (0.292, 0.956) \quad N(\cos 190^\circ, \sin 190^\circ) \div (-0.985, -0.174) \\
 & P(\cos 307^\circ, \sin 307^\circ) \div (0.602, -0.799) \\
 \mathbf{3} & \text{The } x\text{-coordinate of A} = -0.222 \\
 & \therefore \cos \theta = -0.222 \\
 & \therefore \theta = \cos^{-1}(-0.222) \\
 & \therefore \theta \div 102.8^\circ \\
 \mathbf{4} \quad \mathbf{a} & \sin 120^\circ = \sin(180 - 120)^\circ = \sin 60^\circ \\
 & \therefore \theta = 60^\circ \\
 \mathbf{b} & \sin 165^\circ = \sin(180 - 165)^\circ = \sin 15^\circ \\
 & \therefore \theta = 15^\circ \\
 \mathbf{c} & \sin 95^\circ = \sin(180 - 95)^\circ = \sin 85^\circ \\
 & \therefore \theta = 85^\circ \\
 \mathbf{5} \quad \mathbf{a} & \sin 47^\circ \\
 & = \sin(180 - 47)^\circ \\
 & = \sin 133^\circ \\
 & \therefore \theta = 133^\circ \\
 \mathbf{b} & \sin 8^\circ \\
 & = \sin(180 - 8)^\circ \\
 & = \sin 172^\circ \\
 & \therefore \theta = 172^\circ \\
 \mathbf{c} & \sin 86^\circ \\
 & = \sin(180 - 86)^\circ \\
 & = \sin 94^\circ \\
 & \therefore \theta = 94^\circ \\
 \mathbf{6} \quad \mathbf{a} & \sin 159^\circ \\
 & = \sin(180 - 159)^\circ \\
 & = \sin 21^\circ \\
 & \div 0.358 \\
 \mathbf{b} & \cos 92^\circ \\
 & = -\cos(180 - 92)^\circ \\
 & = -\cos 88^\circ \\
 & \div -0.035 \\
 \mathbf{c} & \cos 75^\circ \\
 & = -\cos(180 - 75)^\circ \\
 & = -\cos 105^\circ \\
 & \div 0.259
 \end{array}$$

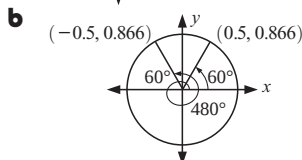
d $\sin 227^\circ = \sin(-47^\circ)$
 $= -\sin 47^\circ$
 $\div -0.731$



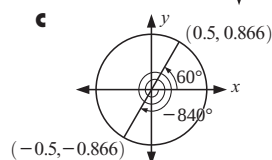
e $\cos 320^\circ$
 $= \cos 40^\circ$
 $\div 0.766$



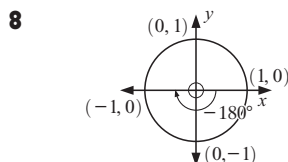
$\therefore \cos 240^\circ = -0.5$
 $\sin 240^\circ = -0.866$



$\therefore \cos 480^\circ = -0.5$
 $\sin 480^\circ = 0.866$



$\therefore \cos(-840^\circ) = -0.5$
 $\sin(-840^\circ) = -0.866$



a $\cos 360^\circ = 1, \sin 360^\circ = 0$

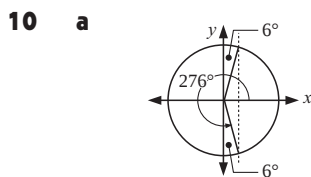
b $\cos(-180^\circ) = -1, \sin(-180^\circ) = 0$

c $\cos(630^\circ) = 0, \sin(630^\circ) = -1$

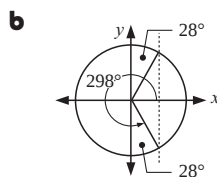
9 a $\sin 101^\circ$
 $= \sin(180 - 101)^\circ$
 $= \sin 79^\circ$
 $\therefore \theta = 79^\circ$

b $\sin 127^\circ$
 $= \sin(180 - 127)^\circ$
 $= \sin 53^\circ$
 $\therefore \theta = 53^\circ$

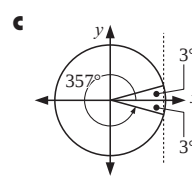
c $\sin 168^\circ$
 $= \sin(180 - 168)^\circ$
 $= \sin 12^\circ$
 $\therefore \theta = 12^\circ$



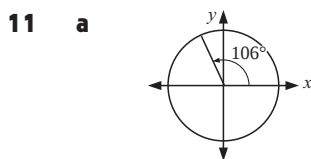
$\therefore \theta = (90 - 6)^\circ = 84^\circ$



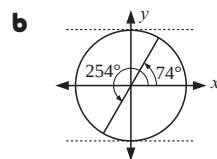
$\therefore \theta = (90 - 28)^\circ = 62^\circ$



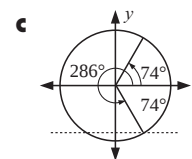
$\therefore \theta = 3^\circ$



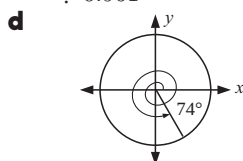
$\sin 106^\circ$
 $= \sin(180 - 106)^\circ$
 $= \sin 74^\circ$
 $\div 0.961$



$254^\circ = 74^\circ + 180^\circ$
 $\therefore \sin 254^\circ = -\sin 74^\circ$
 $\div -0.961$

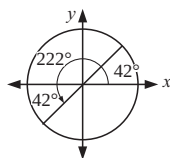


$\sin 286^\circ = -\sin 74^\circ$
 $= -0.961$

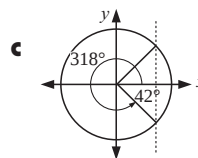


$646^\circ = 360^\circ + 286^\circ$
 $\therefore \sin 646^\circ = \sin 286^\circ$
 $= -0.961 \quad \{\text{from c}\}$

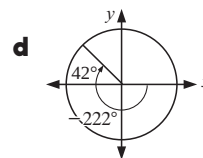
12 a $\cos 138^\circ$
 $= -\cos(180 - 138)^\circ$
 $= -\cos 42^\circ$
 $\div -0.743$



$\cos 222^\circ$
 $= -\cos 42^\circ$
 $= -0.743$



$\cos 318^\circ$
 $= \cos 42^\circ$
 $= 0.743$



$\cos(-222)^\circ$
 $= -\cos 42^\circ$
 $= -0.743$

Chapter 12

NON RIGHT ANGLED TRIANGLE TRIGONOMETRY

EXERCISE 12A

1 a Area

$$= \frac{1}{2} \times 9 \times 10 \times \sin 40^\circ$$

$$\div 28.9 \text{ cm}^2$$

b Area

$$= \frac{1}{2} \times 25 \times 31 \times \sin 82^\circ$$

$$\div 384 \text{ km}^2$$

c Area

$$= \frac{1}{2} \times 10.2 \times 6.4 \times \sin 125^\circ$$

$$\div 26.7 \text{ cm}^2$$

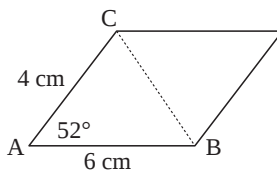
2 Area = 150 cm^2

$$\therefore \frac{1}{2} \times 17 \times x \times \sin 68^\circ = 150$$

$$\therefore x = \frac{2 \times 150}{17 \times \sin 68^\circ}$$

$$\therefore x \div 19.0$$

3



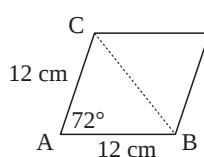
$$\text{Area}$$

$$= 2 \times \text{area } \triangle ABC$$

$$= 2 \times \frac{1}{2} \times 4 \times 6 \times \sin 52^\circ$$

$$\div 18.9 \text{ cm}^2$$

4

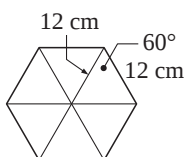


$$\text{Area} = 2 \times \text{area } \triangle ABC$$

$$= 2 \times \frac{1}{2} \times 12^2 \times \sin 72^\circ$$

$$\div 137 \text{ cm}^2$$

5



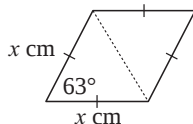
$$\text{Area}$$

$$= 6 \times \text{area of } \triangle$$

$$= 6 \times \frac{1}{2} \times 12^2 \times \sin 60^\circ$$

$$\div 374 \text{ cm}^2$$

6



$$\text{Area} = 2 \times \frac{1}{2} x^2 \sin 63^\circ$$

$$\therefore x^2 \sin 63^\circ = 50$$

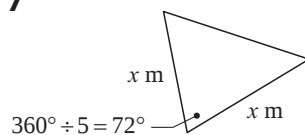
$$\therefore x^2 = \frac{50}{\sin 63^\circ}$$

$$\therefore x = \sqrt{\frac{50}{\sin 63^\circ}}$$

$$\therefore x \div 7.49$$

So, sides are 7.49 cm long.

7



$$\text{Area of } \triangle = \frac{338}{5}$$

$$\therefore \frac{1}{2} x^2 \sin 72^\circ = \frac{338}{5}$$

$$\therefore x^2 = \frac{2 \times 338}{5 \times \sin 72^\circ}$$

$$\therefore x = \sqrt{\frac{2 \times 338}{5 \times \sin 72^\circ}}$$

$$\therefore x \div 11.9$$

So, OA $\div$ 11.9 m long.

8 a If the included angle is θ

$$\text{then } \frac{1}{2} \times 5 \times 8 \times \sin \theta = 15$$

$$\therefore 20 \sin \theta = 15$$

$$\therefore \sin \theta = \frac{3}{4}$$

$$\therefore \theta = \sin^{-1} \left(\frac{3}{4} \right)$$

$$\therefore \theta \div 48.6^\circ \text{ or } (180 - 48.6)^\circ$$

$$\text{i.e., } \theta \div 48.6^\circ \text{ or } 131.4^\circ$$

b

Likewise,

$$\frac{1}{2} \times 45 \times 53 \times \sin \theta = 800$$

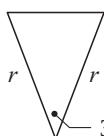
$$\therefore \sin \theta = \frac{800 \times 2}{45 \times 53}$$

$$\therefore \theta = \sin^{-1} \left(\frac{1600}{45 \times 53} \right)$$

$$\therefore \theta \div 42.1^\circ \text{ or } (180 - 42.1)^\circ$$

$$\therefore \theta \div 42.1^\circ \text{ or } 137.9^\circ$$

9



Total area of 8 coins

$$= 8 \times 12 \times \frac{1}{2} r^2 \sin 30^\circ$$

$$= 48 r^2 \left(\frac{1}{2} \right)$$

$$= 24 r^2$$

Area of \$10 note

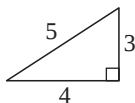
$$= 8r \times 4r$$

$$= 32 r^2$$

Fraction covered

$$= \frac{24 r^2}{32 r^2}$$

$$= \frac{3}{4} \therefore \frac{1}{4} \text{ is uncovered}$$

10 a i

$$\begin{aligned}\text{Area} &= \frac{1}{2} \text{ base} \times \text{alt} \\ &= \frac{1}{2} \times 4 \times 3 \\ &= 6 \text{ cm}^2\end{aligned}$$

$$\text{ii} \quad s = \frac{3+4+5}{2} = 6$$

$$\begin{aligned}\therefore \text{area} &= \sqrt{6(6-3)(6-4)(6-5)} \\ &= \sqrt{6 \times 3 \times 2 \times 1} \\ &= 6 \text{ cm}^2\end{aligned}$$

$$\text{b i} \quad s = \frac{6+8+12}{2} = 13$$

$$\begin{aligned}\therefore A &= \sqrt{13(13-6)(13-8)(13-12)} \\ &= \sqrt{13 \times 7 \times 5 \times 1} \\ &\div 21.3 \text{ cm}^2\end{aligned}$$

$$\text{ii} \quad s = \frac{7.2+8.9+9.7}{2} = 12.9$$

$$\begin{aligned}\therefore A &= \sqrt{12.9(12.9-7.2)(12.9-8.9)(12.9-9.7)} \\ &= \sqrt{12.9 \times 5.7 \times 4 \times 3.2} \\ &\div 30.7 \text{ cm}^2\end{aligned}$$

EXERCISE 12B**1 a i** arc length

$$\begin{aligned}&= \left(\frac{41.6}{360}\right) \times 2\pi \times 9 \\ &\div 6.53 \text{ cm}\end{aligned}$$

ii area

$$\begin{aligned}&= \left(\frac{41.6}{360}\right) \times \pi \times 9^2 \\ &\div 29.4 \text{ cm}^2\end{aligned}$$

b i arc length

$$\begin{aligned}&= \left(\frac{122}{360}\right) \times 2\pi \times 4.93 \\ &\div 10.5 \text{ cm}\end{aligned}$$

ii area

$$\begin{aligned}&= \left(\frac{122}{360}\right) \times \pi \times 4.93^2 \\ &\div 25.9 \text{ cm}^2\end{aligned}$$

2 a $\theta = 107.9^\circ$, $l = 5.92$

$$\therefore \left(\frac{107.9}{360}\right) \times 2\pi \times r = 5.92$$

$$\therefore r = \frac{5.92 \times 360}{107.9 \times 2 \times \pi}$$

$$\therefore r \div 3.14 \text{ m}$$

b Area = $\left(\frac{107.9}{360}\right) \times \pi \times (3.1436)^2$

$$\div 9.30 \text{ m}^2$$

3 a

$$\text{Area} = \left(\frac{\theta}{360}\right) \times \pi r^2$$

$$\therefore 20.8 = \left(\frac{68.2}{360}\right) \times \pi r^2$$

$$\therefore \frac{20.8 \times 360}{68.2 \times \pi} = r^2$$

$$\therefore r = \sqrt{\frac{20.8 \times 360}{68.2 \times \pi}}$$

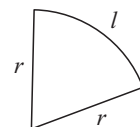
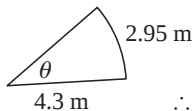
$$\therefore r \div 5.91 \text{ cm}$$

b Perimeter

$$= l + 2r$$

$$= \left(\frac{68.2}{360}\right) \times 2\pi \times 5.912 + 2 \times 5.912$$

$$\div 18.9 \text{ cm}$$

**4 a**

$$l = \left(\frac{\theta}{360}\right) \times 2\pi \times r$$

$$\therefore 2.95 = \left(\frac{\theta}{360}\right) \times 2\pi \times 4.3$$

$$\therefore \frac{2.95 \times 360}{2 \times \pi \times 4.3} = \theta$$

$$\therefore \theta \div 39.3^\circ$$

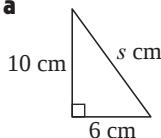
b

$$\text{Area} = \left(\frac{\theta}{360}\right) \times \pi r^2$$

$$\therefore 30 = \left(\frac{\theta}{360}\right) \times \pi \times 10^2$$

$$\therefore \frac{30 \times 360}{\pi \times 100} = \theta$$

$$\therefore \theta \div 34.4^\circ$$

5 a

$$s^2 = 6^2 + 10^2 \quad \{\text{Pythagoras}\}$$

$$\therefore s = \sqrt{6^2 + 10^2}$$

$$\therefore s \div 11.6619$$

$$\therefore s \div 11.7 \text{ cm}$$

b $r = s \div 11.7 \text{ cm}$ **c** arc length = $2\pi \times 6 \div 37.6991 \dots$
 $\div 37.7 \text{ cm}$ **d** arc length = $\left(\frac{\theta}{360}\right) \times 2\pi r$

$$\therefore \frac{\theta}{360} \times 2 \times \pi \times 11.6619 \div 37.6991$$

$$\therefore \frac{37.6991 \times 360}{2 \times \pi \times 11.6619} \div \theta$$

$$\therefore \theta \div 185^\circ$$

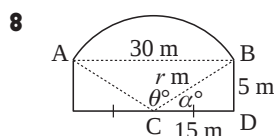
$$\begin{aligned}
 \text{6 a } l &= \left(\frac{\theta}{360}\right) \times 2\pi r \\
 &= \frac{1}{360} \times 2 \times \pi \times 6370 \text{ km} \\
 &\div 1.852957 \dots \text{ km} \\
 &\div 1.853 \text{ km}
 \end{aligned}$$

$$\begin{aligned}
 \text{b speed} &= \frac{\text{distance}}{\text{time}} \quad \therefore \text{time} = \frac{\text{distance}}{\text{speed}} \\
 &= \frac{2130 \text{ km}}{480 \text{ n miles/h}} \\
 &= \frac{2130 \text{ km}}{480 \times 1.853 \text{ km/h}} \\
 &\div 2.3947 \dots \text{ hours} \\
 &\div 2 \text{ hours } 23.68 \text{ min} \\
 &\div 2 \text{ hours } 24 \text{ min}
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a Area} &= \frac{1}{2} \times 12 \times 12 \times \sin 110^\circ \\
 &\div 67.657 \dots \\
 &\div 67.7 \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{b Area of sector} &= \left(\frac{110}{360}\right) \times \pi \times 12^2 \\
 &\div 138 \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{c Area of segment} &= \text{area of sector} - \text{area of } \Delta \\
 &\div 138.2 - 67.7 \\
 &\div 70.5 \text{ cm}^2
 \end{aligned}$$

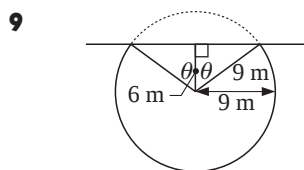


$$\begin{aligned}
 \text{a } \tan \alpha &= \frac{5}{15} \\
 \therefore \alpha &= \tan^{-1}\left(\frac{1}{3}\right) \\
 \therefore \alpha &\div 18.43
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \theta + 2\alpha &= 180 \quad \{\text{angles on a line}\} \\
 \therefore \theta &= 180 - 2 \times 18.43 \\
 \therefore \theta &\div 143.1
 \end{aligned}$$

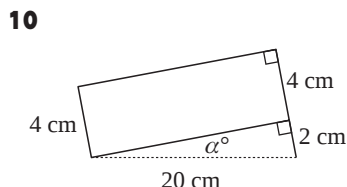
Note: $r^2 = 5^2 + 15^2$
 $\therefore r^2 = 250$

$$\begin{aligned}
 \text{c Area} &= 2 \times \text{area } \Delta CDB + \text{area sector} \\
 &= 2 \times \frac{1}{2} \times 15 \times 5 + \left(\frac{143.1}{360}\right) \times \pi \times 250 \\
 &\div 387.3 \text{ m}^2
 \end{aligned}$$



$$\begin{aligned}
 \cos \theta &= \frac{6}{9} = \frac{2}{3} \\
 \therefore \theta &= \cos^{-1}\left(\frac{2}{3}\right) \\
 \therefore \theta &\div 48.19^\circ \\
 \text{So, } 360 - 2\theta &\div 263.62^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{Area} &= \text{area of } \Delta + \text{area of sector} \\
 &= \frac{1}{2} \times 9^2 \times \sin 96.38^\circ \\
 &\quad + \left(\frac{263.62}{360}\right) \times \pi \times 9^2 \\
 &\div 227 \text{ m}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{a } \sin \alpha &= \frac{2}{20} = 0.1 \\
 \therefore \alpha &= \sin^{-1}(0.1) \\
 \therefore \alpha &\div 5.7392 \dots \\
 \therefore \alpha &\div 5.739
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \phi + \theta &= 360 \\
 \therefore \phi &\div 360 - 168.5 \\
 \therefore \phi &\div 191.5
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \theta + 90 + 90 + 2\alpha &= 360 \\
 \therefore \theta &= 180 - 2\alpha \\
 &\div 180 - 2 \times 5.739 \\
 &\div 168.5
 \end{aligned}$$

$$\begin{aligned}
 \text{d length of belt} &= 2 \times 20 + \frac{\theta}{360} \times 2\pi \times 4 \\
 &\quad + \frac{\phi}{360} \times 2\pi \times 6 \\
 &\div 71.82 \text{ cm}
 \end{aligned}$$

EXERCISE 12C

$$\begin{aligned}
 \text{1 a } BC^2 &= 21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^\circ \\
 \therefore BC &= \sqrt{21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^\circ} \div 28.8 \text{ cm}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } PQ^2 &= 6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^\circ \\
 \therefore PQ &= \sqrt{6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^\circ} \div 3.38 \text{ km}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } KM^2 &= 6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^\circ \\
 \therefore KM &= \sqrt{6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^\circ} \div 14.2 \text{ m}
 \end{aligned}$$

$$2 \quad \cos A = \frac{12^2 + 13^2 - 11^2}{2 \times 12 \times 13}$$

$$\therefore A = \cos^{-1} \left(\frac{192}{312} \right)$$

$$\therefore A \doteq 52.0^\circ$$

$$\cos B = \frac{13^2 + 11^2 - 12^2}{2 \times 13 \times 11}$$

$$\therefore B = \cos^{-1} \left(\frac{146}{286} \right)$$

$$\therefore B \doteq 59.3^\circ$$

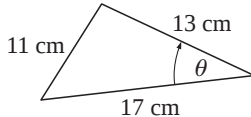
$$C = (180 - A - B)^\circ \\ \doteq 68.7^\circ$$

$$3 \quad \cos Q = \frac{5^2 + 7^2 - 10^2}{2 \times 5 \times 7}$$

$$\therefore Q = \cos^{-1} \left(\frac{-26}{70} \right)$$

$$\therefore Q \doteq 112^\circ$$

4 a



The smallest angle is opposite the shortest side.

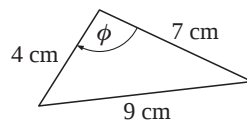
$$\cos \theta = \frac{13^2 + 17^2 - 11^2}{2 \times 13 \times 17}$$

$$\therefore \theta = \cos^{-1} \left(\frac{337}{442} \right)$$

$$\therefore \theta \doteq 40.3$$

So, the smallest angle measures 40.3° .

b



The largest angle is opposite the longest side.

$$\cos \phi = \frac{4^2 + 7^2 - 9^2}{2 \times 4 \times 7}$$

$$\therefore \phi = \cos^{-1} \left(-\frac{16}{56} \right)$$

$$\therefore \phi \doteq 106.60$$

So, the largest angle measures 107° , approx.

$$5 \quad a \quad \cos \theta = \frac{2^2 + 5^2 - 4^2}{2 \times 2 \times 5}$$

$$= \frac{13}{20}$$

$$= 0.65$$

$$b \quad x^2 = 5^2 + 3^2 - 2 \times 5 \times 3 \times \cos \theta$$

$$\therefore x = \sqrt{5^2 + 3^2 - 2 \times 5 \times 3 \times 0.65}$$

$$\therefore x \doteq 3.81$$

EXERCISE 12D.1

1 a By the sine rule,

$$\frac{x}{\sin 48^\circ} = \frac{23}{\sin 37^\circ}$$

$$\therefore x = \frac{23 \times \sin 48^\circ}{\sin 37^\circ}$$

$$\therefore x \doteq 28.4$$

b By the sine rule,

$$\frac{x}{\sin 115^\circ} = \frac{11}{\sin 48^\circ}$$

$$\therefore x = \frac{11 \times \sin 115^\circ}{\sin 48^\circ}$$

$$\therefore x \doteq 13.4$$

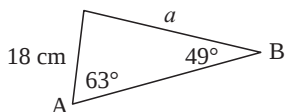
c By the sine rule,

$$\frac{x}{\sin 51^\circ} = \frac{4.8}{\sin 80^\circ}$$

$$\therefore x = \frac{4.8 \times \sin 51^\circ}{\sin 80^\circ}$$

$$\therefore x \doteq 3.79$$

2 a

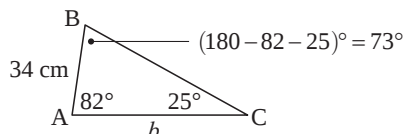


$$\frac{a}{\sin 63^\circ} = \frac{18}{\sin 49^\circ} \quad \{\text{sine rule}\}$$

$$\therefore a = \frac{18 \times \sin 63^\circ}{\sin 49^\circ}$$

$$\therefore a \doteq 21.25 \text{ cm}$$

b

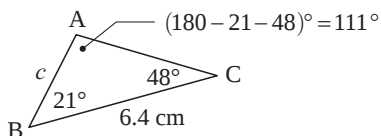


$$\text{By the sine rule, } \frac{b}{\sin 73^\circ} = \frac{34}{\sin 25^\circ}$$

$$\therefore b = \frac{34 \times \sin 73^\circ}{\sin 25^\circ}$$

$$\therefore b \doteq 76.9 \text{ cm}$$

c



$$\text{By the sine rule, } \frac{c}{\sin 48^\circ} = \frac{6.4}{\sin 111^\circ}$$

$$\therefore c = \frac{6.4 \times \sin 48^\circ}{\sin 111^\circ}$$

$$\therefore c \doteq 5.09 \text{ cm}$$

EXERCISE 12D.2**1** By the sine rule

$$\frac{\sin C}{11} = \frac{\sin 40^\circ}{8}$$

$$\therefore \sin C = \frac{11 \times \sin 40^\circ}{8}$$

$$\therefore C = \sin^{-1} \left(\frac{11 \times \sin 40^\circ}{8} \right)$$

$$\therefore C \doteq 62.1^\circ \text{ or } (180 - 62.1)^\circ$$

$$\therefore C \doteq 62.1^\circ \text{ or } 117.9^\circ$$

$$\mathbf{b} \quad \frac{\sin B}{43.8} = \frac{\sin 43^\circ}{31.4}$$

$$\therefore \sin B = \frac{43.8 \times \sin 43^\circ}{31.4}$$

$$\therefore B = \sin^{-1} \left(\frac{43.8 \times \sin 43^\circ}{31.4} \right)$$

$$\therefore B \doteq 72.0^\circ \text{ or } 108^\circ$$

both of which are possible as
 $108 + 43 = 151$ which is < 180 .

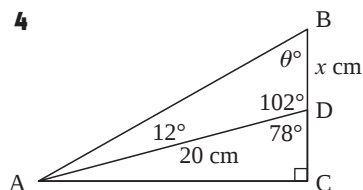
$$\mathbf{3} \quad \text{The third angle is } \frac{\sin 85^\circ}{11.4} \quad \text{and} \quad \frac{\sin 27^\circ}{9.8}$$

$$= 27^\circ \quad = 0.08738\dots \quad = 0.04632\dots$$

$$\therefore \text{it is not possible as}$$

$$\frac{\sin 85^\circ}{11.4} \neq \frac{\sin 27^\circ}{9.8}$$

i.e., the sine rule is violated.

4

In $\triangle ABD$,

$$\theta = (180 - 12 - 102)^\circ$$

$$\therefore \theta = 66^\circ$$

Now $\frac{x}{\sin 12^\circ} = \frac{20}{\sin 66^\circ}$

$$\therefore x = \frac{20 \times \sin 12^\circ}{\sin 66^\circ}$$

$$\therefore x \doteq 4.55$$

$\therefore$ BD is 4.55 cm long

5 First we find the length of the diagonal, d m.

$$\frac{d}{\sin 118^\circ} = \frac{22}{\sin 30^\circ}$$

$$\therefore d = \frac{22 \times \sin 118^\circ}{\sin 30^\circ}$$

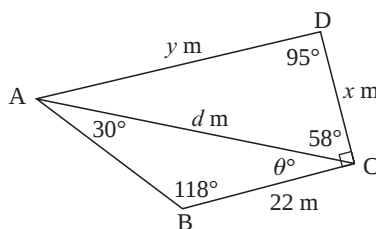
$$\therefore d \doteq 38.85$$

Using the sine rule

$$\frac{y}{\sin 58^\circ} = \frac{38.85}{\sin 95^\circ}$$

$$\therefore y = \frac{38.85 \times \sin 58^\circ}{\sin 95^\circ}$$

$$\therefore y \doteq 33.1$$



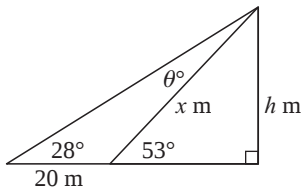
$$\theta = 180^\circ - 30^\circ - 118^\circ = 32^\circ$$

$$\therefore \angle ACD = 58^\circ$$

and $\frac{x}{\sin(180 - 95 - 58)} = \frac{38.85}{\sin 95^\circ}$

$$\therefore x = \frac{38.85 \times \sin 27^\circ}{\sin 95^\circ}$$

$$\therefore x \doteq 17.7$$

EXERCISE 12E
1


$$\begin{aligned}\theta^\circ + 28^\circ &= 53^\circ \\ \{\text{exterior angle of a } \triangle \text{ theorem}\} \\ \therefore \theta &= 25^\circ\end{aligned}$$

By the Sine Rule,

$$\begin{aligned}\frac{x}{\sin 28^\circ} &= \frac{20}{\sin 25^\circ} \\ \therefore x &= \frac{20 \times \sin 28^\circ}{\sin 25^\circ} \\ \therefore x &\doteq 22.22\end{aligned}$$

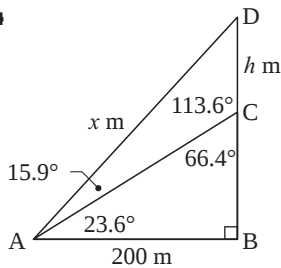
$$\begin{aligned}\text{and } \sin 53^\circ &= \frac{h}{x} \\ \therefore h &= x \sin 53^\circ \\ &= 22.22 \times \sin 53^\circ \\ &\doteq 17.7 \text{ m} \\ \therefore \text{the pole is } 17.7 \text{ m high}\end{aligned}$$

2

$$\begin{aligned}\text{PR}^2 &= 63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^\circ \\ \therefore \text{PR} &= \sqrt{63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^\circ} \\ \therefore \text{PR} &\doteq 207 \text{ m}\end{aligned}$$

3

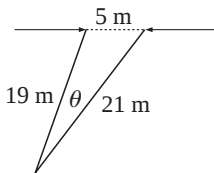
$$\begin{aligned}\cos T &= \frac{220^2 + 340^2 - 165^2}{2 \times 220 \times 340} \\ \therefore T &= \cos^{-1} \left(\frac{136\,775}{149\,600} \right) \\ \therefore T &\doteq 23.9 \\ \therefore \text{was } 23.9^\circ \text{ off line.}\end{aligned}$$

4

 In $\triangle ABD$,

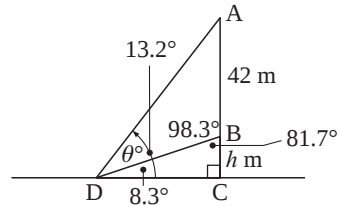
$$\begin{aligned}\cos(23.6 + 15.9)^\circ &= \frac{200}{x} \\ \therefore x &= \frac{200}{\cos 39.5^\circ} \\ \therefore x &\doteq 259.2\end{aligned}$$

 In $\triangle ACD$,

$$\begin{aligned}\frac{h}{\sin 15.9^\circ} &= \frac{259.2}{\sin 113.6^\circ} \\ \therefore h &= \frac{259.2 \times \sin 15.9^\circ}{\sin 113.6^\circ} \\ \therefore h &\doteq 77.5 \\ \therefore \text{is } 77.5 \text{ m high.}\end{aligned}$$

5


$$\begin{aligned}\cos \theta &= \frac{19^2 + 21^2 - 5^2}{2 \times 19 \times 21} \\ \therefore \theta &= \cos^{-1} \left(\frac{777}{798} \right) \\ \therefore \theta &= 13.2 \\ \therefore \text{angle of view is } 13.2^\circ\end{aligned}$$

6


$$\theta = 13.2^\circ - 8.3^\circ = 4.9^\circ$$

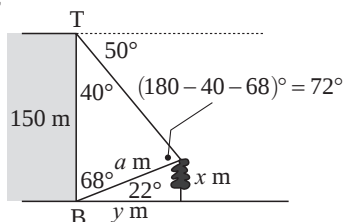
 In $\triangle ABD$,

$$\begin{aligned}\frac{AD}{\sin 98.3^\circ} &= \frac{42}{\sin 4.9^\circ} \\ \therefore AD &= \frac{42 \times \sin 98.3^\circ}{\sin 4.9^\circ} \\ \therefore AD &\doteq 486.56 \text{ m}\end{aligned}$$

 In $\triangle ADC$,

$$\begin{aligned}\sin 13.2^\circ &= \frac{h + 42}{486.56} \\ \therefore h + 42 &= 486.56 \times \sin 13.2^\circ \\ \therefore h + 42 &\doteq 111.1 \\ \therefore h &\doteq 69.1 \\ \therefore \text{the hill is } 69.1 \text{ m high}\end{aligned}$$

7



By the sine rule

$$\frac{a}{\sin 40^\circ} = \frac{150}{\sin 72^\circ}$$

$$\therefore a = \frac{150 \times \sin 40^\circ}{\sin 72^\circ}$$

$$\therefore a \doteq 101.38$$

$$\text{Now } \sin 22^\circ = \frac{x}{101.38}$$

$$\therefore x = 101.38 \times \sin 22^\circ$$

$$\therefore x \doteq 38.0$$

$$\text{and } \cos 22^\circ = \frac{y}{101.38}$$

$$\therefore y = 101.38 \times \cos 22^\circ$$

$$\therefore y \doteq 94.0$$

$\therefore$ the tree is 38.0 m high and 94.0 m from the building.

8 Using Pythagoras' theorem

$$RQ = \sqrt{4^2 + 7^2} = \sqrt{65} \text{ cm}$$

$$PQ = \sqrt{8^2 + 7^2} = \sqrt{113} \text{ cm}$$

$$PR = \sqrt{8^2 + 4^2} = \sqrt{80} \text{ cm}$$

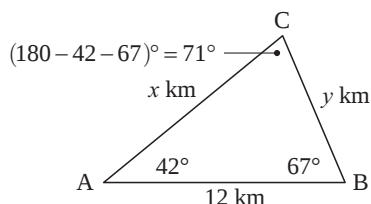
$$\text{Now } \cos Q = \frac{(\sqrt{113})^2 + (\sqrt{65})^2 - (\sqrt{80})^2}{2 \times \sqrt{113} \times \sqrt{65}}$$

$$\therefore \cos \theta \doteq \left(\frac{98}{171.4} \right)$$

$$\therefore \theta = \cos^{-1} \left(\frac{98}{171.4} \right)$$

$$\therefore \theta \doteq 55.1 \quad \text{So } \angle PQR \text{ measures } 55.1^\circ$$

9



$$\frac{x}{\sin 67^\circ} = \frac{12}{\sin 71^\circ} = \frac{y}{\sin 42^\circ}$$

$$\therefore x = \frac{12 \times \sin 67^\circ}{\sin 71^\circ} \quad \text{and} \quad y = \frac{12 \times \sin 42^\circ}{\sin 71^\circ}$$

$$\therefore x \doteq 11.7 \quad \therefore y \doteq 8.49$$

So, C is 11.7 km from A and 8.49 km from B.

$$\text{10 a} \quad QS = \sqrt{8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70^\circ}$$

$$\doteq 11.93$$

$$\therefore \text{area} = \frac{1}{2} \times 8 \times 12 \times \sin 70^\circ + \frac{1}{2} \times 10 \times 11.93 \times \sin 30^\circ$$

$$\doteq 74.9 \text{ km}^2$$

$$\text{b} \quad 1 \text{ ha is } 100 \text{ m} \times 100 \text{ m}$$

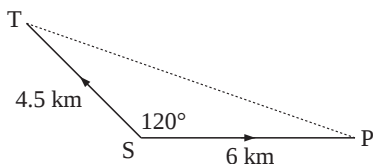
$$= 0.1 \text{ km} \times 0.1 \text{ km}$$

$$= 0.01 \text{ km}^2$$

$$\therefore 1 \text{ km}^2 = 100 \text{ ha}$$

$$\therefore \text{area} \doteq 7490 \text{ ha}$$

11


 Distance = speed $\times$ time

So, after 45 min = 0.75 h

$$ST = 6 \times 0.75 = 4.5 \text{ km}$$

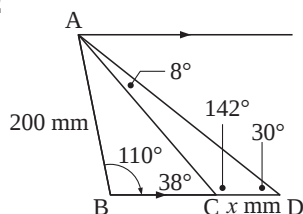
$$SP = 8 \times 0.75 = 6 \text{ km}$$

$$\text{Now } PT = \sqrt{4.5^2 + 6^2 - 2 \times 4.5 \times 6 \times \cos 120^\circ}$$

$$\therefore PT \doteq 9.12$$

So, they are 9.12 km apart.

12



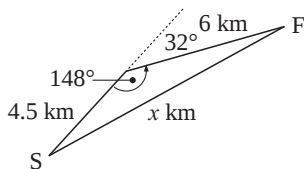
$$\text{In } \triangle ABC \quad \frac{AC}{\sin 110^\circ} = \frac{200}{\sin 38^\circ}$$

$$\therefore AC = \frac{200 \times \sin 110^\circ}{\sin 38^\circ} \doteq 305.26$$

$$\text{and in } \triangle ACD \quad \frac{x}{\sin 8^\circ} = \frac{305.26}{\sin 30^\circ}$$

$$\therefore x = \frac{305.26 \times \sin 8^\circ}{\sin 30^\circ} \doteq 84.968$$

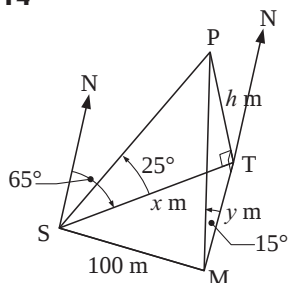
$\therefore$ the metal strip is 85.0 mm wide.

13

$$x = \sqrt{6^2 + (4.5)^2 - 2 \times 6 \times 4.5 \times \cos 148^\circ}$$

$$\therefore x \doteq 10.1$$

$$\therefore \text{ is 10.1 km from the start.}$$

14

$$\text{In } \triangle PST, \tan 25^\circ = \frac{h}{x}$$

$$\text{In } \triangle PMT, \tan 15^\circ = \frac{h}{y}$$

$$\therefore x = \frac{h}{\tan 25^\circ}$$

$$\doteq 2.145h$$

$$\therefore y = \frac{h}{\tan 15^\circ}$$

$$\doteq 3.732h$$

But $\angle STM = 65^\circ$ {equal alternate angles}

$$\text{and } 100^2 = x^2 + y^2 - 2xy \cos 65^\circ$$

$$\therefore 10\,000 = (2.145h)^2 + (3.732h)^2 - 2 \times (2.145)(3.732)h^2 \cos 65^\circ$$

$$\therefore 10\,000 \doteq 11.763h^2$$

$$\therefore h^2 \doteq 850.15$$

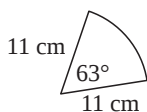
$$\therefore h \doteq 29.2 \quad \text{So, the tree is 29.2 m high.}$$

REVIEW SET 12A

1 Area = $\frac{1}{2} \times 7.3 \times 9.4 \times \sin 38^\circ$
 $\doteq 21.1 \text{ km}^2$

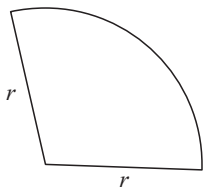
2 a Area = $\left(\frac{80}{360}\right) \times T \times 13^2 \doteq 118 \text{ cm}^2$

b Area = $\frac{1}{2} \times 11 \times 9 \times \sin 65^\circ \doteq 44.9 \text{ cm}^2$

3

a Perimeter = $2 \times 11 + \left(\frac{63}{360}\right) \times 2\pi \times 11$
 $\doteq 34.1 \text{ cm}$

b Area = $\left(\frac{63}{360}\right) \times \pi \times 11^2$
 $\doteq 66.5 \text{ cm}^2$

4

$$\text{Perimeter} = 2r + \left(\frac{120}{360}\right) \times 2\pi r \quad \text{Area} = \left(\frac{120}{360}\right) \times \pi r^2$$

$$\therefore 36 = 2r + \frac{1}{3} \times 2\pi r \quad = \frac{1}{3} \times \pi \times (8.7925)^2$$

$$\therefore 36 = r \left(2 + \frac{2\pi}{3}\right) \quad \doteq 81.0 \text{ cm}^2$$

$$\therefore r = \frac{36}{2 + \frac{2\pi}{3}} \text{ cm}$$

$$\therefore r \doteq 8.7925 \dots$$

$$\therefore r \doteq 8.79 \text{ cm}$$

5 Area = $42 \text{ cm}^2 \quad \therefore \frac{1}{2} \times 7 \times 13 \times \sin x = 42$

$$\therefore \sin x = \frac{42 \times 2}{7 \times 13}$$

$$\therefore x = \sin^{-1} \left(\frac{84}{91}\right)$$

$$\therefore x \doteq 67.4 \quad \text{or} \quad 180 - 67.4$$

$$\therefore x = 67.4 \quad \text{or} \quad 112.6$$

{assuming the figure is not drawn accurately}

6

$$\text{Area} = 80 \text{ cm}^2$$

$$\therefore \frac{1}{2} \times 11.3 \times 19.2 \sin x^\circ = 80$$

$$\therefore \sin x^\circ = \frac{160}{11.3 \times 19.2}$$

$$\therefore x = \sin^{-1} \left(\frac{160}{11.3 \times 19.2}\right)$$

$$\therefore x \doteq 47.5 \quad \text{or} \quad 180 - 47.5$$

$$\therefore x \doteq 47.5 \quad \text{or} \quad 132.5$$

$$\text{When } x \div 47.5, \quad AC \div \sqrt{19.2^2 + 11.3^2 - 2 \times 19.2 \times 11.3 \times \cos 47.5^\circ} \div 14.3 \text{ cm}$$

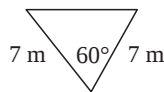
$$\text{When } x \div 132.5, \quad AC \div \sqrt{19.2^2 + 11.3^2 - 2 \times 19.2 \times 11.3 \times \cos 132.5^\circ} \div 28.1 \text{ cm}$$

$$\begin{aligned} \mathbf{7} \quad & \text{Non shaded area} & \therefore & \text{shaded area} \\ & = \text{area of } \Delta + \text{area of sector} & & \div \pi \times 7^2 - 117.12 \text{ cm}^2 \\ & = \frac{1}{2} \times 7 \times 7 \times \sin 130^\circ + \left(\frac{230}{360}\right) \times \pi \times 7^2 & & \div 36.8 \text{ cm}^2 \\ & \div 117.12 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad & BD = \sqrt{125^2 + 120^2 - 2 \times 125 \times 120 \times \cos 75^\circ} \div 149.2 \\ \therefore \text{ area} &= \frac{1}{2} \times 120 \times 125 \sin 75^\circ + \frac{1}{2} \times 149.2 \times 90 \times \sin 30^\circ \\ &\div 10\,600 \text{ m}^2 \end{aligned}$$

$$\mathbf{b} \quad \div 1.06 \text{ ha} \quad (10\,000 \text{ m}^2 = 1 \text{ ha})$$

$$\begin{aligned} \mathbf{9} \quad \text{Shaded area} &= \text{area of circle} - \text{area of hexagon} \\ &= \pi \times 7^2 - 6 \times \text{area of one } \nabla \\ &= 49\pi - 6 \times \frac{1}{2} \times 7 \times 7 \times \sin 60^\circ \\ &\div 26.6 \text{ m}^2 \end{aligned}$$



REVIEW SET 12B

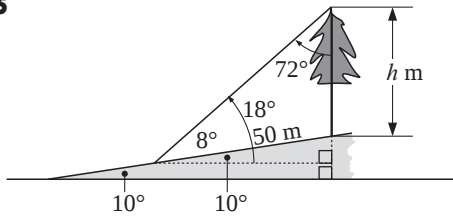
$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \cos x = \frac{13^2 + 19^2 - 11^2}{2 \times 13 \times 19} & \mathbf{b} \quad & x = \sqrt{15^2 + 17^2 - 2 \times 15 \times 17 \times \cos 72^\circ} \\ & \therefore \cos x = \frac{409}{494} & & \therefore x \div 18.9 \\ & \therefore x = \cos^{-1}\left(\frac{409}{494}\right) \\ & \therefore x \div 34.1 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & \cos x = \frac{11^2 + 19^2 - 13^2}{2 \times 11 \times 19} & \mathbf{b} \quad & x = \sqrt{14^2 + 21^2 - 2 \times 14 \times 21 \times \cos 47^\circ} \\ & \therefore \cos x = \frac{313}{418} & & \therefore x \div 15.4 \\ & \therefore x = \cos^{-1}\left(\frac{313}{418}\right) \\ & \therefore x \div 41.5 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad & AC = \sqrt{11^2 + 9.8^2 - 2 \times 11 \times 9.8 \times \cos 74^\circ} & \text{Now } & \frac{\sin C}{11} = \frac{\sin 74^\circ}{12.554} \\ \therefore AC &\div 12.554 \dots & & \therefore \sin C = \frac{11 \times \sin 74^\circ}{12.554} \\ \therefore AC &\div 12.6 \text{ cm} & & \therefore C = \sin^{-1}\left(\frac{11 \times \sin 74^\circ}{12.554}\right) \\ & & & \therefore C = 57.4^\circ \text{ or } 122.6^\circ \\ & & & \quad \quad \quad \uparrow \\ & & & \text{impossible as } 122.6 + 74 > 180 \\ & & & \therefore \angle C \text{ measures } 57.4^\circ \\ & & & \therefore \angle A \text{ measures } 48.6^\circ \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad & DB = \sqrt{7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 110^\circ} \\ & \div 14.922 \\ \therefore \text{ total area} &= \frac{1}{2} \times 7 \times 11 \times \sin 110^\circ + \frac{1}{2} \times 16 \times 14.922 \times \sin 40^\circ \\ &\div 113 \text{ cm}^2 \end{aligned}$$

5



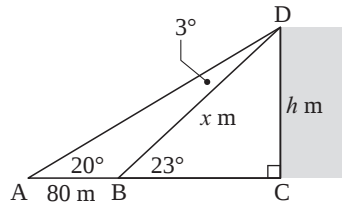
$$\frac{h}{\sin 8^\circ} = \frac{50}{\sin 72^\circ}$$

$$\therefore h = \frac{50 \times \sin 8^\circ}{\sin 72^\circ}$$

$$\therefore h \doteq 7.32$$

i.e., the tree is 7.32 m high

6



In $\triangle ABD$, $\frac{x}{\sin 20^\circ} = \frac{80}{\sin 3^\circ}$

$$\therefore x = \frac{80 \times \sin 20^\circ}{\sin 3^\circ}$$

$$\doteq 522.8$$

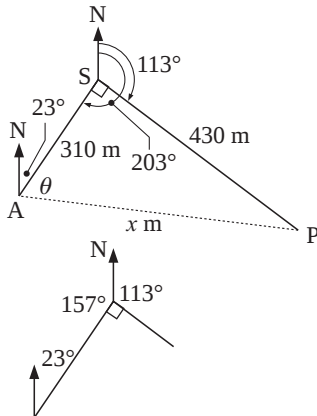
Now $\sin 23^\circ = \frac{h}{x}$

$$\therefore h = 522.8 \times \sin 23^\circ$$

$$\therefore h \doteq 204$$

So the building is 204 m tall.

7



$$\angle ASP = 203^\circ - 113^\circ = 90^\circ$$

Now $x^2 = 310^2 + 430^2$ {Pythagoras}

$$\therefore x = \sqrt{310^2 + 430^2}$$

$$\therefore x \doteq 530$$

$\therefore$ they are 530 m apart.

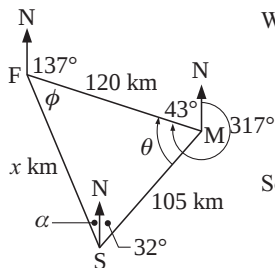
and $\tan \theta = \frac{430}{310}$

$$\therefore \theta = \tan^{-1} \left(\frac{430}{310} \right) \doteq 54.2$$

and $23 + \theta = 77.2$

$\therefore$ bearing of P from A is 077.2°

8



In 45 minutes, $140 \times \frac{3}{4} = 105$ km is travelled.

In 40 minutes, $180 \times \frac{2}{3} = 120$ km is travelled.

We notice that $\theta + 43 + 32 = 180$ {allied angles add to 180° }

$$\therefore \theta = 105$$

and so, $x = \sqrt{120^2 + 105^2 - 2 \times 120 \times 105 \times \cos 105^\circ}$

$$\therefore x \doteq 178.74$$

So, is 179 km from the start.

Now $\frac{\sin \phi}{105} = \frac{\sin 105^\circ}{178.74}$

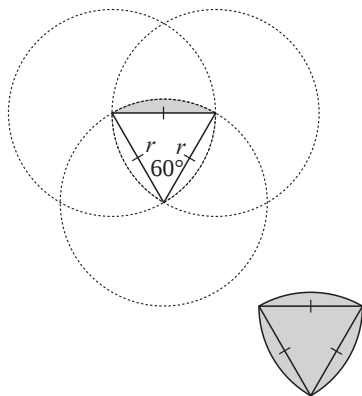
$$\therefore \sin \phi = \frac{105 \times \sin 105^\circ}{178.74}$$

$$\therefore \phi \doteq 34.6$$

$$\therefore \theta = 180 - 105 - 34.6 - 32 = 8.4 \doteq 8$$

So, the bearing is 352° .

9



shaded area of sector

= area of sector – area of Δ

$$= \frac{1}{6}\pi r^2 - \frac{1}{2} \times r \times r \times \sin 60^\circ$$

$$= \frac{\pi}{6}r^2 - \frac{1}{2}r^2 \left(\frac{\sqrt{3}}{2} \right)$$

 $\therefore$ shaded area of figure

$$= 3 \left[\frac{\pi}{6}r^2 - \frac{\sqrt{3}}{4}r^2 \right] + \frac{1}{2}r^2 \left(\frac{\sqrt{3}}{2} \right)$$

$$= \frac{\pi}{2}r^2 - \frac{3\sqrt{3}}{4}r^2 + \frac{\sqrt{3}}{4}r^2$$

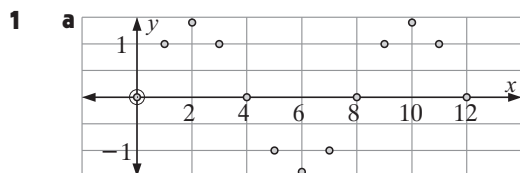
$$= \frac{\pi}{2}r^2 - \frac{1}{2}\sqrt{3}r^2$$

$$= \frac{r^2}{2}(\pi - \sqrt{3})$$

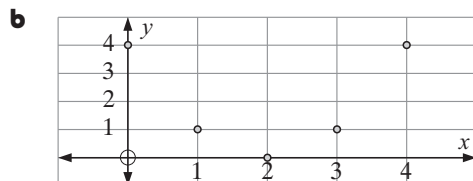
Chapter 13

PERIODIC PHENOMENA

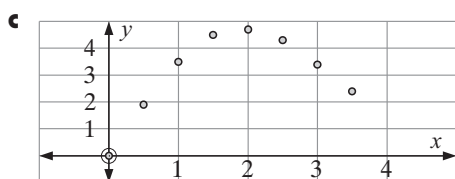
EXERCISE 13A



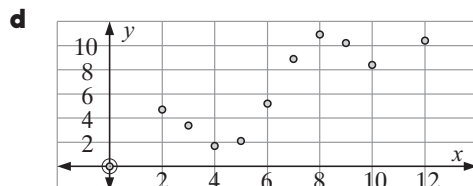
Data exhibits periodic behaviour.



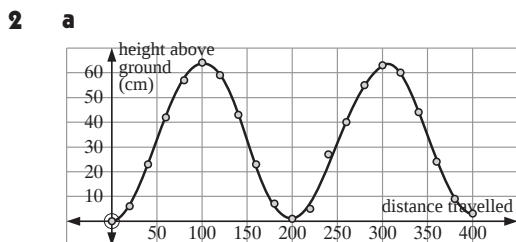
Not enough information to say data is periodic. It may in fact be quadratic.



Not enough information to say data is periodic. It may in fact be quadratic.



Not enough information to say data is periodic.



c A curve can be fitted to the data as the distance travelled is continuous.

3 a periodic **b** periodic **c** periodic **d** not periodic **e** periodic **f** periodic

EXERCISE 13B.1

1 a $\pi \equiv 180^\circ$ **b** $-\frac{\pi}{4} \equiv -45^\circ$ **c** $\pi \equiv 180^\circ$ **d** $-\frac{3\pi}{2} \equiv -270^\circ$
 $\therefore \frac{\pi}{4} \equiv 45^\circ$ $\therefore \frac{3\pi}{2} \equiv \frac{3}{2} \times 180^\circ$
 $\equiv 270^\circ$
e $\pi \equiv 180^\circ$ **f** $\pi \equiv 180^\circ$
 $\therefore \frac{\pi}{3} \equiv 60^\circ$ $\therefore \frac{\pi}{6} \equiv 30^\circ$

2 a 30° is $\frac{1}{12}$ of 360° **b** 60° is $\frac{1}{6}$ of 360° **c** 150° is $\frac{15}{36}$ of 360°
 $\therefore d = \frac{1}{12} \times 2\pi$ $\therefore d = \frac{1}{6} \times 2\pi$ $\therefore d = \frac{15}{36} \times 2\pi$
 $\therefore d = \frac{\pi}{6}$ $\therefore d = \frac{\pi}{3}$ $\therefore d = \frac{5\pi}{6}$
d 135° is $\frac{135}{360}$ of 360° **e** 240° is $\frac{2}{3}$ of 360° **f** 45° is $\frac{1}{8}$ of 360°
 $\therefore d = \frac{135}{360} \times 2\pi$ $\therefore d = \frac{2}{3} \times 2\pi$ $\therefore d = -\frac{1}{8} \times 2\pi$
 $\therefore d = \frac{3\pi}{4}$ $\therefore d = \frac{4\pi}{3}$ $\therefore d = -\frac{\pi}{4}$

g From **d**,
 $d = -\frac{3\pi}{4}$

h 270° is $\frac{27}{36}$ of 360°
 $\therefore d = -\frac{3}{4} \times 2\pi$
 $\therefore d = -\frac{3\pi}{2}$

3 a $\triangle OAP$ is isosceles, so $\angle AOP = 60^\circ \therefore \theta = 60$.

b 60° is $\frac{1}{6}$ of 360° , $\therefore d = \frac{1}{6} \times 2\pi$ i.e., $d = \frac{\pi}{3}$.

c As $AP < \text{arc } AP$, then $AP < 1 \therefore \theta < 60^\circ$, i.e., θ decreases.

d When $d = \pi$, $\theta = 180^\circ$

So, when $d = 1$, $\theta = \frac{180^\circ}{\pi} \div 57.3$

EXERCISE 13B.2

1 a $180^\circ = \pi$ radians
 $\therefore 90^\circ = \frac{\pi}{2}$ radians

b $180^\circ = \pi$ radians
 $\therefore 60^\circ = \frac{\pi}{3}$ radians

c $180^\circ = \pi$ radians
 $\therefore 30^\circ = \frac{\pi}{6}$ radians

d $180^\circ = \pi$ radians
 $\therefore 18^\circ = \frac{\pi}{10}$ radians

e $180^\circ = \pi$ radians
 $\therefore 9^\circ = \frac{\pi}{20}$ radians

f $180^\circ = \pi$ radians
 $\therefore 45^\circ = \frac{\pi}{4}$ radians
 $\therefore 135^\circ = \frac{3\pi}{4}$ radians

g $180^\circ = \pi$ radians
 $\therefore 45^\circ = \frac{\pi}{4}$ radians
 $\therefore 225^\circ = \frac{5\pi}{4}$ radians

h $180^\circ = \pi$ radians
 $\therefore 90^\circ = \frac{\pi}{2}$ radians
 $\therefore 270^\circ = \frac{3\pi}{2}$ radians

i $360^\circ = 2 \times 180^\circ$
 $= 2\pi$ radians

j $720^\circ = 4 \times 180^\circ$
 $= 4\pi$ radians

k $180^\circ = \pi$ radians
 $\therefore 45^\circ = \frac{\pi}{4}$ radians
 $\therefore 315^\circ = \frac{7\pi}{4}$ radians

l $180^\circ = \pi$ radians
 $\therefore 540^\circ = 3\pi$ radians

m $180^\circ = \pi$ radians
 $\therefore 36^\circ = \frac{\pi}{5}$ radians

n $180^\circ = \pi$ radians
 $\therefore 10^\circ = \frac{\pi}{18}$ radians
 $\therefore 80^\circ = \frac{8\pi}{18}$ radians
 $= \frac{4\pi}{9}$ radians

o $180^\circ = \pi$ radians
 $\therefore 10^\circ = \frac{\pi}{18}$ radians
 $\therefore 230^\circ = \frac{23\pi}{18}$ radians

2 a 36.7°
 $= 36.7 \times \frac{\pi}{180}$ radians
 $\div 0.641$ radians

b 137.2°
 $= 137.2 \times \frac{\pi}{180}$ radians
 $\div 2.39$ radians

c 317.9°
 $= 317.9 \times \frac{\pi}{180}$ radians
 $\div 5.55$ radians

d 219.6°
 $= 219.6 \times \frac{\pi}{180}$ radians
 $\div 3.83$ radians

e 396.7°
 $= 396.7 \times \frac{\pi}{180}$ radians
 $\div 6.92$ radians

3 a $\frac{\pi}{5}$
 $= \frac{180^\circ}{5}$
 $= 36^\circ$

b $\frac{3\pi}{5}$
 $= \frac{3 \times 180^\circ}{5}$
 $= 108^\circ$

c $\frac{3\pi}{4}$
 $= \frac{3 \times 180^\circ}{4}$
 $= 135^\circ$

d $\frac{\pi}{18}$
 $= \frac{180^\circ}{18}$
 $= 10^\circ$

e $\frac{\pi}{9}$
 $= \frac{180^\circ}{9}$
 $= 20^\circ$

f $\frac{7\pi}{9}$
 $= \frac{7 \times 180^\circ}{9}$
 $= 140^\circ$

g $\frac{\pi}{10}$
 $= \frac{180^\circ}{10}$
 $= 18^\circ$

h $\frac{3\pi}{20}$
 $= \frac{3 \times 180^\circ}{20}$
 $= 27^\circ$

i $\frac{5\pi}{6}$
 $= \frac{5 \times 180^\circ}{6}$
 $= 150^\circ$

j $\frac{\pi}{8}$
 $= \frac{180^\circ}{8}$
 $= 22\frac{1}{2}^\circ$

4 a 2^c
 $= 2 \times \frac{180}{\pi}$ degrees
 $\div 114.59^\circ$

b 1.53^c
 $= 1.53 \times \frac{180}{\pi}$ degrees
 $\div 87.66^\circ$

c 0.867^c
 $= 0.867 \times \frac{180}{\pi}$ degrees
 $\div 49.68^\circ$

d 3.179^c
 $= 3.179 \times \frac{180}{\pi}$ degrees
 $\div 182.14^\circ$

e 5.267^c
 $= 5.267 \times \frac{180}{\pi}$ degrees
 $\div 301.78^\circ$

5 a

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

6 a i arc length, AB

$$l = \frac{\theta^c}{2\pi^c} \times 2\pi r$$

$$= r\theta$$

i.e., $l = r\theta$

$\left\{ \frac{\theta^c}{2\pi^c} \right\}$ is the fraction of the whole circle}

ii sector area

$$A = \frac{\theta^c}{2\pi^c} \times \pi r^2$$

$$= \frac{1}{2} r^2 \theta$$

i.e., $A = \frac{1}{2} r^2 \theta$

b i $l = r\theta$
 $= 12 \text{ cm} \times 0.8$
 $= 9.6 \text{ cm}$

ii $l = r\theta$
 $= 8 \text{ cm} \times 1.75$
 $= 14 \text{ cm}$

iii $l = r\theta$
 $= 11 \text{ cm} \times (2\pi - 2)$
 $\div 47.1 \text{ cm}$

c i $l = r\theta$
 $\therefore 6 = 8\theta$
 $\therefore \theta = 0.75^c$

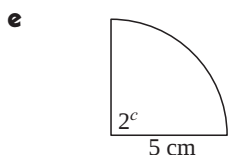
ii $l = r\theta$
 $\therefore 8.4 = 5\theta$
 $\therefore \theta = \frac{8.4}{5}$
 $\therefore \theta = 1.68^c$

iii $l = r\theta$
 $\therefore 31.7 = 8(2\pi - \theta)$
 $\therefore 2\pi - \theta = \frac{31.7}{8}$
 $\therefore \theta = 2\pi - \frac{31.7}{8}$
 $\therefore \theta \div 2.32^c$

d i $A = \frac{1}{2} r^2 \theta$
 $= \frac{1}{2} \times 5^2 \times 0.7$
 $= 8.75 \text{ cm}^2$

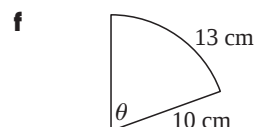
ii $A = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta$
 $= \frac{1}{2} r^2 (\theta - \sin \theta)$
 $= \frac{1}{2} \times 12^2 (1.5 - \sin 1.5)$
 $\div 36.2 \text{ cm}^2$

iii $A = \frac{1}{2} (\text{OA})(\text{OP}) \sin(0.66)$
 $- \frac{1}{2} (\text{OA})^2 \times (0.66)$
 $= \frac{1}{2} (12)(30) \sin(0.66)$
 $- \frac{1}{2} (12)^2 \times (0.66)$
 $\div 62.8 \text{ cm}^2$



arc length
 $= r\theta$
 $= 5 \times 2^c$
 $= 10 \text{ cm}$

area
 $= \frac{1}{2} r^2 \theta$
 $= \frac{1}{2} \times 5^2 \times 2$
 $= 25 \text{ cm}^2$

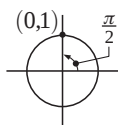


$l = r\theta$
 $\therefore 13 = 10\theta$
 $\therefore \theta = 1.3$

and $A = \frac{1}{2} r^2 \theta$
 $= \frac{1}{2} \times 10^2 \times 1.3$
 $= 65 \text{ cm}^2$

EXERCISE 13C.1

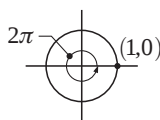
1 a



$$\cos\left(\frac{\pi}{2}\right) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

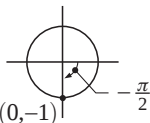
b



$$\cos 2\pi = 1$$

$$\sin 2\pi = 0$$

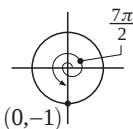
c



$$\cos\left(-\frac{\pi}{2}\right) = 0$$

$$\sin\left(-\frac{\pi}{2}\right) = -1$$

d



$$\cos\left(\frac{7\pi}{2}\right) = 0$$

$$\sin\left(\frac{7\pi}{2}\right) = -1$$

2

$$\text{a } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + \frac{1}{4} = 1$$

$$\therefore \cos^2 \theta = \frac{3}{4}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{3}}{2}$$

$$\text{b } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + \frac{1}{9} = 1$$

$$\therefore \cos^2 \theta = \frac{8}{9}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{8}}{3}$$

$$= \pm \frac{2\sqrt{2}}{3}$$

$$\text{c } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + 0 = 1$$

$$\therefore \cos \theta = \pm 1$$

$$\text{d } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + 1 = 1$$

$$\therefore \cos \theta = 0$$

3

$$\text{a } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \frac{16}{25} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{9}{25}$$

$$\therefore \sin \theta = \pm \frac{3}{5}$$

$$\text{b } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \frac{9}{16} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{7}{16}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$$

$$\text{c } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore 1 + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = 0$$

$$\therefore \sin \theta = 0$$

$$\text{d } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = 1$$

$$\therefore 0 + \sin^2 \theta = 1$$

$$\therefore \sin \theta = \pm 1$$

4

a

Quadrant	Degree measure	Radian measure	$\cos \theta$	$\sin \theta$
1	$0 < \theta < 90$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve
2	$90 < \theta < 180$	$\frac{\pi}{2} < \theta < \pi$	-ve	+ve
3	$180 < \theta < 270$	$\pi < \theta < \frac{3\pi}{2}$	-ve	-ve
4	$270 < \theta < 360$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	-ve

b i 1 and 4

ii 2 and 3

iii 3

iv 2

5

$$\text{a } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \frac{4}{9} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{5}{9}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{5}}{3}$$

But θ is in quadrant 1where $\sin \theta > 0$

$$\therefore \sin \theta = \frac{\sqrt{5}}{3}$$

$$\text{b } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + \frac{4}{25} = 1$$

$$\therefore \cos^2 \theta = \frac{21}{25}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{21}}{5}$$

But θ is in quadrant 2where $\cos \theta < 0$

$$\therefore \cos \theta = -\frac{\sqrt{21}}{5}$$

$$\text{c } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \cos^2 \theta + \frac{9}{25} = 1$$

$$\therefore \cos^2 \theta = \frac{16}{25}$$

$$\therefore \cos \theta = \pm \frac{4}{5}$$

But θ is in quadrant 4where $\cos \theta > 0$

$$\therefore \cos \theta = \frac{4}{5}$$

$$\text{d } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \frac{25}{169} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{144}{169}$$

$$\therefore \sin \theta = \pm \frac{12}{13}$$

But θ is in quadrant 3where $\sin \theta < 0$

$$\therefore \sin \theta = -\frac{12}{13}$$

$$\text{e } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \frac{1}{4} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{3}{4}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{3}}{2}$$

But θ is in quadrant 4where $\sin \theta < 0$

$$\therefore \sin \theta = -\frac{\sqrt{3}}{2}$$

$$\text{f } \cos^2 \theta + \sin^2 \theta = 1$$

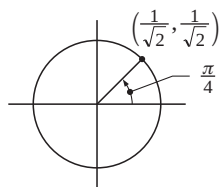
$$\therefore \cos^2 \theta + \frac{1}{2} = 1$$

$$\therefore \cos^2 \theta = \frac{1}{2}$$

$$\therefore \cos \theta = \pm \frac{1}{\sqrt{2}}$$

But θ is in quadrant 3where $\cos \theta < 0$

$$\therefore \cos \theta = -\frac{1}{\sqrt{2}}$$

EXERCISE 13C.2**1**

$$\text{So } \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Draw separate unit circle diagrams for each case.

	a	b	c	d	e
$\sin \theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos \theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$

2 Likewise, draw separate unit circle diagrams for each angle.

	a	b	c	d	e
$\sin \beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos \beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

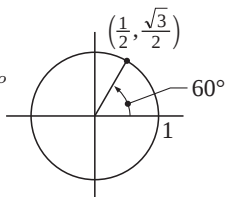
3

a $\sin^2 60^\circ$

$$= \sin 60^\circ \times \sin 60^\circ$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$

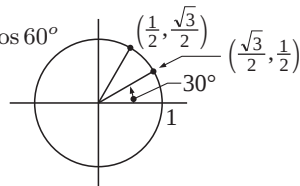
$$= \frac{3}{4}$$

**b**

$\sin 30^\circ \cos 60^\circ$

$$= \frac{1}{2} \times \frac{1}{2}$$

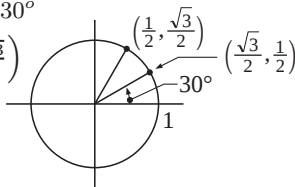
$$= \frac{1}{4}$$



c $4 \sin 60^\circ \cos 30^\circ$

$$= 4 \left(\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right)$$

$$= 3$$

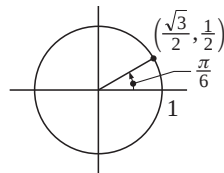
**d**

$1 - \cos^2\left(\frac{\pi}{6}\right)$

$$= 1 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}$$

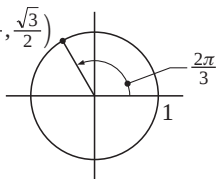


e $\sin^2\left(\frac{2\pi}{3}\right) - 1$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= \frac{3}{4} - 1$$

$$= -\frac{1}{4}$$

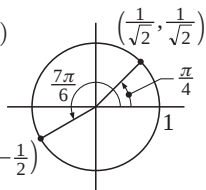
**f**

$\cos^2\left(\frac{\pi}{4}\right) - \sin\left(\frac{7\pi}{6}\right)$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$



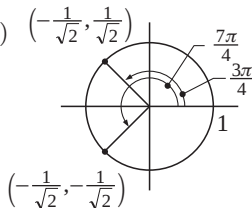
g $\sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right)$

$$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

**h**

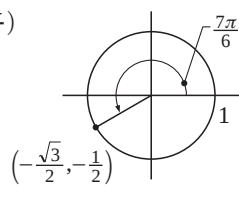
$1 - 2 \sin^2\left(\frac{7\pi}{6}\right)$

$$= 1 - 2\left(-\frac{1}{2}\right)^2$$

$$= 1 - 2 \times \frac{1}{4}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

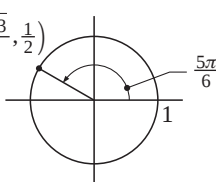


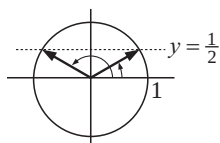
i $\cos^2\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)$

$$= \left(-\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

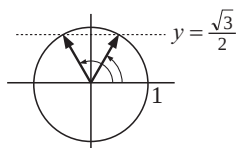
$$= \frac{3}{4} - \frac{1}{4}$$

$$= \frac{1}{2}$$

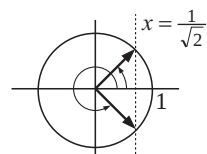


4 a


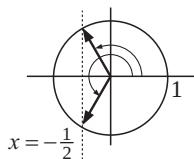
$$\theta = 30^\circ \text{ or } 150^\circ$$

b


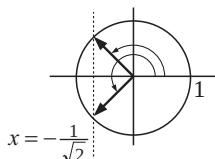
$$\theta = 60^\circ \text{ or } 120^\circ$$

c


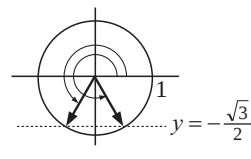
$$\theta = 45^\circ \text{ or } 315^\circ$$

d


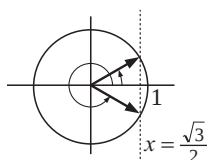
$$\theta = 120^\circ \text{ or } 240^\circ$$

e


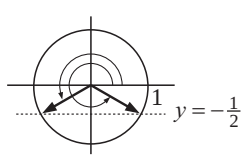
$$\theta = 135^\circ \text{ or } 225^\circ$$

f


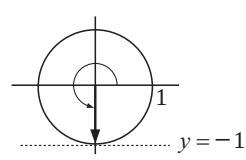
$$\theta = 240^\circ \text{ or } 300^\circ$$

5 a


$$\theta = 30^\circ, 330^\circ, 390^\circ, 690^\circ$$

b


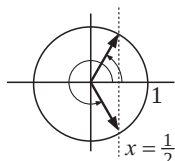
$$\theta = 210^\circ, 330^\circ, 570^\circ, 690^\circ$$

c


$$\theta = 270^\circ, 630^\circ$$

6 a

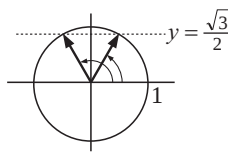
$$\cos \theta = \frac{1}{2}$$



$$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

b

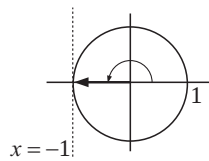
$$\sin \theta = \frac{\sqrt{3}}{2}$$



$$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}$$

c

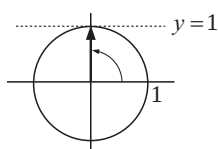
$$\cos \theta = -1$$



$$\therefore \theta = \pi$$

d

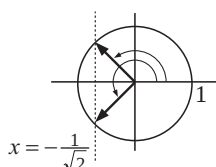
$$\sin \theta = 1$$



$$\therefore \theta = \frac{\pi}{2}$$

e

$$\cos \theta = -\frac{1}{\sqrt{2}}$$

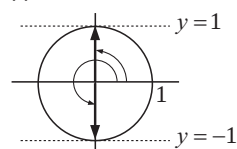


$$\therefore \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$$

f

$$\sin^2 \theta = 1$$

$$\therefore \sin \theta = \pm 1$$

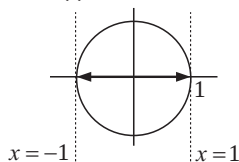


$$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

g

$$\cos^2 \theta = 1$$

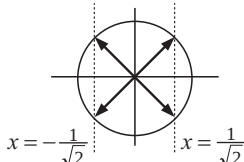
$$\therefore \cos \theta = \pm 1$$



$$\therefore \theta = 0, \pi, 2\pi$$

h

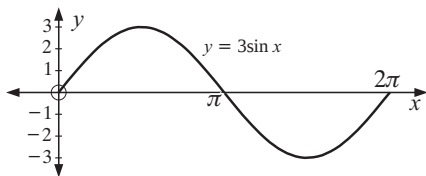
$$\cos^2 \theta = \frac{1}{2} \therefore \cos \theta = \pm \frac{1}{\sqrt{2}}$$



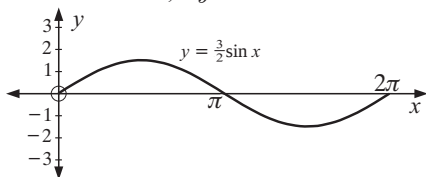
$$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

EXERCISE 13D.1

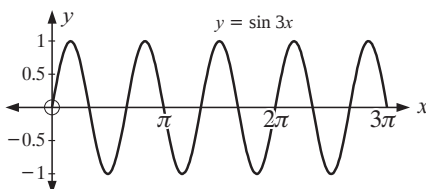
- 1 a** $y = 3 \sin x$
has amplitude 3 and period $\frac{2\pi}{1} = 2\pi$
When $x = 0$, $y = 0$.



- c** $y = \frac{3}{2} \sin x$
has amplitude $\frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.
When $x = 0$, $y = 0$.

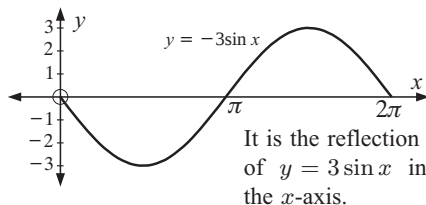


- 2 a** $y = \sin 3x$
has amplitude 1 and period $\frac{2\pi}{3}$.
When $x = 0$, $y = 0$.

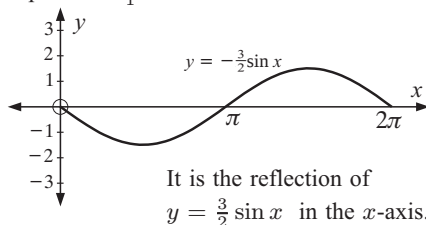


- c** $y = \sin(-2x)$
has amplitude 1 and period $\frac{2\pi}{|2|} = \pi$.
When $x = 0$, $y = 0$.

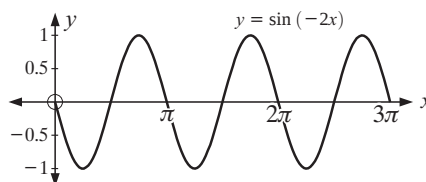
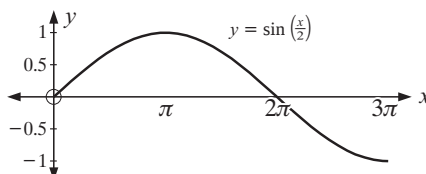
- b** $y = -3 \sin x$
has amplitude $|-3| = 3$ and period $\frac{2\pi}{1} = 2\pi$.
When $x = 0$, $y = 0$.



- d** $y = -\frac{3}{2} \sin x$
has amplitude $|\frac{-3}{2}| = \frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.



- b** $y = \sin(\frac{x}{2})$
has amplitude 1 and period $\frac{2\pi}{\frac{1}{2}} = 4\pi$.
When $x = 0$, $y = 0$.



It is the reflection of $y = \sin 2x$ in the x -axis.

3 a period $= \frac{2\pi}{|4|}$
 $= \frac{\pi}{2}$

b period $= \frac{2\pi}{|-4|}$
 $= \frac{\pi}{2}$

c period $= \frac{2\pi}{|\frac{1}{3}|}$
 $= 6\pi$

d period $= \frac{2\pi}{0.6}$
 $= \frac{20\pi}{6}$
 $= \frac{10\pi}{3}$

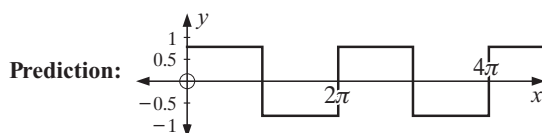
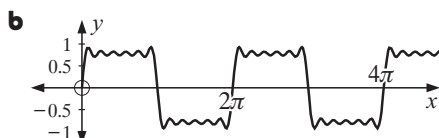
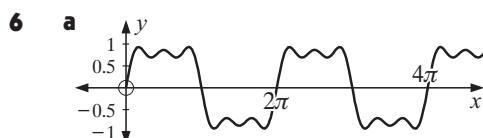
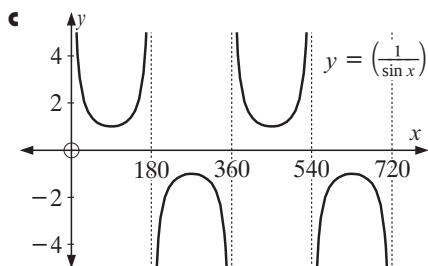
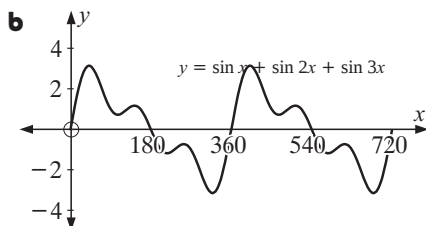
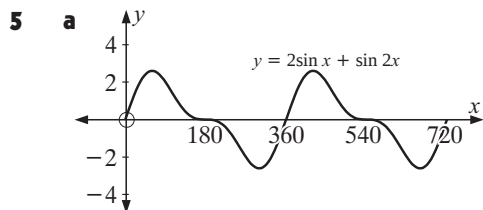
4 a $\frac{2\pi}{B} = 5\pi$
 $\therefore B = \frac{2}{5}$

b $\frac{2\pi}{B} = \frac{2\pi}{3}$
 $\therefore B = 3$

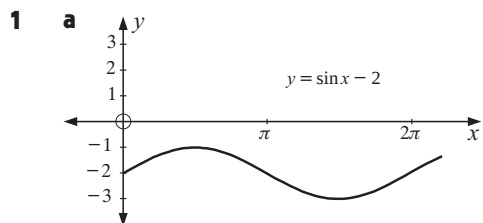
c $\frac{2\pi}{B} = 12\pi$
 $\therefore B = \frac{1}{6}$

d $\frac{2\pi}{B} = 4$
 $\therefore B = \frac{\pi}{2}$

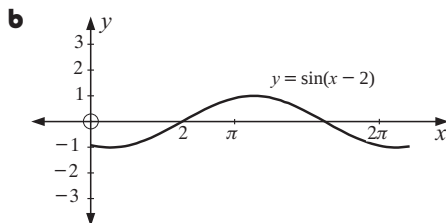
e $\frac{2\pi}{B} = 100$
 $\therefore B = \frac{2\pi}{100} = \frac{\pi}{50}$



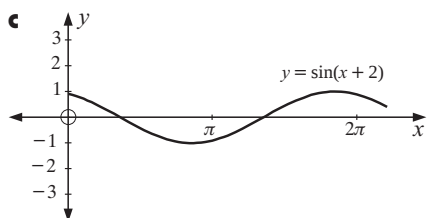
EXERCISE 13D.2



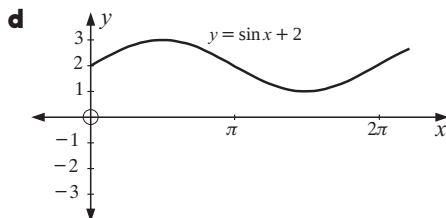
This is the graph of $y = \sin x$
translated by $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$.



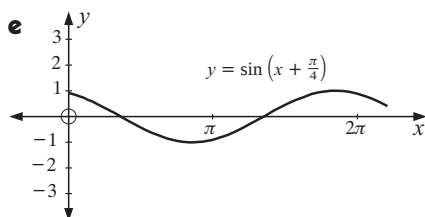
This is the graph of $y = \sin x$
translated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.



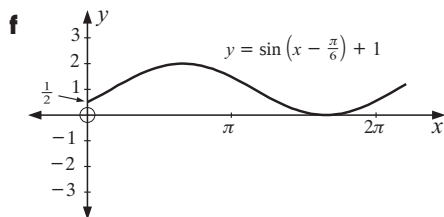
This is the graph of $y = \sin x$
translated by $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$.



This is the graph of $y = \sin x$
translated by $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.



This is the graph of $y = \sin x$
translated by $\begin{bmatrix} -\frac{\pi}{4} \\ 0 \end{bmatrix}$.



This is the graph of $y = \sin x$
translated by $\begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$.

3 a period = $\frac{2\pi}{|5|} = \frac{2\pi}{5}$

b period = $\frac{2\pi}{|\frac{1}{4}|} = 8\pi$

c period = $\frac{2\pi}{|-2|} = \pi$

4 a $\frac{2\pi}{B} = 3\pi$
 $\therefore B = \frac{2}{3}$

b $\frac{2\pi}{B} = \frac{\pi}{10}$
 $\therefore B = 20$

c $\frac{2\pi}{B} = 100\pi$
 $\therefore B = \frac{2}{100} = \frac{1}{50}$

d $\frac{2\pi}{B} = 50$
 $\therefore B = \frac{2\pi}{50} = \frac{\pi}{25}$

5 a A translation of $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, i.e., vertically down 1 unit.

b A translation of $\begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$, i.e., horizontally $\frac{\pi}{4}$ units right.

c A vertical dilation of factor 2.

d A horizontal dilation of factor $\frac{1}{4}$.

e A vertical dilation of factor $\frac{1}{2}$.

f A horizontal dilation of factor $\frac{1}{\frac{1}{4}} = 4$.

g A reflection in the x -axis.

h A translation of $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

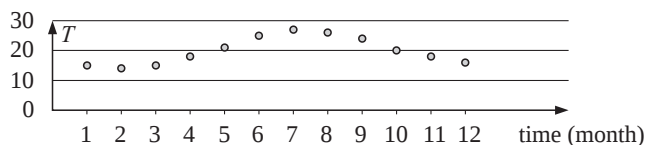
i A vertical dilation of factor 2 followed by a horizontal dilation of factor $\frac{1}{3}$.

j A translation of $\begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$.

EXERCISE 13E

1 a

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	14	15	18	21	25	27	26	24	20	18	16



The period is 12 months so $\frac{2\pi}{B} = 12 \therefore B = \frac{\pi}{6}$ {assuming $B > 0$ }.

Amplitude, $A \doteq \frac{\text{max.} - \text{min.}}{2} \doteq \frac{27 - 14}{2} \doteq 6.5$

As the principal axis is midway between min. and max., then $D \doteq \frac{27 + 14}{2} \doteq 20.5$
When T is 20.5 (midway between min. and max.)

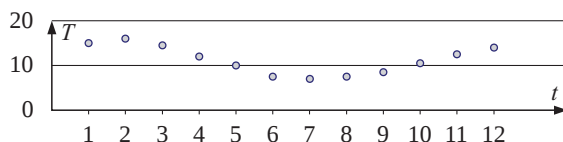
$C \doteq \frac{2 + 7}{2} \doteq 4.5$ {average of t values}

$\therefore T \doteq 6.5 \sin \frac{\pi}{6}(t - 4.5) + 20.5$ (Note: $\frac{\pi}{6} \doteq 0.524$)

b Using technology, $T \doteq 6.14 \sin(0.575t - 2.70) + 20.4$
i.e., $T \doteq 6.14 \sin 0.575(t - 4.70) + 20.4$

2 a

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14



The period is $\frac{2\pi}{B} = 12 \quad \therefore B = \frac{\pi}{6} \quad \{B > 0\}$

Amplitude, $A \div \frac{\text{max.} - \text{min.}}{2} \div \frac{16 - 7}{2} \div 4.5$

As the principal axis is midway between min. and max. then $D \div \frac{16 + 7}{2} \div 11.5$

At min., $t = 7$ and at max., $t = 2 + 12 = 14 \quad \therefore C = \frac{7 + 14}{2} = 10.5$

So, $T \div 4.5 \sin \frac{\pi}{6}(t - 10.5) + 11.5$

b Using technology, $T \div 4.29 \sin(0.533t + 0.769) + 11.2$

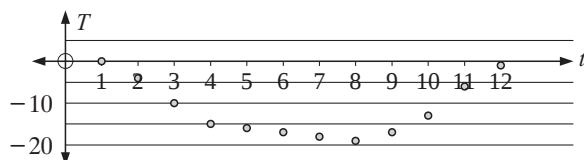
i.e., $T \div 4.29 \sin 0.533(t + 1.44) + 11.2$

Note: (1) $\frac{\pi}{6} \div 0.524 \quad \checkmark$

(2) $\frac{\pi}{6}(1.44 - (-10.5)) \div 6.25 \div 2\pi$

3

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	0	-4	-10	-15	-16	-17	-18	-19	-17	-13	-6	-1



The period is $\frac{2\pi}{B} = 12 \quad \therefore B = \frac{\pi}{6} \quad \{B > 0\}$

Amplitude, $A \div \frac{\text{max.} - \text{min.}}{2} \div \frac{0 - (-19)}{2} \div 9.5$

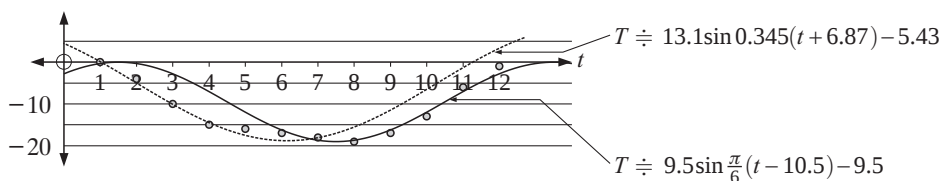
$D \div \frac{\text{max.} + \text{min.}}{2} \div \frac{0 + (-19)}{2} \div -9.5$

At min., $t = 8$ and at max., $t = 1 + 12 = 13 \quad \therefore C = \frac{8 + 13}{2} = 10.5$

So, $T \div 9.5 \sin \frac{\pi}{6}(t - 10.5) - 9.5 \quad \dots\dots (1)$

From technology, $T \div 13.1 \sin(0.345t + 2.37) - 5.43$

i.e., $T \div 13.1 \sin 0.345(t + 6.87) - 5.43 \quad \dots\dots (2)$



The model does not seem appropriate.

- 5 a** For the model $H = A \sin B(t - C) + D$

$$\text{period} = \frac{2\pi}{B} = 2 \times 12.4 \text{ hours} \quad \therefore B = \frac{2\pi}{24.8} \doteq 0.253$$

We let the principal axis be 0, i.e., $D = 0$

$\therefore A$, the amplitude = 7, i.e., min. is -7 , max. is $+7$

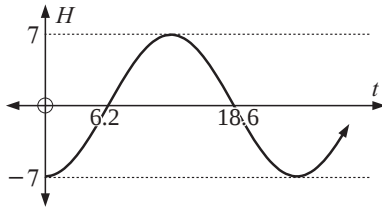
Let $t = 0$ correspond to 'low tide' $\therefore t = 12.4$ corresponds to 'high tide'

$$\therefore C = \frac{0 + 12.4}{2} = 6.2$$

$$\text{So, } H \doteq 7 \sin 0.253(t - 6.2) + 0$$

$$\text{i.e., } H \doteq 7 \sin 0.253(t - 6.2)$$

b



- 6** Let the model be $H = A \sin B(t - C) + D$ metres

When $t = 0$, $H = 2$ and when $t = 50$, $H = 22$
 $\quad \quad \quad \uparrow \quad \quad \quad \uparrow$
 $\quad \quad \quad \text{min.} \quad \quad \quad \text{max.}$

$$\text{period} = \frac{2\pi}{B} = 100 \quad \therefore B = \frac{2\pi}{100} = \frac{\pi}{50}$$

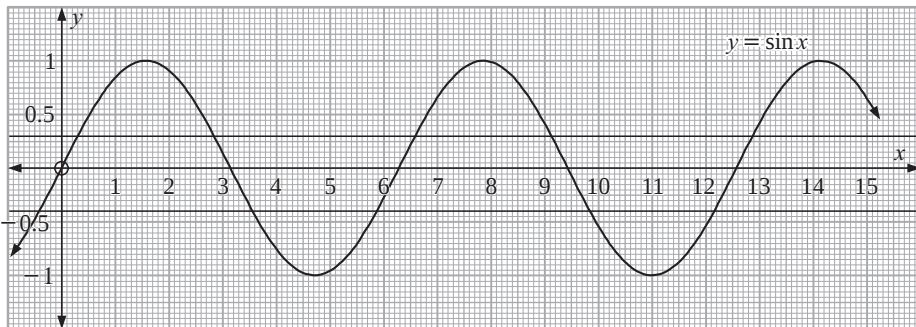
$$A = 10 \quad \{\text{from the diagram}\} \quad D = \frac{\text{max.} + \text{min.}}{2} = \frac{24}{2} = 12$$

$$C = \frac{0 + 50}{2} = 25 \quad \{\text{values of } t \text{ at max. and min.}\}$$

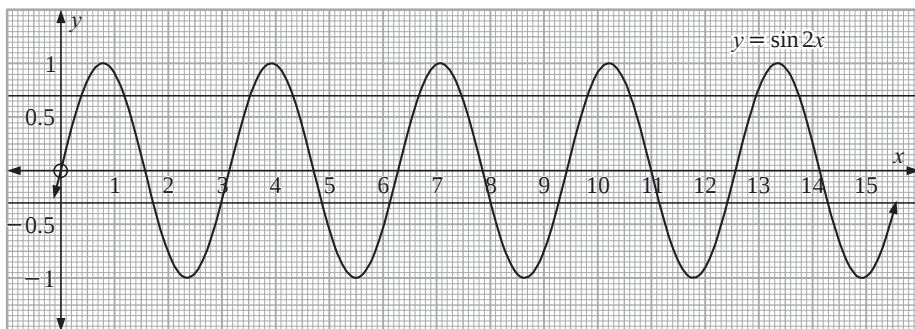
$$\therefore H = 10 \sin \frac{\pi}{50}(t - 25) + 12$$

EXERCISE 13F.1

1



- a** When $\sin x = 0.3$, $x \doteq 0.3, 2.8, 6.6, 9.1, 12.9$
b When $\sin x = -0.4$, $x \doteq 5.9, 9.8, 12.2$

2

a When $\sin(2x) = 0.7$, $x \doteq 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.8$

b When $\sin(2x) = -0.3$, $x \doteq 1.7, 3.0, 4.8, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3$

EXERCISE 13F.2

1 a $\sin x = 0.414$

$\therefore x \doteq 0.4268, 2.715, 6.710$

c $\sin x = 1.289$

There are no solutions as all values of $\sin x$ lie between -1 and $+1$.

e $\sin\left(\frac{x}{2}\right) = -0.606$

$\therefore x \doteq 7.585$

g $\sin(x - 1.3) = 0.866$

$\therefore x \doteq 2.347, 3.394$

i $\sin\left(\frac{2x}{3}\right) = -0.9367 \quad \therefore x \doteq 6.532, 7.605$

b $\sin x = -0.673$

$\therefore x \doteq 3.880, 5.545$

d $\sin 2x = 0.162$

$\therefore x \doteq 0.08136, 1.489, 3.223, 4.631, 6.365, 7.773$

f $\sin(x + 2) = 0.0652$

$\therefore x \doteq 1.076, 4.348, 7.360$

h $\sin\left(x - \frac{\pi}{3}\right) = 0.7063$

$\therefore x \doteq 1.831, 3.405$

EXERCISE 13F.3

1 a $x = \frac{\pi}{6} + \frac{k12\pi}{6}$ and $0 \leq x \leq \frac{36\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$

c $x = -\frac{\pi}{2} + \frac{k2\pi}{2}$ and $-\frac{8\pi}{2} \leq x \leq \frac{8\pi}{2}$

$\therefore x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, -\frac{7\pi}{2}$

b $x = -\frac{\pi}{3} + \frac{k6\pi}{3}$ and $-\frac{6\pi}{3} \leq x \leq \frac{6\pi}{3}$

$\therefore x = -\frac{\pi}{3}, \frac{5\pi}{3}$

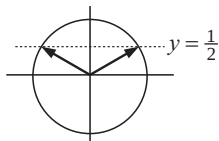
d $x = \frac{5\pi}{6} + \frac{k3\pi}{6}$ and $0 \leq x \leq \frac{24\pi}{6}$

$\therefore x = \frac{2\pi}{6}, \frac{5\pi}{6}, \frac{8\pi}{6}, \frac{11\pi}{6}, \frac{14\pi}{6}, \frac{17\pi}{6}, \frac{20\pi}{6}, \frac{23\pi}{6}$

i.e., $x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{7\pi}{3}, \frac{17\pi}{6}, \frac{10\pi}{3}, \frac{23\pi}{6}$

2 a $2 \sin x = 1, \quad 0 \leq x \leq 6\pi$

$\therefore \sin x = \frac{1}{2}$

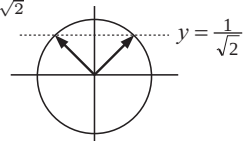


$\therefore x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$

$\therefore x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{29\pi}{6}$

b $\sqrt{2} \sin x = 1, \quad 0 \leq x \leq 4\pi$

$\therefore \sin x = \frac{1}{\sqrt{2}}$

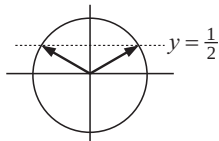


$\therefore x = \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\} + k2\pi$

$\therefore x = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{4}$

c $2 \sin x - 1 = 0, \quad -2\pi \leq x \leq 2\pi$

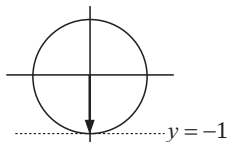
$$\therefore \sin x = \frac{1}{2}$$



$$\therefore x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$$

$$\therefore x = \frac{\pi}{6}, -\frac{11\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}$$

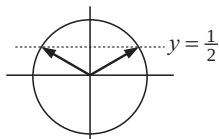
e $\sin x = -1, \quad 0 \leq x \leq 6\pi$



$$\therefore x = \frac{3\pi}{2} + k2\pi$$

$$\therefore x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

g $\sin 2x = \frac{1}{2}, \quad 0 \leq x \leq 3\pi$



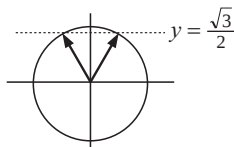
$$\therefore 2x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$$

$$\therefore x = \left\{ \frac{\pi}{12}, \frac{5\pi}{12} \right\} + k\pi$$

$$\therefore x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}, \frac{29\pi}{12}$$

i $2 \sin 2x - \sqrt{3} = 0, \quad 0 \leq x \leq 3\pi$

$$\therefore \sin 2x = \frac{\sqrt{3}}{2}$$



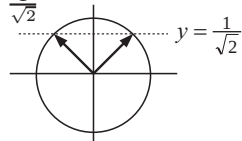
$$\therefore 2x = \left\{ \frac{\pi}{3}, \frac{2\pi}{3} \right\} + k2\pi$$

$$\therefore x = \left\{ \frac{\pi}{6}, \frac{\pi}{3} \right\} + k\pi$$

$$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}$$

d $\sqrt{2} \sin x - 1 = 0, \quad -4\pi \leq x \leq 0$

$$\therefore \sin x = \frac{1}{\sqrt{2}}$$

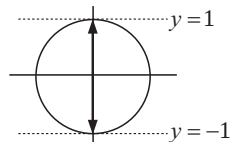


$$\therefore x = \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\} + k2\pi$$

$$\therefore x = -\frac{7\pi}{4}, -\frac{15\pi}{4}, -\frac{5\pi}{4}, -\frac{13\pi}{4}$$

f $\sin^2 x = 1, \quad 0 \leq x \leq 4\pi$

$$\therefore \sin x = \pm 1$$

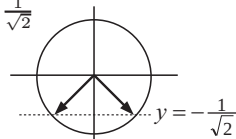


$$\therefore x = \frac{\pi}{2} + k\pi$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

h $\sqrt{2} \sin 3x + 1 = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \sin 3x = -\frac{1}{\sqrt{2}}$$



$$\therefore 3x = \left\{ \frac{5\pi}{4}, \frac{7\pi}{4} \right\} + k2\pi$$

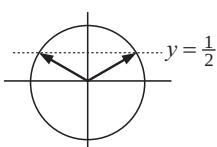
$$\therefore x = \left\{ \frac{5\pi}{12}, \frac{7\pi}{12} \right\} + \frac{k2\pi}{3}$$

$$\therefore x = \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{21\pi}{12}, \frac{7\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$$

$$\text{i.e., } x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$$

j $2 \sin(x + \frac{\pi}{3}) = 1, \quad -3\pi \leq x \leq 3\pi$

$$\therefore \sin(x + \frac{\pi}{3}) = \frac{1}{2}$$



$$\therefore x + \frac{\pi}{3} = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$$

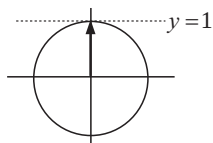
$$\therefore x = \left\{ -\frac{\pi}{6}, \frac{\pi}{2} \right\} + k2\pi$$

$$\therefore x = -\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{13\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{2}, -\frac{3\pi}{2}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & \sin^2 x + \sin x - 2 = 0 \\ & \therefore (\sin x - 1)(\sin x + 2) = 0 \\ & \therefore \sin x = 1 \text{ or } -2 \end{aligned}$$

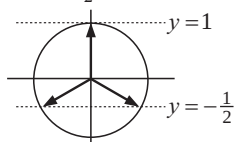
But values of sine can only lie between -1 and 1 inclusive.

$$\therefore \sin x = 1$$



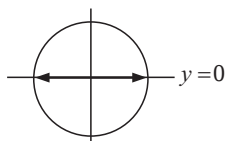
$$\therefore x = \frac{\pi}{2}$$

$$\begin{aligned} \mathbf{c} \quad & 2\sin^2 x = \sin x + 1 \\ & \therefore 2\sin^2 x - \sin x - 1 = 0 \\ & \therefore (2\sin x + 1)(\sin x - 1) = 0 \\ & \therefore \sin x = -\frac{1}{2} \text{ or } 1 \end{aligned}$$



$$\therefore x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

- $\mathbf{4} \quad \mathbf{a}$ The zeros of $y = \sin 2x$ are the solutions of $\sin 2x = 0$ ($0 \leq x \leq \pi$)

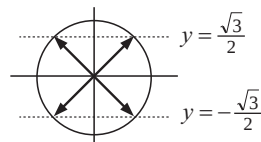


$$\therefore 2x = 0 + k\pi$$

$$\therefore x = 0 + k\frac{\pi}{2}$$

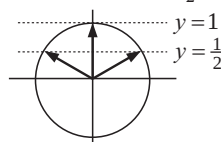
$$\therefore x = 0, \frac{\pi}{2}, \pi$$

$$\begin{aligned} \mathbf{b} \quad & 4\sin^2 x = 3 \\ & \therefore \sin^2 x = \frac{3}{4} \\ & \therefore \sin x = \pm \frac{\sqrt{3}}{2} \end{aligned}$$



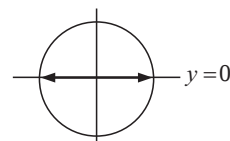
$$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$$

$$\begin{aligned} \mathbf{d} \quad & 2\sin^2 x + 1 = 3\sin x \\ & \therefore 2\sin^2 x - 3\sin x + 1 = 0 \\ & \therefore (2\sin x - 1)(\sin x - 1) = 0 \\ & \therefore \sin x = \frac{1}{2} \text{ or } 1 \end{aligned}$$



$$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

- $\mathbf{b}$ The zeros of $y = \sin(x - \frac{\pi}{4})$ are the solutions of $\sin(x - \frac{\pi}{4}) = 0$ ($0 \leq x \leq 3\pi$)



$$\therefore x - \frac{\pi}{4} = 0 + k\pi$$

$$\therefore x = \frac{\pi}{4} + k\pi$$

$$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$$

EXERCISE 13F.4

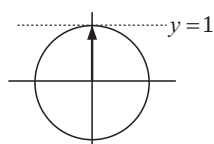
$\mathbf{1} \quad \mathbf{a} \quad P(t) = 7500 + 3000 \sin(\frac{\pi t}{8}), \quad 0 \leq t \leq 12$

$$\begin{aligned} \mathbf{i} \quad & P(0) = 7500 + 3000 \sin 0 \\ & = 7500 + 0 \\ & = 7500 \text{ grass-hoppers} \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & P(5) = 7500 + 3000 \sin(\frac{5\pi}{8}) \\ & \div 10\,271.63 \dots \\ & \div 10\,300 \text{ grass-hoppers} \end{aligned}$$

- $\mathbf{b}$ The greatest value of $P(t)$ occurs when $\sin(\frac{\pi t}{8}) = 1$

i.e., is $7500 + 3000 = 10\,500$ grass-hoppers when $\frac{\pi t}{8} = \frac{\pi}{2} + k2\pi$



$$\therefore \frac{t}{8} = \frac{1}{2} + 2k$$

$$\therefore t = 4 + 16k$$

$$\therefore t = 4 \quad \{\text{as } 0 \leq t \leq 12\}$$

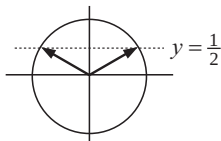
i.e., after 4 weeks

c i When $P(t) = 9000$,

$$7500 + 3000 \sin\left(\frac{\pi t}{8}\right) = 9000$$

$$\therefore 3000 \sin\left(\frac{\pi t}{8}\right) = 1500$$

$$\therefore \sin\left(\frac{\pi t}{8}\right) = \frac{1}{2}$$



$$\therefore \frac{\pi t}{8} = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$$

$$\therefore \frac{t}{8} = \left\{ \frac{1}{6}, \frac{5}{6} \right\} + k2$$

$$\therefore t = \left\{ \frac{4}{3}, \frac{20}{3} \right\} + k16$$

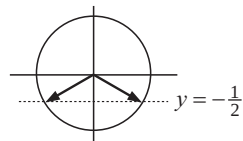
$$\therefore t = 1\frac{1}{3} \text{ or } 6\frac{2}{3}$$

i.e., at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks**ii** When $P(t) = 6000$,

$$7500 + 3000 \sin\left(\frac{\pi t}{8}\right) = 6000$$

$$\therefore 3000 \sin\left(\frac{\pi t}{8}\right) = -1500$$

$$\therefore \sin\left(\frac{\pi t}{8}\right) = -\frac{1}{2}$$



$$\therefore \frac{\pi t}{8} = \left\{ \frac{7\pi}{6}, \frac{11\pi}{6} \right\} + k2\pi$$

$$\therefore \frac{t}{8} = \left\{ \frac{7}{6}, \frac{11}{6} \right\} + k2$$

$$\therefore t = \left\{ \frac{28}{3}, \frac{44}{3} \right\} + k16$$

$$\therefore t = 9\frac{1}{3}$$

i.e., at $9\frac{1}{3}$ weeks**d** If $P(t) > 10\,000$, then

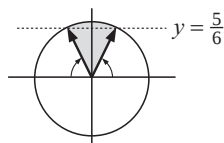
$$7500 + 3000 \sin\left(\frac{\pi t}{8}\right) > 10\,000$$

$$\therefore 3000 \sin\left(\frac{\pi t}{8}\right) > 2500$$

$$\therefore \sin\left(\frac{\pi t}{8}\right) > \frac{5}{6}$$

Solving $\sin\left(\frac{\pi t}{8}\right) = \frac{5}{6}$ using technology

$$t \doteq 2.51 \text{ or } 5.49 \quad \text{So } 2.51 \leq t \leq 5.49 \text{ weeks.}$$

**2** $H(t) = 20 - 19 \sin\left(\frac{2\pi t}{3}\right)$

a $H(0) = 20 - 19(0)$

$= 20 \text{ m}$

i.e., 20 m above the ground

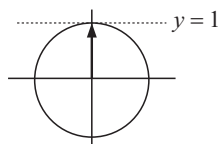
b H is smallest when $\sin\left(\frac{2\pi t}{3}\right) = 1$

$$\therefore \frac{2\pi t}{3} = \frac{\pi}{2} + k2\pi$$

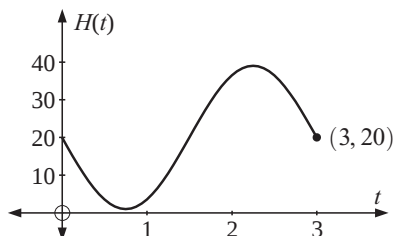
$$\therefore \frac{2t}{3} = \frac{1}{2} + k2$$

$$\therefore t = \frac{3}{4} + k3$$

$$\therefore t = \frac{3}{4} \text{ min } \{\text{as } k = 0\}$$



c period $= \frac{2\pi}{\frac{2\pi}{3}} = 3 \text{ min}$

 $\therefore$ one revolution takes 3 min**d**

3 $P(t) = 400 + 250 \sin\left(\frac{\pi t}{2}\right)$ years

a $P(0) = 400 + 250(0)$
 $= 400$ water buffalo

b i $P\left(\frac{1}{2}\right) = 400 + 250 \sin\left(\frac{\pi(\frac{1}{2})}{2}\right)$
 $= 400 + 250 \sin\left(\frac{\pi}{4}\right)$
 $= 400 + 250 \times \frac{1}{\sqrt{2}}$
 $\div 577$ water buffalo

c $P(1) = 400 + 250 \sin\left(\frac{\pi}{2}\right)$
 $= 400 + 250 \times 1$
 $= 650$ water buffalo

ii $P(2) = 400 + 250 \sin \pi$
 $= 400 + 250(0)$
 $= 400$ water buffalo

This is the maximum herd size.

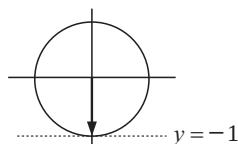
d $P(t)$ is smallest when $\sin\left(\frac{\pi t}{2}\right) = -1$ and is $400 - 250 = 150$ water buffalo.

It occurs when

$$\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$$

$$\therefore \frac{t}{2} = \frac{3}{2} + k2$$

$$\therefore t = 3 + 4k \quad \text{So, the first time is after 3 years.}$$



e If $P(t) > 500$ then

$$400 + 250 \sin\left(\frac{\pi t}{2}\right) > 500$$

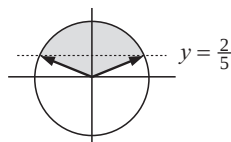
$$\therefore 250 \sin\left(\frac{\pi t}{2}\right) > 100$$

$$\therefore \sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$$

$$\sin\left(\frac{\pi t}{2}\right) = \frac{2}{5} \quad \text{when} \quad \frac{\pi t}{2} = 0.4115 \quad \text{or} \quad \pi - 0.4115$$

$$\therefore t \div 0.262 \quad \text{or} \quad 1.74$$

$$\text{So, for } \sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}, \quad 0.26 < t < 1.74$$



4 $C(t) = 9.2 \sin \frac{\pi}{7}(t - 4) + 107.8$ cents/L

a i 107.8 is the median value. Values are between $107.8 - 9.2$ and $107.8 + 9.2$
 i.e., 98.6 cents/L and 117.0 cents/L
 $\uparrow$ $\uparrow$
 min. max.
 $\therefore$ the statement is true.

ii period $= \frac{2\pi}{\frac{\pi}{7}} = 14$ days $\therefore$ true

b $C(7) = 9.2 \sin \frac{\pi}{7}(3) + 107.8 \div 116.8$ cents/L

c When $C(t) = \$1.10/\text{L}$ then $9.2 \sin \frac{\pi}{7}(t - 4) + 107.8 = 110$

$$\therefore \sin \frac{\pi}{7}(t - 4) = \frac{2.2}{9.2} \div 0.23913$$

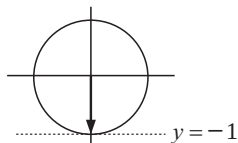
$$\therefore \frac{\pi}{7}(t - 4) \div 0.23913 \quad \text{or} \quad \pi - 0.23913$$

$$\therefore t - 4 \div 0.533 \quad \text{or} \quad 6.467$$

$$\therefore t \div 4.53 \quad \text{or} \quad 10.47$$

i.e., on the 5th and 11th days

- d** The min. cost/litre is $-9.2 + 107.8 = 98.6$ cents when $\sin \frac{\pi}{7}(t - 4) = -1$



$$\text{i.e., } \frac{\pi}{7}(t - 4) = \frac{3\pi}{2}$$

$$\therefore \frac{t - 4}{7} = \frac{3}{2}$$

$$\therefore 2t - 8 = 21$$

$$\therefore 2t = 29$$

$$\therefore t = 14.5$$

i.e., on the 15th day

5 a $S(10) = 3 \sin \frac{\pi}{12}(4) + 23$

$$= 3 \sin\left(\frac{\pi}{3}\right) + 23$$

$$\div 25.6^\circ\text{C}$$

So, the inside temperature is 25.6°C .

b $D = S - T$

$$= 3 \sin \frac{\pi}{12}(t - 6) + 23 - 5 \sin \frac{\pi}{12}(t - 6) - 24$$

$$= -2 \sin \frac{\pi}{12}(t - 6) - 1^\circ\text{C}$$

- c** The outside and inside temperatures are the same when $D(t) = 0$

i.e., where the graph cuts the t -axis,

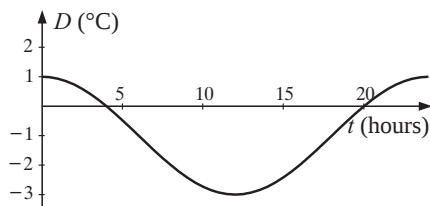
i.e., at about $t = 4$ and $t = 20$ i.e., 4 am and 8 pm.

$$T(10) = 5 \sin \frac{\pi}{12}(10) + 24$$

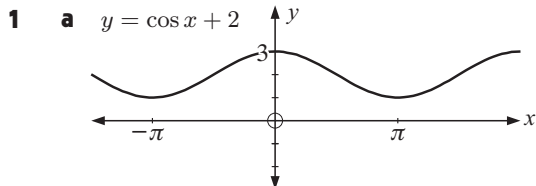
$$= 5 \sin\left(\frac{5\pi}{6}\right) + 24$$

$$\div 28.3^\circ\text{C}$$

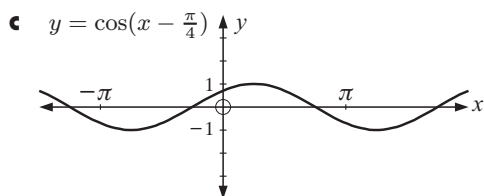
So, the outside temperature is 28.3°C .



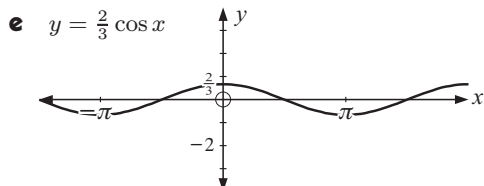
EXERCISE 13G



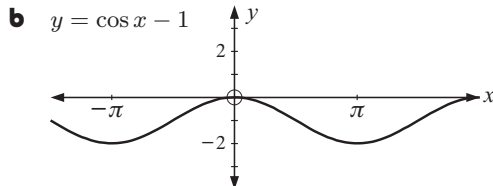
i.e., a vertical translation of $y = \cos x$ through $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.



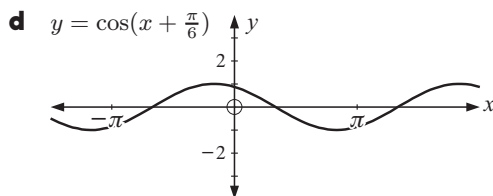
i.e., a horizontal translation of $y = \cos x$ through $\begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$.



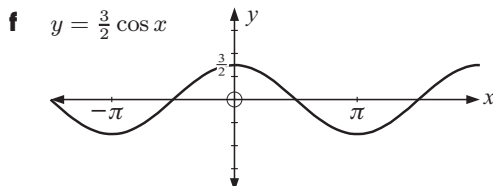
i.e., a vertical dilation of $y = \cos x$ with factor $\frac{2}{3}$.



i.e., a vertical translation of $y = \cos x$ through $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

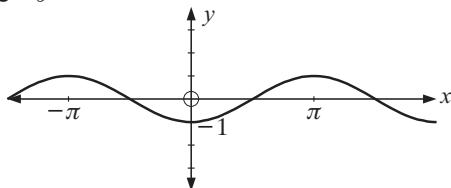


i.e., a horizontal translation of $y = \cos x$ through $\begin{bmatrix} -\frac{\pi}{6} \\ 0 \end{bmatrix}$.



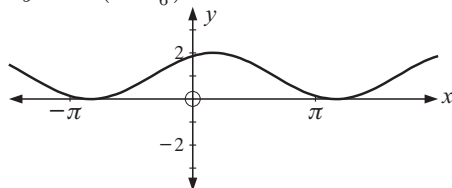
i.e., a vertical dilation of $y = \cos x$ with factor $\frac{3}{2}$.

g $y = -\cos x$



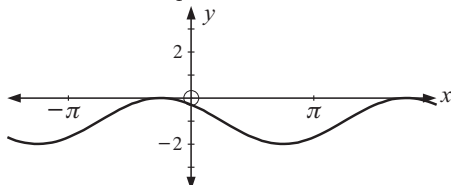
i.e., a reflection of $y = \cos x$ in the x -axis.

h $y = \cos(x - \frac{\pi}{6}) + 1$



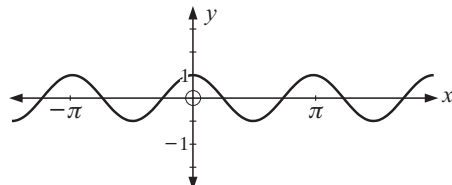
i.e., a translation of $\begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$.

i $y = \cos(x + \frac{\pi}{4}) - 1$



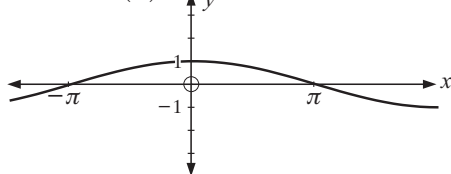
i.e., a translation of $\begin{bmatrix} -\frac{\pi}{4} \\ -1 \end{bmatrix}$.

j $y = \cos 2x$



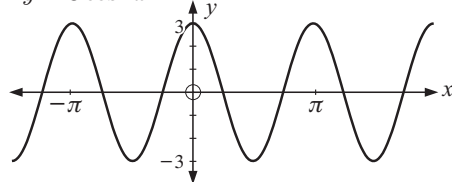
i.e., a horizontal dilation of factor $\frac{1}{2}$.

k $y = \cos(\frac{x}{2})$



i.e., a horizontal dilation of factor 2.

l $y = 3 \cos 2x$



i.e., a horizontal dilation of factor $\frac{1}{2}$ followed by a vertical dilation of factor 3.

2 a period = $\frac{2\pi}{3}$ **b** period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$ **c** period = $\frac{2\pi}{\frac{\pi}{50}} = 100$

3 A controls the amplitude B controls the period, period = $\frac{2\pi}{|B|}$

C controls the horizontal translation D controls the vertical translation

4 a If $y = A \cos B(x - C) + D$, then $A = 2$, $\pi = \frac{2\pi}{B}$ $\therefore B = 2$

C and D are 0 as there is no horizontal or vertical shift. $\therefore y = 2 \cos 2x$

b If $y = A \cos B(x - C) + D$, then $A = 1$, $4\pi = \frac{2\pi}{B}$ $\therefore B = \frac{1}{2}$

A vertical shift of 2 units, no horizontal shift $\therefore D = 2$, $C = 0$.

So, $y = \cos \frac{1}{2}(x) + 2$ i.e., $y = \cos(\frac{x}{2}) + 2$.

c If $y = A \cos B(x - C) + D$, then $A = 3$, $8 = \frac{2\pi}{B}$ $\therefore B = \frac{\pi}{4}$

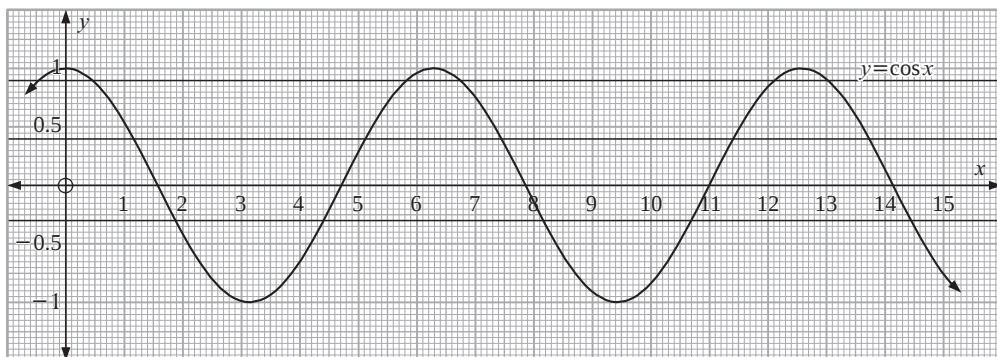
$C = D = 0$ {as no translation} $\therefore y = 3 \cos(\frac{\pi}{4}x)$

d If $y = A \cos B(x - C) + D$, then $A = -5$, $6 = \frac{2\pi}{B}$ $\therefore B = \frac{\pi}{3}$

$C = D = 0$ {as no translation} $\therefore y = -5 \cos(\frac{\pi}{3}x)$

EXERCISE 13H

1

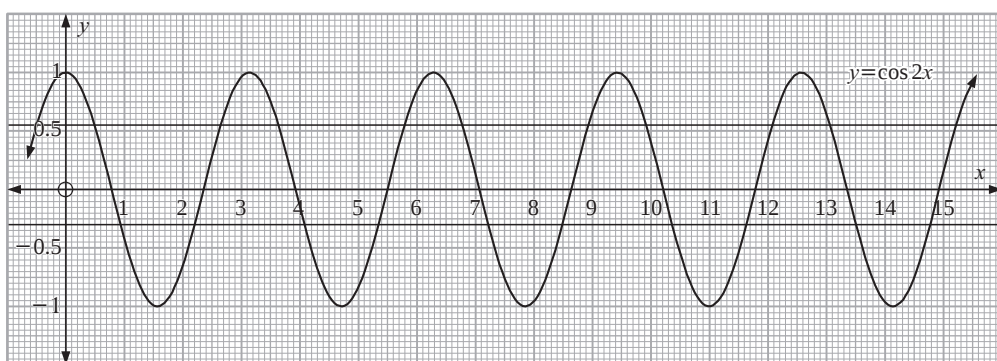


a $x \div 1.2, 5.1, 7.4$

b $x \div 4.4, 8.2, 10.7$

c $x \div 0.5, 5.8, 6.7$

2



a $x \div 0.5 + k\pi$ or $2.6 + k\pi$

i.e., $x \div \left. \begin{matrix} 0.5 \\ 2.6 \end{matrix} \right\} + k\pi$

b $2 \cos 2x + 0.6 = 0$

$\therefore 2 \cos 2x = -0.6$

$\therefore \cos 2x = -0.3$

$\therefore x \div \left. \begin{matrix} 0.9 \\ 2.2 \end{matrix} \right\} + k\pi$

3 a $\cos x = 0.561, 0 \leq x \leq 10$

$\therefore x \div 0.975, 5.308, 7.258$

c $\cos(x - 1.3) = -0.609, 0 \leq x \leq 12$

$\therefore x \div 3.526, 5.358, 9.809, 11.641$

e $5 \cos 2x + 2 = 0$ for all x

$\therefore \cos 2x = -\frac{2}{5} = -0.4$

$\therefore x = \left. \begin{matrix} 0.991 \\ 2.150 \end{matrix} \right\} + k\pi$

b $\cos 2x = 0.782, 0 \leq x \leq 6$

$\therefore x \div 0.336, 2.805, 3.478, 5.947$

d $4 \cos 3x + 1 = 0, 0 \leq x \leq 5$

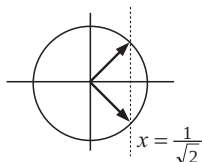
$\therefore \cos 3x = -0.25, 0 \leq x \leq 5$

$\therefore x \div 0.608, 1.487, 2.702, 3.581, 4.797$

4 a $\cos x = \frac{1}{\sqrt{2}}, 0 \leq x \leq 4\pi$

$\therefore x = \left. \begin{matrix} \frac{\pi}{4} \\ \frac{7\pi}{4} \end{matrix} \right\} + k2\pi$

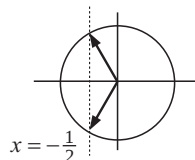
$\therefore x = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}$



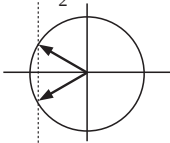
b $\cos x = -\frac{1}{2}, 0 \leq x \leq 5\pi$

$\therefore x = \left. \begin{matrix} \frac{2\pi}{3} \\ \frac{4\pi}{3} \end{matrix} \right\} + k2\pi$

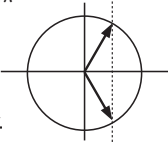
$\therefore x = \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \frac{4\pi}{3}, \frac{10\pi}{3}$



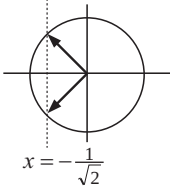
$$\mathbf{c} \quad 2 \cos x + \sqrt{3} = 0, \quad 0 \leq x \leq 3\pi$$

$$\begin{aligned} \therefore \cos x &= -\frac{\sqrt{3}}{2} & x &= -\frac{\sqrt{3}}{2} \\ \therefore x &= \frac{5\pi}{6} \Bigg\} + k2\pi & & \\ \therefore x &= \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{7\pi}{6} \end{aligned}$$


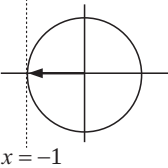
$$\mathbf{d} \quad \cos(x - \frac{2\pi}{3}) = \frac{1}{2}, \quad -2\pi \leq x \leq 2\pi$$

$$\begin{aligned} \therefore x - \frac{2\pi}{3} &= \frac{\pi}{3} \Bigg\} + k2\pi & x &= \frac{1}{2} \\ \therefore x &= \frac{\pi}{3} \Bigg\} + k2\pi & & \\ \therefore x &= \pi, -\pi, \frac{\pi}{3}, -\frac{5\pi}{3} \end{aligned}$$


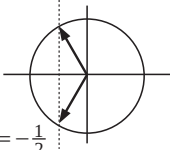
$$\mathbf{e} \quad \sqrt{2} \cos(x - \frac{\pi}{4}) + 1 = 0, \quad 0 \leq x \leq 3\pi$$

$$\begin{aligned} \therefore \cos(x - \frac{\pi}{4}) &= -\frac{1}{\sqrt{2}} \\ \therefore x - \frac{\pi}{4} &= \frac{3\pi}{4} \Bigg\} + k2\pi \\ \therefore x &= \frac{\pi}{2} \Bigg\} + k2\pi \\ \therefore x &= \pi, 3\pi, \frac{3\pi}{2} \end{aligned}$$


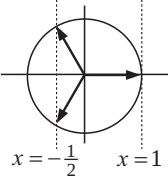
$$\mathbf{f} \quad \cos 2x + 1 = 0, \quad 0 \leq x \leq 2\pi$$

$$\begin{aligned} \therefore \cos 2x &= -1 \\ \therefore 2x &= \pi + k2\pi \\ \therefore x &= \frac{\pi}{2} + k\pi \\ \therefore x &= \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$


$$\mathbf{g} \quad 2 \cos 3x + 1 = 0, \quad 0 \leq x \leq \pi$$

$$\begin{aligned} \therefore \cos 3x &= -\frac{1}{2} \\ \therefore 3x &= \frac{2\pi}{3} \Bigg\} + k2\pi \\ \therefore x &= \frac{2\pi}{9} \Bigg\} + k \frac{2\pi}{3} \\ \therefore x &= \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{4\pi}{9} \end{aligned}$$


$$\mathbf{h} \quad 2 \cos^2 x = \cos x + 1, \quad 0 \leq x \leq 3\pi$$

$$\begin{aligned} \therefore 2 \cos^2 x - \cos x - 1 &= 0 \\ \therefore (2 \cos x + 1)(\cos x - 1) &= 0 \\ \therefore \cos x &= -\frac{1}{2} \text{ or } 1 \\ \therefore x &= \frac{2\pi}{3} \Bigg\} + k2\pi \\ \therefore x &= 0, 2\pi, \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{4\pi}{3} \end{aligned}$$


5 a The period is 4 seconds.

$$\therefore \frac{2\pi}{B} = 4$$

$$\therefore B = \frac{\pi}{2}$$

Amplitude is 3

$$\therefore A = 3$$

$$D = 1 + 3 = 4$$

$$C = 0$$

$$\therefore H(t) = 3 \cos \frac{\pi}{2}(t - 0) + 4 \text{ metres}$$

$$\text{i.e., } H(t) = 3 \cos(\frac{\pi}{2}t) + 4 \text{ metres}$$

Check: When $t = 0$, $H(0) = 3 \cos 0 + 4 = 7$ ✓

b X enters the water when $H(t) = 2$

$$\therefore 3 \cos(\frac{\pi t}{2}) + 4 = 2$$

$$\therefore \cos(\frac{\pi t}{2}) = -\frac{2}{3}$$

Using technology, $t \div 1.46 \text{ sec}$

EXERCISE 13I

1 a $\sin \theta + \sin \theta = 2 \sin \theta$

b $2 \cos \theta + \cos \theta = 3 \cos \theta$

c $3 \sin \theta - \sin \theta = 2 \sin \theta$

d $3 \sin \theta - 2 \sin \theta = \sin \theta$

e $\cos \theta - 3 \cos \theta = -2 \cos \theta$

f $2 \cos \theta - 5 \cos \theta = -3 \cos \theta$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & 3 \sin^2 \theta + 3 \cos^2 \theta \\
 &= 3(\sin^2 \theta + \cos^2 \theta) \\
 &= 3(1) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 3 - 3 \sin^2 \theta \\
 &= 3(1 - \sin^2 \theta) \\
 &= 3 \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \cos^2 \theta - 1 \\
 &= 1 - \sin^2 \theta - 1 \\
 &= -\sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad & \frac{1 - \sin^2 \theta}{\cos^2 \theta} \\
 &= \frac{\cos^2 \theta}{\cos^2 \theta} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & -2 \sin^2 \theta - 2 \cos^2 \theta \\
 &= -2(\sin^2 \theta + \cos^2 \theta) \\
 &= -2(1) \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & 4 - 4 \cos^2 \theta \\
 &= 4(1 - \cos^2 \theta) \\
 &= 4 \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \sin^2 \theta - 1 \\
 &= 1 - \cos^2 \theta - 1 \\
 &= -\cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad & \frac{1 - \cos^2 \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta}{\sin \theta} \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & -\cos^2 \theta - \sin^2 \theta \\
 &= -(\cos^2 \theta + \sin^2 \theta) \\
 &= -(1) \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \sin^3 \theta + \sin \theta \cos^2 \theta \\
 &= \sin \theta (\sin^2 \theta + \cos^2 \theta) \\
 &= \sin \theta (1) \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & 2 \cos^2 \theta - 2 \\
 &= -2(1 - \cos^2 \theta) \\
 &= -2 \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \frac{\cos^2 \theta - 1}{-\sin \theta} \\
 &= \frac{1 - \sin^2 \theta - 1}{-\sin \theta} \\
 &= \frac{-\sin^2 \theta}{-\sin \theta} \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & (1 + \sin \theta)^2 \\
 &= 1 + 2 \sin \theta + \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (\cos \alpha - 1)^2 \\
 &= \cos^2 \alpha - 2 \cos \alpha + 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (\sin \beta - \cos \beta)^2 \\
 &= \sin^2 \beta - 2 \sin \beta \cos \beta + \cos^2 \beta \\
 &= 1 - 2 \sin \beta \cos \beta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (\sin \alpha - 2)^2 \\
 &= \sin^2 \alpha - 4 \sin \alpha + 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & (\sin \alpha + \cos \alpha)^2 \\
 &= \sin^2 \alpha + 2 \sin \alpha \cos \alpha + \cos^2 \alpha \\
 &= 1 + 2 \sin \alpha \cos \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & -(2 - \cos \alpha)^2 \\
 &= -[4 - 4 \cos \alpha + \cos^2 \alpha] \\
 &= -4 + 4 \cos \alpha - \cos^2 \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & 1 - \sin^2 \theta \\
 &= (1 + \sin \theta)(1 - \sin \theta)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 2 \sin^2 \beta - \sin \beta \\
 &= \sin \beta (2 \sin \beta - 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \sin^2 \theta + 5 \sin \theta + 6 \\
 &= (\sin \theta + 2)(\sin \theta + 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \sin^2 \alpha - \cos^2 \alpha \\
 &= (\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & 2 \cos \phi + 3 \cos^2 \phi \\
 &= \cos \phi (2 + 3 \cos \phi)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 2 \cos^2 \theta + 7 \cos \theta + 3 \\
 &= (2 \cos \theta + 1)(\cos \theta + 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \cos^2 \alpha - 1 \\
 &= (\cos \alpha + 1)(\cos \alpha - 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 3 \sin^2 \theta - 6 \sin \theta \\
 &= 3 \sin \theta (\sin \theta - 2)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & 6 \cos^2 \alpha - \cos \alpha - 1 \\
 &= (3 \cos \alpha + 1)(2 \cos \alpha - 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & \frac{1 - \sin^2 \alpha}{1 - \sin \alpha} \\
 &= \frac{(1 + \sin \alpha) \cancel{(1 - \sin \alpha)}^1}{\cancel{1 - \sin \alpha}_1} \\
 &= 1 + \sin \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi + \sin \phi} \\
 &= \frac{\cancel{(\cos \phi + \sin \phi)}^1 (\cos \phi - \sin \phi)}{\cancel{1 - \cos \phi + \sin \phi}_1} \\
 &= \cos \phi - \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{\cos^2 \beta - 1}{\cos \beta + 1} \\
 &= \frac{\cancel{(\cos \beta + 1)}^1 (\cos \beta - 1)}{\cancel{\cos \beta + 1}_1} \\
 &= \cos \beta - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi - \sin \phi} \\
 &= \frac{(\cos \phi + \sin \phi) \cancel{(\cos \phi - \sin \phi)}^1}{\cancel{1 - \cos \phi + \sin \phi}_1} \\
 &= \cos \phi + \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{\sin \alpha + \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\
 &= \frac{1 \cancel{\sin \alpha + \cos \alpha}}{1(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)} \\
 &= \frac{1}{\sin \alpha - \cos \alpha}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \text{a} \quad & (\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 \\
 &= \cos^2 \theta + \cancel{2 \cos \theta \sin \theta} + \sin^2 \theta \\
 &\quad + \cos^2 \theta - \cancel{2 \cos \theta \sin \theta} + \sin^2 \theta \\
 &= 2 \cos^2 \theta + 2 \sin^2 \theta \\
 &= 2(\cos^2 \theta + \sin^2 \theta) \\
 &= 2(1) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (1 - \cos \theta) \left(1 + \frac{1}{\cos \theta} \right) \\
 &= 1 + \frac{1}{\cos \theta} - \cos \theta - 1 \\
 &= \frac{1}{\cos \theta} - \cos \theta \\
 &= \frac{1}{\cos \theta} - \cos \theta \left(\frac{\cos \theta}{\cos \theta} \right) \\
 &= \frac{1 - \cos^2 \theta}{\cos \theta} \\
 &= \frac{\sin^2 \theta}{\cos \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta + (1 + \cos \theta)(1 + \cos \theta)}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{\sin^2 \theta + 1 + 2 \cos \theta + \cos^2 \theta}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{1 + 1 + 2 \cos \theta}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{\cancel{2(1 + \cos \theta)}^1}{\sin \theta \cancel{(1 + \cos \theta)}_1} \\
 &= \frac{2}{\sin \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} \\
 &= \frac{1 + \sin \theta + 1 - \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} \\
 &= \frac{2}{1 - \sin^2 \theta} \\
 &= \frac{2}{\cos^2 \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{3 - 3 \sin^2 \theta}{6 \cos \theta} = \frac{3(1 - \sin^2 \theta)}{6 \cos \theta} \\
 &= \frac{3 \cos^2 \theta}{6 \cos \theta} \\
 &= \frac{\cos \theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (2 \sin \theta + 3 \cos \theta)^2 + (3 \sin \theta - 2 \cos \theta)^2 \\
 &= 4 \sin^2 \theta + \cancel{12 \sin \theta \cos \theta} + 9 \cos^2 \theta \\
 &\quad + 9 \sin^2 \theta - \cancel{12 \sin \theta \cos \theta} + 4 \cos^2 \theta \\
 &= 13 \sin^2 \theta + 13 \cos^2 \theta \\
 &= 13(\sin^2 \theta + \cos^2 \theta) \\
 &= 13(1) \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \left(1 + \frac{1}{\sin \theta} \right) (\sin \theta - \sin^2 \theta) \\
 &= \sin \theta - \sin^2 \theta + 1 - \sin \theta \\
 &= 1 - \sin^2 \theta \\
 &= \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{\sin \theta}{1 - \cos \theta} - \frac{\sin \theta}{1 + \cos \theta} \\
 &= \frac{\sin \theta(1 + \cos \theta) - \sin \theta(1 - \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} \\
 &= \frac{\cancel{\sin \theta} + \sin \theta \cos \theta - \cancel{\sin \theta} + \sin \theta \cos \theta}{1 - \cos^2 \theta} \\
 &= \frac{2 \sin \theta \cos \theta}{\sin^2 \theta} \\
 &= \frac{\cancel{2 \sin \theta} \cos \theta}{\cancel{\sin \theta} \sin \theta} \\
 &= \frac{2 \cos \theta}{\sin \theta}
 \end{aligned}$$

EXERCISE 13J

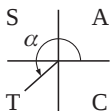
$$\begin{aligned} 1 \quad a \quad \sin 2A &= 2 \sin A \cos A \\ &= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) \\ &= \frac{24}{25} \end{aligned}$$

$$\begin{aligned} b \quad \cos 2A &= \cos^2 A - \sin^2 A \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25} \end{aligned}$$

$$\begin{aligned} 2 \quad \cos 2A &= 2 \cos^2 A - 1 \\ &= 2\left(\frac{1}{3}\right)^2 - 1 \\ &= 2 \times \frac{1}{9} - 1 \\ &= \frac{2}{9} - 1 \\ &= -\frac{7}{9} \end{aligned}$$

$$\begin{aligned} 3 \quad \cos 2\phi &= 1 - 2 \sin^2 \phi \\ &= 1 - 2\left(-\frac{2}{3}\right)^2 \\ &= 1 - 2\left(\frac{4}{9}\right) \\ &= 1 - \frac{8}{9} \\ &= \frac{1}{9} \end{aligned}$$

$$\begin{aligned} 4 \quad a \quad \sin \alpha &= -\frac{2}{3} \\ \alpha \text{ is in Q3} \\ \therefore \cos \alpha &< 0 \end{aligned}$$



$$\begin{aligned} \cos^2 \alpha + \sin^2 \alpha &= 1 \\ \therefore \cos^2 \alpha + \frac{4}{9} &= 1 \\ \therefore \cos^2 \alpha &= \frac{5}{9} \\ \therefore \cos \alpha &= -\frac{\sqrt{5}}{3} \end{aligned}$$

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= 2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) \\ &= \frac{4\sqrt{5}}{9} \end{aligned}$$

$$\begin{aligned} b \quad \cos \beta &= \frac{2}{5} \\ \beta \text{ is in Q4} \\ \therefore \sin \beta &< 0 \end{aligned}$$



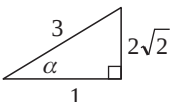
$$\begin{aligned} \cos^2 \beta + \sin^2 \beta &= 1 \\ \therefore \frac{4}{25} + \sin^2 \beta &= 1 \\ \therefore \sin^2 \beta &= \frac{21}{25} \\ \therefore \sin \beta &= -\frac{\sqrt{21}}{5} \end{aligned}$$

$$\begin{aligned} \sin 2\beta &= 2 \sin \beta \cos \beta \\ &= 2\left(-\frac{\sqrt{21}}{5}\right)\left(\frac{2}{5}\right) \\ &= -\frac{4\sqrt{21}}{25} \end{aligned}$$

5 α is acute $\therefore \cos \alpha$ and $\sin \alpha$ are positive

$$\begin{aligned} a \quad \cos 2\alpha &= 2 \cos^2 \alpha - 1 \\ \therefore -\frac{7}{9} &= 2 \cos^2 \alpha - 1 \\ \therefore 2 \cos^2 \alpha &= \frac{2}{9} \\ \therefore \cos^2 \alpha &= \frac{1}{9} \\ \therefore \cos \alpha &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} b \quad \cos 2\alpha &= 1 - 2 \sin^2 \alpha \\ \therefore -\frac{7}{9} &= 1 - 2 \sin^2 \alpha \\ \therefore 2 \sin^2 \alpha &= 1\frac{7}{9} = \frac{16}{9} \\ \therefore \sin^2 \alpha &= \frac{8}{9} \\ \therefore \sin \alpha &= \frac{2\sqrt{2}}{3} \end{aligned}$$

or 

$$\sin \alpha = \frac{2\sqrt{2}}{3}$$

$$\begin{aligned} 6 \quad a \quad 2 \sin \alpha \cos \alpha \\ &= \sin 2\alpha \end{aligned}$$

$$\begin{aligned} b \quad 4 \cos \alpha \sin \alpha \\ &= 2(2 \sin \alpha \cos \alpha) \\ &= 2 \sin 2\alpha \end{aligned}$$

$$\begin{aligned} c \quad \sin \alpha \cos \alpha \\ &= \frac{1}{2}(2 \sin \alpha \cos \alpha) \\ &= \frac{1}{2} \sin 2\alpha \end{aligned}$$

$$\begin{aligned} d \quad 2 \cos^2 \beta - 1 \\ &= \cos 2\beta \end{aligned}$$

$$\begin{aligned} e \quad 1 - 2 \cos^2 \phi \\ &= -(2 \cos^2 \phi - 1) \\ &= -\cos 2\phi \end{aligned}$$

$$\begin{aligned} f \quad 1 - 2 \sin^2 N \\ &= \cos 2N \end{aligned}$$

$$\begin{aligned} g \quad 2 \sin^2 M - 1 \\ &= -(1 - 2 \sin^2 M) \\ &= -\cos 2M \end{aligned}$$

$$\begin{aligned} h \quad \cos^2 \alpha - \sin^2 \alpha \\ &= \cos 2\alpha \end{aligned}$$

$$\begin{aligned} i \quad \sin^2 \alpha - \cos^2 \alpha \\ &= -(\cos^2 \alpha - \sin^2 \alpha) \\ &= -\cos 2\alpha \end{aligned}$$

$$\begin{aligned} j \quad 2 \sin 2A \cos 2A \\ &= \sin 2(2A) \\ &= \sin 4A \end{aligned}$$

$$\begin{aligned} k \quad 2 \cos 3\alpha \sin 3\alpha \\ &= \sin 2(3\alpha) \\ &= \sin 6\alpha \end{aligned}$$

$$\begin{aligned} l \quad 2 \cos^2 4\theta - 1 \\ &= \cos 2(4\theta) \\ &= \cos 8\theta \end{aligned}$$

$$\begin{aligned} m \quad 1 - 2 \cos^2 3\beta \\ &= -(2 \cos^2 3\beta - 1) \\ &= -\cos 2(3\beta) \\ &= -\cos 6\beta \end{aligned}$$

$$\begin{aligned} n \quad 1 - 2 \sin^2 5\alpha \\ &= \cos 2(5\alpha) \\ &= \cos 10\alpha \end{aligned}$$

$$\begin{aligned} o \quad 2 \sin^2 3D - 1 \\ &= -(1 - 2 \sin^2 3D) \\ &= -\cos 2(3D) \\ &= -\cos 6D \end{aligned}$$

$$\begin{aligned} \mathbf{p} \quad & \cos^2 2A - \sin^2 2A \\ &= \cos 2(2A) \\ &= \cos 4A \end{aligned}$$

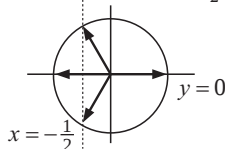
$$\begin{aligned} \mathbf{q} \quad & \cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right) \\ &= \cos 2\left(\frac{\alpha}{2}\right) \\ &= \cos \alpha \end{aligned}$$

$$\begin{aligned} \mathbf{r} \quad & 2 \sin^2 3P - 2 \cos^2 3P \\ &= -2[\cos^2 3P - \sin^2 3P] \\ &= -2 \cos 2(3P) \\ &= -2 \cos 6P \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad & (\sin \theta + \cos \theta)^2 \\ &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= \underbrace{\sin^2 \theta + \cos^2 \theta}_{=1} + 2 \sin \theta \cos \theta \\ &= 1 + \sin 2\theta \end{aligned}$$

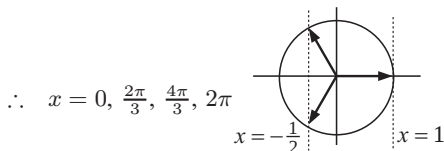
$$\begin{aligned} \mathbf{b} \quad & \cos^4 \theta - \sin^4 \theta \\ &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\ &= 1 \times \cos 2\theta \\ &= \cos 2\theta \end{aligned}$$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad & \sin 2x + \sin x = 0 \\ \therefore & 2 \sin x \cos x + \sin x = 0 \\ \therefore & \sin x(2 \cos x + 1) = 0 \\ \therefore & \sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2} \end{aligned}$$



$$\therefore x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$$

$$\begin{aligned} \mathbf{c} \quad & \cos 2x - \cos x = 0 \\ \therefore & 2 \cos^2 x - 1 - \cos x = 0 \\ \therefore & 2 \cos^2 x - \cos x - 1 = 0 \\ \therefore & (2 \cos x + 1)(\cos x - 1) = 0 \\ \therefore & \cos x = -\frac{1}{2} \quad \text{or} \quad 1 \end{aligned}$$



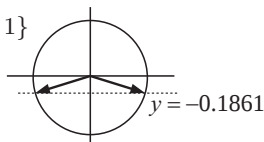
$$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

$$\begin{aligned} \mathbf{e} \quad & \cos 2x + 5 \sin x = 0 \\ \therefore & 1 - 2 \sin^2 x + 5 \sin x = 0 \\ \therefore & 2 \sin^2 x - 5 \sin x - 1 = 0 \\ \therefore \sin x = & \frac{5 \pm \sqrt{25 - 4(2)(-1)}}{4} \\ &= \frac{5 \pm \sqrt{33}}{4} \\ &\div 2.6861 \quad \text{or} \quad -0.1861 \end{aligned}$$

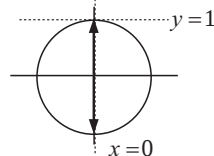
$$\therefore \sin x = -0.1861$$

{as $-1 \leq \sin x \leq 1$ }

$$\begin{aligned} \therefore x &\div \pi + 0.1872 \\ &\quad \text{or} \quad 2\pi - 0.1872 \\ \therefore x &\div 3.33 \quad \text{or} \quad 6.10 \end{aligned}$$

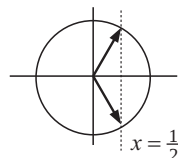


$$\begin{aligned} \mathbf{b} \quad & \sin 2x - 2 \cos x = 0 \\ \therefore & 2 \sin x \cos x - 2 \cos x = 0 \\ \therefore & 2 \cos x(\sin x - 1) = 0 \\ \therefore & \cos x = 0 \quad \text{or} \quad \sin x = 1 \end{aligned}$$



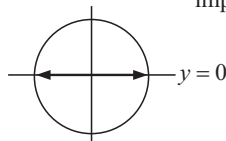
$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\begin{aligned} \mathbf{d} \quad & \cos 2x + 3 \cos x = 1 \\ \therefore & 2 \cos^2 x - 1 + 3 \cos x - 1 = 0 \\ \therefore & 2 \cos^2 x + 3 \cos x - 2 = 0 \\ \therefore & (2 \cos x - 1)(\cos x + 2) = 0 \\ \therefore \cos x = & \frac{1}{2} \quad \text{or} \quad -2 \\ \therefore \cos x = & \frac{1}{2} \quad \{\text{as } -1 \leq \cos x \leq 1\} \end{aligned}$$



$$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\begin{aligned} \mathbf{f} \quad & \sin 2x + 3 \sin x = 0 \\ \therefore & 2 \sin x \cos x + 3 \sin x = 0 \\ \therefore & \sin x(2 \cos x + 3) = 0 \\ \therefore \sin x = & 0 \quad \text{or} \quad \cos x = -\frac{3}{2} \\ & \quad \quad \quad \uparrow \\ & \quad \quad \text{impossible} \end{aligned}$$



$$\therefore x = 0, \pi, 2\pi$$

EXERCISE 13K.1

1 a $\tan 0^\circ$
 $= 0$

b $\tan 15^\circ$
 $\div 0.27$

c $\tan 20^\circ$
 $\div 0.36$

d $\tan 25^\circ$
 $\div 0.47$

e $\tan 35^\circ$
 $\div 0.70$

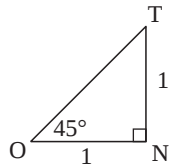
f $\tan 45^\circ$
 $= 1.00$

g $\tan 50^\circ$
 $\div 1.19$

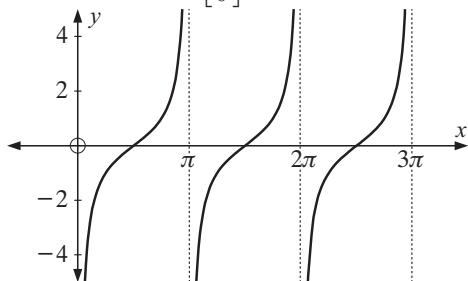
h $\tan 55^\circ$
 $\div 1.43$

3 In $\triangle TON$, $ON = NT = 1$ ($\triangle$ is isosceles) $\tan 45^\circ = \frac{NT}{ON} = \frac{1}{1} = 1$

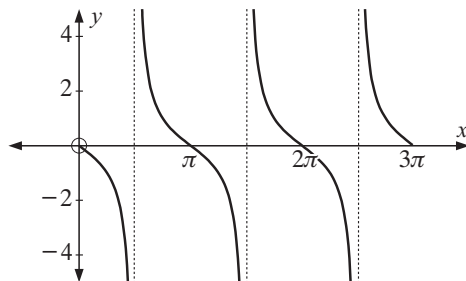
4 The graph does not extend far enough to find $\tan 85^\circ$.
 $\tan 85^\circ \div 11.43$

**EXERCISE 13K.2**

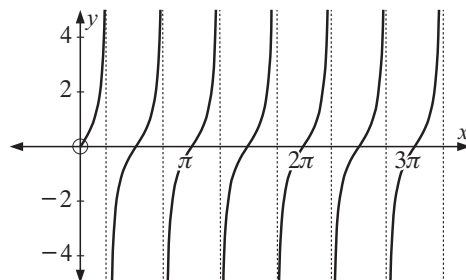
1 a i $y = \tan(x - \frac{\pi}{2})$ is $y = \tan x$
 translated $\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$.



ii $y = -\tan x$ is $y = \tan x$ reflected in the x -axis.



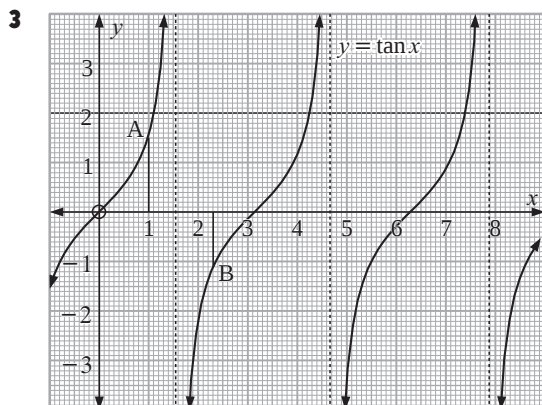
iii $y = \tan 2x$ comes from $y = \tan x$ under a horizontal dilation of factor $\frac{1}{2}$.



2 a translation through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b reflection in x -axis

c horizontal dilation, factor $k = 2$



a i $\tan 1 \div 1.6$ {point A}

ii $\tan 2.3 \div -1.1$ {point B}

b i $\tan 1 \div 1.557$

ii $\tan 2.3 \div -1.119$

c i When $\tan x = 2$, $x \div 1.1, 4.2, 7.4$

ii When $\tan x = -1.4$, $x \div 2.2, 5.3$

4 **a** period = $\frac{\pi}{1} = \pi$ **b** period = $\frac{\pi}{2}$ **c** period = $\frac{\pi}{n}$

EXERCISE 13L.1

1 $X = \tan^{-1} 2 \quad \therefore X \div 1.107 + k\pi$

a If $\tan 2x = 2$

then $2x = 1.107 + k\pi$

$\therefore x = 0.554 + k(\frac{\pi}{2})$

b $\tan(\frac{x}{3}) = 2$

$\therefore \frac{x}{3} = 1.107 + k\pi$

$\therefore x = 3.32 + k3\pi$

c $\tan(x + 1.2) = 2$

$\therefore x + 1.2 = 1.107 + k\pi$

$\therefore x = -0.0929 + k\pi$

2 $X = \tan^{-1}(-3) \div -1.249 + k\pi$

a $\tan(x - 2) = -3$

$\therefore x - 2 \div -1.249 + k\pi$

$\therefore x \div 0.751 + k\pi$

b $\tan(3x) = -3$

$\therefore 3x = -1.249 + k\pi$

$\therefore x \div -0.416 + \frac{k\pi}{3}$

c $\tan(\frac{x}{2}) = -3$

$\therefore \frac{x}{2} = -1.249 + k\pi$

$\therefore x = -2.50 + k2\pi$

3 $X = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} + k\pi$

a $\tan(x - \frac{\pi}{6}) = \sqrt{3}$

$\therefore x - \frac{\pi}{6} = \frac{\pi}{3} + k\pi$

$\therefore x = \frac{\pi}{2} + k\pi$

b $\tan 4x = \sqrt{3}$

$\therefore 4x = \frac{\pi}{3} + k\pi$

$\therefore x = \frac{\pi}{12} + \frac{k\pi}{4}$

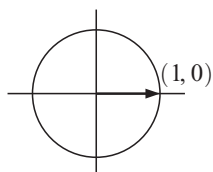
c $\tan^2 x = 3$

$\therefore \tan x = \pm\sqrt{3}$

$\therefore x = \left\{ \frac{\pi}{3}, -\frac{\pi}{3} \right\} + k\pi$

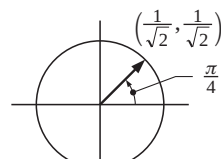
EXERCISE 13L.2

1 a



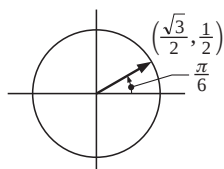
$\tan 0$
 $= \frac{0}{1}$
 $= 0$

b



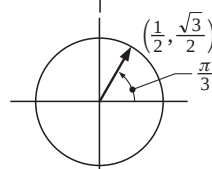
$\tan(\frac{\pi}{4})$
 $= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$
 $= 1$

c



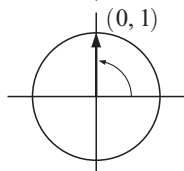
$\tan(\frac{\pi}{6})$
 $= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$
 $= \frac{1}{\sqrt{3}}$

d



$\tan(\frac{\pi}{3})$
 $= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \div \frac{1}{2}$
 $= \sqrt{3}$

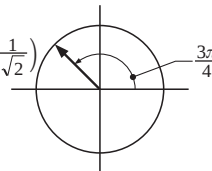
e



$\tan(\frac{\pi}{2})$
 $= \frac{1}{0}$

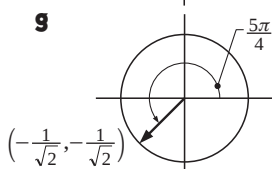
i.e., undefined

f



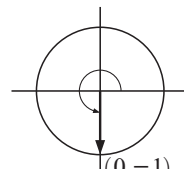
$\tan(\frac{3\pi}{4})$
 $= \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} \div -\frac{1}{\sqrt{2}}$
 $= -1$

g



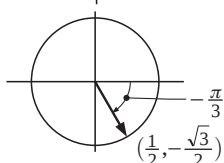
$\tan(\frac{5\pi}{4})$
 $= -\frac{\sqrt{3}}{2} \div \frac{1}{2}$
 $= -\sqrt{3}$

h



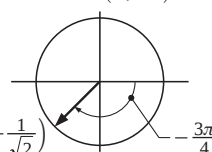
$\tan(\frac{3\pi}{2})$
 $= -\frac{1}{0}$
 i.e., undefined

i



$\tan(-\frac{\pi}{3})$
 $= -\frac{\sqrt{3}}{2} \div \frac{1}{2}$
 $= -\sqrt{3}$

j



$\tan(-\frac{3\pi}{4})$
 $= -\frac{1}{\sqrt{2}} \div -\frac{1}{\sqrt{2}}$
 $= 1$

2 a $\frac{\pi}{4}, \frac{\pi}{4} + \pi$
i.e., $\frac{\pi}{4}, \frac{5\pi}{4}$

d $0, 0 + \pi, 0 + 2\pi$
i.e., $0, \pi, 2\pi$

b $\frac{3\pi}{4}, \frac{3\pi}{4} + \pi$
i.e., $\frac{3\pi}{4}, \frac{7\pi}{4}$

e $\frac{\pi}{6}, \frac{\pi}{6} + \pi$
i.e., $\frac{\pi}{6}, \frac{7\pi}{6}$

c $\frac{\pi}{3}, \frac{\pi}{3} + \pi$
i.e., $\frac{\pi}{3}, \frac{4\pi}{3}$

f $\frac{5\pi}{3}, \frac{5\pi}{3} - \pi$
i.e., $\frac{5\pi}{3}, \frac{2\pi}{3}$

3 a $3 \tan x - \tan x$
 $= 2 \tan x$

b $\tan x - 4 \tan x$
 $= -3 \tan x$

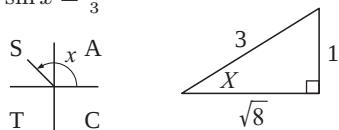
c $\tan x \cos x$
 $= \frac{\sin x}{\cos x} \times \cos x$
 $= \sin x$

d $\frac{\sin x}{\tan x}$
 $= \sin x \div \frac{\sin x}{\cos x}$
 $= \sin x \times \frac{\cos x}{\sin x}$
 $= \cos x$

e $3 \sin x + 2 \cos x \tan x$
 $= 3 \sin x + 2 \cos x \frac{\sin x}{\cos x}$
 $= 3 \sin x + 2 \sin x$
 $= 5 \sin x$

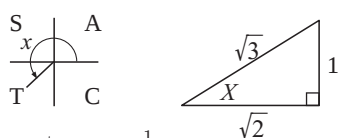
f $\frac{2 \tan x}{\sin x}$
 $= 2 \left(\frac{\sin x}{\cos x} \right) \div \frac{\sin x}{1}$
 $= \frac{2 \sin x}{\cos x} \times \frac{1}{\sin x}$
 $= \frac{2}{\cos x}$

4 a $\sin x = \frac{1}{3}$



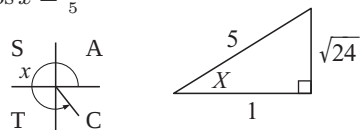
$\therefore \tan x = -\frac{1}{\sqrt{8}} = -\frac{1}{2\sqrt{2}}$

c $\sin x = -\frac{1}{\sqrt{3}}$



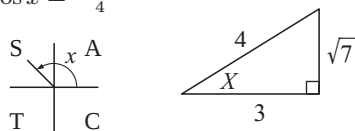
$\therefore \tan x = \frac{1}{\sqrt{2}}$

b $\cos x = \frac{1}{5}$



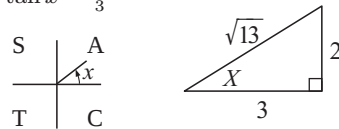
$\therefore \tan x = -\frac{\sqrt{24}}{1} = -2\sqrt{6}$

d $\cos x = -\frac{3}{4}$



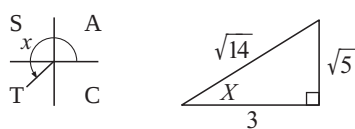
$\therefore \tan x = -\frac{\sqrt{7}}{3}$

5 a $\tan x = \frac{2}{3}$



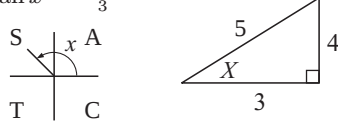
$\therefore \sin x = \frac{2}{\sqrt{13}}, \cos x = \frac{3}{\sqrt{13}}$

c $\tan x = \frac{\sqrt{5}}{3}$



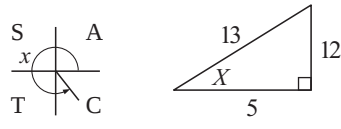
$\therefore \sin x = -\frac{\sqrt{5}}{\sqrt{14}}, \cos x = -\frac{3}{\sqrt{14}}$

b $\tan x = -\frac{4}{3}$

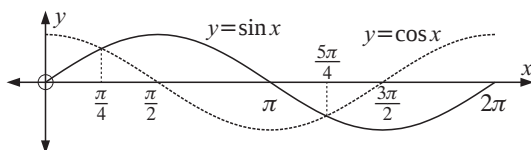


$\therefore \sin x = \frac{4}{5}, \cos x = -\frac{3}{5}$

d $\tan x = -\frac{12}{5}$



$\therefore \sin x = -\frac{12}{13}, \cos x = \frac{5}{13}$

EXERCISE 13M**1 a**

b $x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$

c If $\sin x = \cos x$ then $\frac{\sin x}{\cos x} = 1$
 $\therefore \tan x = 1$
 $\therefore x = \frac{\pi}{4} + k\pi$
 $\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

2 a

$$\begin{aligned}\sin x &= -\cos x \\ \therefore \frac{\sin x}{\cos x} &= \frac{-\cos x}{\cos x} \\ \therefore \tan x &= -1 \\ \therefore x &= \frac{3\pi}{4} + k\pi \\ \therefore x &= \frac{3\pi}{4}, \frac{7\pi}{4}\end{aligned}$$

c $\sin(2x) = \sqrt{3} \cos(2x)$
 $\therefore \frac{\sin(2x)}{\cos(2x)} = \sqrt{3}$
 $\therefore \tan(2x) = \sqrt{3}$
 $\therefore 2x = \frac{\pi}{3} + k\pi$
 $\therefore x = \frac{\pi}{6} + \frac{k\pi}{2}$
 $\therefore x = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, \frac{10\pi}{6}$
 i.e., $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

b

$$\begin{aligned}\sin(3x) &= \cos(3x) \\ \therefore \frac{\sin(3x)}{\cos(3x)} &= 1 \\ \therefore \tan(3x) &= 1 \\ \therefore 3x &= \frac{\pi}{4} + k\pi \\ \therefore x &= \frac{\pi}{12} + \frac{k\pi}{3} \\ \therefore x &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12} \\ \therefore x &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}\end{aligned}$$

4 a

$$\begin{aligned}\sin x &= 5 \cos x \\ \therefore \tan x &= 5 \\ \therefore x &= \tan^{-1}(5) \\ \therefore x &= 1.373 + k\pi \\ \therefore x &\doteq 1.37, 4.51, 7.66\end{aligned}$$

b $4 \sin x + 3 \cos x = 0$

$$\begin{aligned}\therefore 4 \sin x &= -3 \cos x \\ \therefore \frac{\sin x}{\cos x} &= -\frac{3}{4} \\ \therefore \tan x &= -\frac{3}{4} \\ \therefore x &= \tan^{-1}\left(-\frac{3}{4}\right) \\ \therefore x &\doteq -1.081 + k\pi \\ \therefore x &\doteq 2.50, 5.64, 8.78\end{aligned}$$

REVIEW SET 13A

1 a 120°
 $= 2 \times 60^\circ$
 $= 2 \times \frac{\pi}{3}$
 $= \frac{2\pi}{3}$

b 225°
 $= 5 \times 45^\circ$
 $= 5 \times \frac{\pi}{4}$
 $= \frac{5\pi}{4}$

c 150°
 $= 5 \times 30^\circ$
 $= 5 \times \frac{\pi}{6}$
 $= \frac{5\pi}{6}$

d 540°
 $= 3 \times 180^\circ$
 $= 3\pi$

2 a 71°
 $= 71 \times \frac{\pi}{180}$ radians
 $\doteq 1.239^c$

b 124.6°
 $= 124.6 \times \frac{\pi}{180}^c$
 $\doteq 2.175^c$

c -142°
 $= -142 \times \frac{\pi}{180}^c$
 $\doteq -2.478^c$

d -25.3°
 $= -25.3 \times \frac{\pi}{180}^c$
 $\doteq -0.4416^c$

3 a $\frac{2\pi}{5} = \frac{2(180^\circ)}{5}$
 $= 72^\circ$

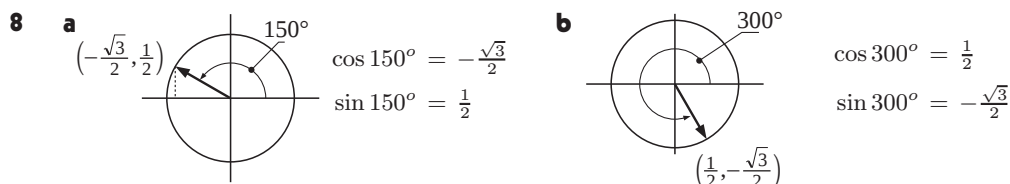
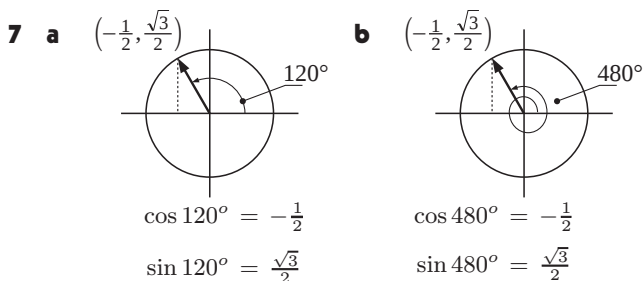
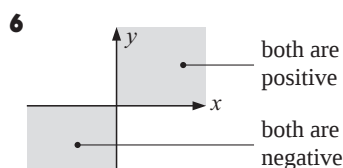
b $\frac{5\pi}{4} = \frac{5(180^\circ)}{4}$
 $= 225^\circ$

c $\frac{7\pi}{9} = \frac{7(180^\circ)}{9}$
 $= 140^\circ$

d $\frac{11\pi}{6} = \frac{11(180^\circ)}{6}$
 $= 330^\circ$

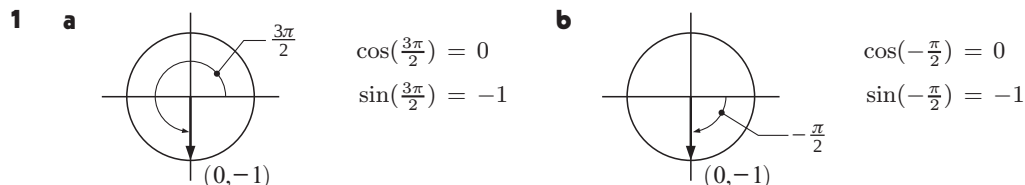
$$\begin{array}{llll}
 \mathbf{4} \quad \mathbf{a} & 3 & \mathbf{b} & 1.46 \\
 & = 3 \times \left(\frac{180}{\pi}\right)^\circ & & = 1.46 \times \left(\frac{180}{\pi}\right)^\circ \\
 & \doteq 171.89^\circ & & \doteq 83.65^\circ \\
 \mathbf{c} & 0.435 & \mathbf{d} & -5.271 \\
 & = 0.435 \times \left(\frac{180}{\pi}\right)^\circ & & = -5.271 \times \left(\frac{180}{\pi}\right)^\circ \\
 & \doteq 24.92^\circ & & \doteq -302.01^\circ
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{5} \quad \mathbf{a} & \text{point is } (\cos 320^\circ, \sin 320^\circ) \\
 & \text{i.e., } (0.766, -0.643) \\
 \mathbf{b} & \text{point is } (\cos 163^\circ, \sin 163^\circ) \\
 & \text{i.e., } (-0.956, 0.292)
 \end{array}$$



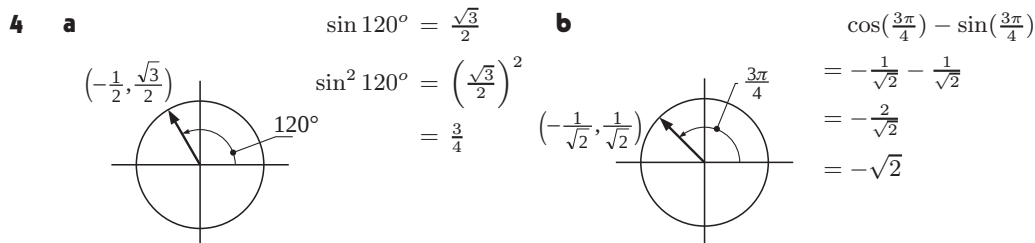
9 Look for points where the y -coordinate is the negative of the x -coordinate
 or notice that if $\cos \theta = -\sin \theta$ then $\frac{\sin \theta}{\cos \theta} = -1$ i.e., $\tan \theta = -1$.

REVIEW SET 13B

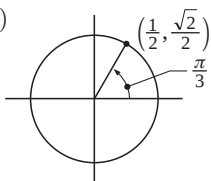


2 $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \frac{9}{16} + \sin^2 \theta = 1$
 $\therefore \sin^2 \theta = \frac{7}{16}$
 $\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$

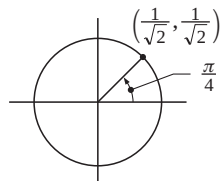
3 $\cos \theta = -\frac{3}{4}$
 $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \frac{9}{16} + \sin^2 \theta = 1$
 $\therefore \sin^2 \theta = \frac{7}{16}$
 $\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$, but $\sin \theta > 0$
 $\therefore \sin \theta = \frac{\sqrt{7}}{4}$



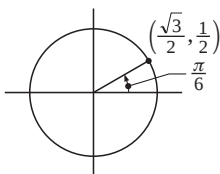
5 a $2 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{3}\right)$
 $= 2 \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right)$
 $= \frac{\sqrt{3}}{2}$



b $\sin^2\left(\frac{\pi}{4}\right) - 1$
 $= \left(\frac{1}{\sqrt{2}}\right)^2 - 1$
 $= \frac{1}{2} - 1$
 $= -\frac{1}{2}$

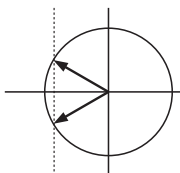


c $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$
 $= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$
 $= \frac{3}{4} - \frac{1}{4}$
 $= \frac{1}{2}$

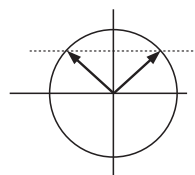


or $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$
 $= \cos 2\left(\frac{\pi}{6}\right)$
 $= \cos\left(\frac{\pi}{3}\right)$
 $= \frac{1}{2}$

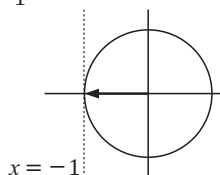
6 a If $\cos \theta = -\frac{\sqrt{3}}{2}$
 $\theta = 150^\circ, 210^\circ$



b If $\sin \theta = \frac{1}{\sqrt{2}}$
 $\theta = 45^\circ, 135^\circ$



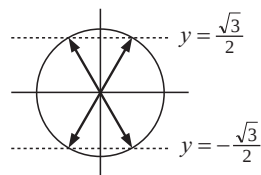
7 a $\cos \theta = -1$



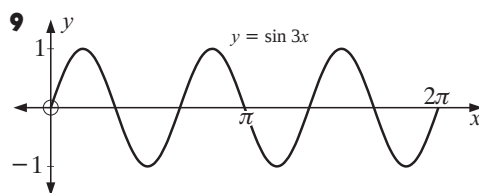
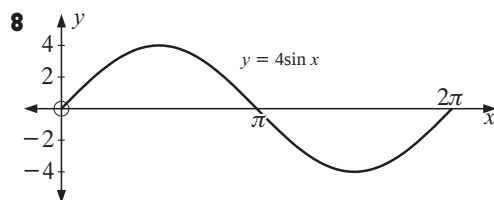
$\therefore \theta = \pi + k2\pi$

b $\sin^2 \theta = \frac{3}{4}$

$\therefore \sin \theta = \pm \frac{\sqrt{3}}{2}$



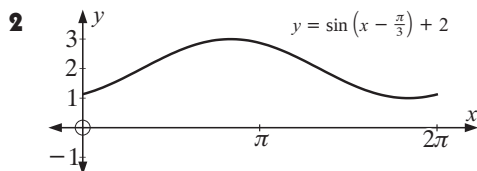
$\therefore \theta = \left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2} \right\} + k\pi$



REVIEW SET 13C

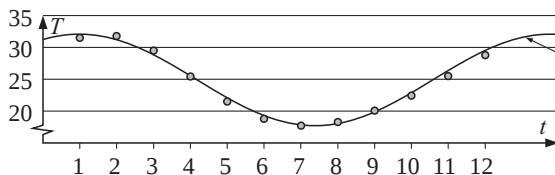
1 a period $= \frac{2\pi}{\frac{1}{3}} = 6\pi$

b period $= \frac{2\pi}{4} = \frac{\pi}{2}$



3

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8



$$T = A \sin B(t - C) + D \quad \text{period} = \frac{2\pi}{B} = 12, \quad \therefore B = \frac{2\pi}{12} = \frac{\pi}{6}$$

$$\text{max.} = 31.8 \quad \therefore A = \frac{\text{max.} - \text{min.}}{2} \div \frac{31.8 - 17.7}{2} \div 7.05$$

$$\text{min.} = 17.7$$

$$D = \frac{\text{max.} + \text{min.}}{2} \div \frac{31.8 + 17.7}{2} \div 24.75$$

$$C = \frac{7 + 14}{2} = 10.5 \quad \{\text{values of } t \text{ at min. and max.}\}$$

$$\text{So, } T \div 7.05 \sin \frac{\pi}{6}(t - 10.5) + 24.75$$

$$\text{From technology, } T \div 7.21 \sin(0.488t + 1.082) + 24.75$$

$$\div 7.21 \sin 0.488(t + 2.22) + 24.75 \quad \text{Note: } 0.488(2.22 - -10.5) \div 6.21 \div 2\pi$$

4 a $\sin x = 0.382$

$$\therefore x \div 0.392, 2.750, 6.675$$

b $\sin\left(\frac{x}{2}\right) = -0.458$

$$\therefore x \div 7.235$$

5 a $\sin(x - 2.4) = 0.754$

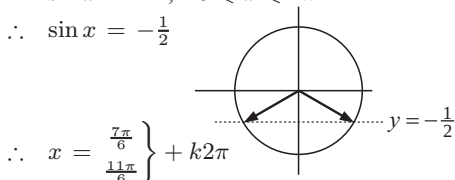
$$\therefore x \div 3.254, 4.687$$

b $\sin\left(x + \frac{\pi}{3}\right) = 0.6049$

$$\therefore x \div 1.445, 5.89, 7.73$$

6 a $2 \sin x = -1, \quad 0 \leq x \leq 4\pi$

$$\therefore \sin x = -\frac{1}{2}$$

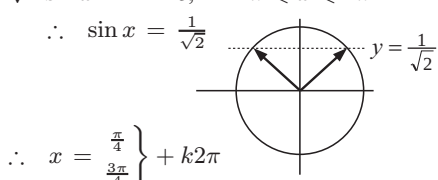


$$\therefore x = \left\{ \frac{7\pi}{6}, \frac{11\pi}{6} \right\} + k2\pi$$

$$\therefore x = \frac{7\pi}{6}, \frac{19\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6}$$

b $\sqrt{2} \sin x - 1 = 0, \quad -2\pi \leq x \leq 2\pi$

$$\therefore \sin x = \frac{1}{\sqrt{2}} \quad y = \frac{1}{\sqrt{2}}$$

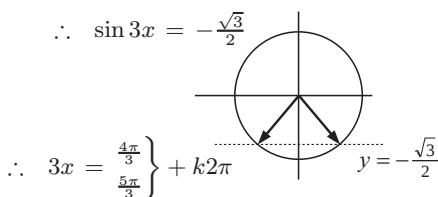


$$\therefore x = \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\} + k2\pi$$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{7\pi}{4}, -\frac{5\pi}{4}$$

7 a $2 \sin 3x + \sqrt{3} = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \sin 3x = -\frac{\sqrt{3}}{2}$$



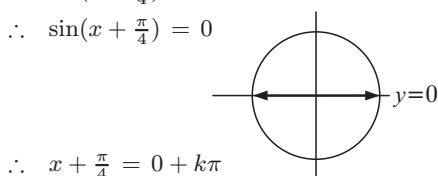
$$\therefore 3x = \left\{ \frac{4\pi}{3}, \frac{5\pi}{3} \right\} + k2\pi$$

$$\therefore x = \left\{ \frac{4\pi}{9}, \frac{5\pi}{9} \right\} + k \frac{2\pi}{3}$$

$$\therefore x = \frac{4\pi}{9}, \frac{10\pi}{9}, \frac{16\pi}{9}, \frac{5\pi}{9}, \frac{11\pi}{9}, \frac{17\pi}{9}$$

b $\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) = 0, \quad 0 \leq x \leq 3\pi$

$$\therefore \sin\left(x + \frac{\pi}{4}\right) = 0$$



$$\therefore x + \frac{\pi}{4} = 0 + k\pi$$

$$\therefore x = -\frac{\pi}{4} + k\pi$$

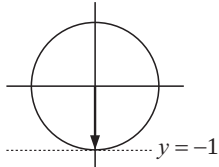
$$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$$

$$\begin{aligned} 8 \quad a \quad & \sin^2 x - \sin x - 2 = 0 \\ & \therefore (\sin x - 2)(\sin x + 1) = 0 \\ & \therefore \sin x = 2 \text{ or } -1 \end{aligned}$$

But $\sin x$ values lie between -1 and 1 inclusive

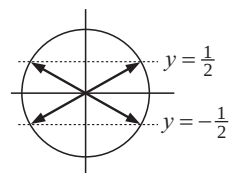
$$\therefore \sin x = -1$$

$$\therefore x = \frac{3\pi}{2} + k2\pi$$



$$\begin{aligned} b \quad & 4 \sin^2 x = 1 \\ & \therefore \sin^2 x = \frac{1}{4} \\ & \therefore \sin x = \pm \frac{1}{2} \end{aligned}$$

$$\therefore x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k\pi$$



$$9 \quad P(t) = 5 + 2 \sin\left(\frac{\pi t}{3}\right), \quad 0 \leq t \leq 8$$

$$a \quad \begin{aligned} P(0) &= 5 + 2 \sin 0 \\ &= 5 \end{aligned}$$

i.e., 5000 water beetles

$$b \quad \text{Smallest } P = 5 + 2(-1) = 3$$

$$\text{Largest } P = 5 + 2(1) = 7$$

$\therefore$ smallest is 3000 water beetles
largest is 7000 water beetles

c If population is > 6000 ,
then $P(t) > 6$

$$\therefore 5 + 2 \sin\left(\frac{\pi t}{3}\right) > 6$$

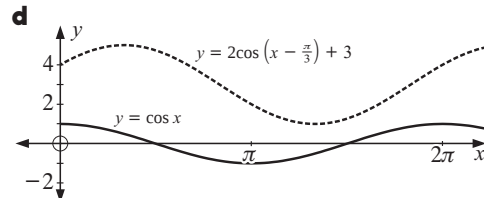
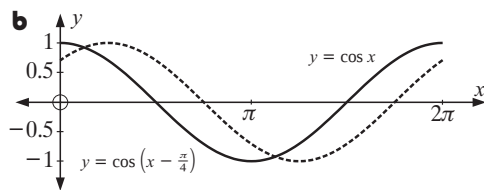
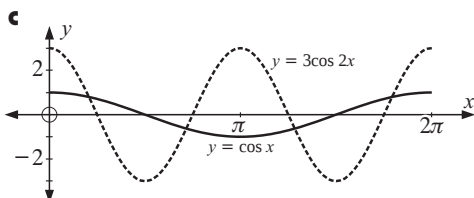
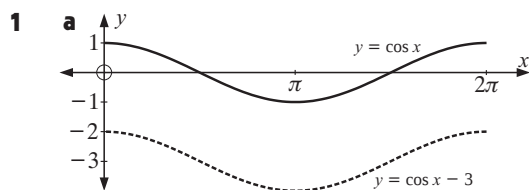
$$\therefore 2 \sin\left(\frac{\pi t}{3}\right) > 1$$

$$\therefore \sin\left(\frac{\pi t}{3}\right) > \frac{1}{2}$$

Using technology,

$$0.5 < t < 2.5 \quad \text{and} \quad 6.5 < t < 8$$

REVIEW SET 13D



$$2 \quad P(t) = 40 + 12 \sin \frac{2\pi}{7} \left(t - \frac{37}{12}\right) \text{ m}^3$$

$$a \quad \begin{aligned} P(t) \text{ is a minimum of } & 40 + 12(-1) \\ & = 28 \text{ mg/m}^3 \end{aligned}$$

$$b \quad \text{when } \sin \frac{2\pi}{7} \left(t - \frac{37}{12}\right) = -1$$

$$\therefore \frac{2\pi}{7} \left(t - \frac{37}{12}\right) = \frac{3\pi}{2} + k2\pi$$

$$\therefore \frac{2}{7} \left(t - \frac{37}{12}\right) = \frac{3}{2} + k2$$

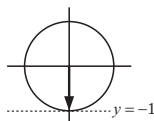
$$\text{So, } t - \frac{37}{12} = \frac{21}{4} + k7$$

$$\therefore t = 8\frac{1}{3} + k7$$

$$\therefore t = 1\frac{1}{3}, 8\frac{1}{3}, 15\frac{1}{3}, \text{ etc.}$$

$\therefore$ on Mondays at 8.00 am

$\{1\frac{1}{3} \text{ days after midnight Sat.}\}$



$$3 \quad a \quad \text{If } y = A \cos B(t - C) + D$$

$$\text{then } A = -4, \quad \frac{2\pi}{B} = \pi$$

$$\therefore B = 2$$

$$C = D = 0$$

$$\therefore y = -4 \cos 2x$$

$$b \quad \text{If } y = A \cos B(x - C) + D$$

$$\text{then } A = 1, \quad \frac{2\pi}{B} = 8 \quad \therefore B = \frac{\pi}{4}$$

$$D = \frac{\text{max.} + \text{min.}}{2} = \frac{3 + 1}{2} = 2$$

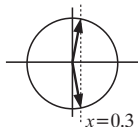
$$C = 0$$

$$\text{So, } y = \cos\left(\frac{\pi}{4}x\right) + 2$$

4 a $\cos x = 0.4379, \quad 0 \leq x \leq 10$
 $\therefore x \doteq 1.12, 5.17, 7.40$

b $\cos(x - 2.4) = -0.6014, \quad 0 \leq x \leq 6$
 $\therefore x \doteq 0.184, 4.62$

5 a $\cos 4x = 0.3, \quad \text{for all } x$
 $\therefore 4x = \cos^{-1}(0.3)$

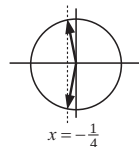


$$\therefore 4x = \left. \begin{matrix} 1.2661 \\ 2\pi - 1.2661 \end{matrix} \right\} + k2\pi$$

$$\therefore x = \left. \begin{matrix} 0.317 \\ 1.254 \end{matrix} \right\} + k\left(\frac{\pi}{2}\right)$$

b $4 \cos 2x + 1 = 0, \quad 0 \leq x \leq 5$
 $\therefore \cos 2x = -\frac{1}{4}$

$$\therefore 2x = \left. \begin{matrix} \pi - 1.318 \\ \pi + 1.318 \end{matrix} \right\} + k2\pi$$

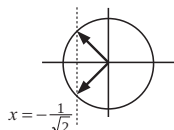


$$\therefore x = \left. \begin{matrix} 0.912 \\ 2.230 \end{matrix} \right\} + k\pi$$

$$\therefore x = 0.912, 4.05, 2.23$$

6 a $\cos x = -\frac{1}{\sqrt{2}}, \quad 0 \leq x \leq 4\pi$

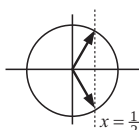
$$x = \left. \begin{matrix} \frac{3\pi}{4} \\ \frac{5\pi}{4} \end{matrix} \right\} + k2\pi$$



$$\therefore x = \frac{3\pi}{4}, \frac{11\pi}{4}, \frac{5\pi}{4}, \frac{13\pi}{4}$$

b $\cos\left(x + \frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leq x \leq 2\pi$

$$x + \frac{2\pi}{3} = \left. \begin{matrix} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{matrix} \right\} + k2\pi$$



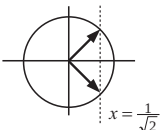
$$\therefore x = \left. \begin{matrix} -\frac{\pi}{3} \\ \pi \end{matrix} \right\} + k2\pi$$

$$\therefore x = -\frac{\pi}{3}, \frac{5\pi}{3}, \pi, -\pi$$

7 a $\sqrt{2} \cos\left(x + \frac{\pi}{4}\right) - 1 = 0, \quad 0 \leq x \leq 4\pi$

$$\therefore \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore x + \frac{\pi}{4} = \left. \begin{matrix} \frac{\pi}{4} \\ \frac{7\pi}{4} \end{matrix} \right\} + k2\pi$$



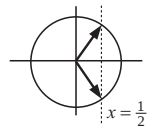
$$\therefore x = \left. \begin{matrix} 0 \\ \frac{3\pi}{2} \end{matrix} \right\} + k2\pi$$

$$\therefore x = 0, 2\pi, 4\pi, \frac{3\pi}{2}, \frac{7\pi}{2}$$

b $2 \cos 2x - 1 = 0, \quad \text{for all } x$

$$\therefore \cos 2x = \frac{1}{2}$$

$$\therefore 2x = \left. \begin{matrix} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{matrix} \right\} + k2\pi$$



$$\therefore x = \left. \begin{matrix} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{matrix} \right\} + k\pi$$

8 a $\cos^3 \theta + \sin^2 \theta \cos \theta$
 $= \cos \theta (\cos^2 \theta + \sin^2 \theta)$
 $= \cos \theta (1)$
 $= \cos \theta$

b $\frac{\cos^2 \theta - 1}{\sin \theta}$
 $= \frac{-(1 - \cos^2 \theta)}{\sin \theta}$
 $= -\frac{\sin^2 \theta}{\sin \theta}$
 $= -\sin \theta$

c $3 \cos \theta - \cos \theta$
 $= 2 \cos \theta$

d $5 - 5 \sin^2 \theta$
 $= 5(1 - \sin^2 \theta)$
 $= 5 \cos^2 \theta$

e $\frac{\sin^2 \theta - 1}{\cos \theta}$
 $= -\frac{(1 - \sin^2 \theta)}{\cos \theta}$
 $= -\frac{\cos^2 \theta}{\cos \theta}$
 $= -\cos \theta$

$$\begin{aligned} 9 \quad \mathbf{a} \quad & (2 \sin \alpha - 1)^2 \\ & = 4 \sin^2 \alpha - 4 \sin \alpha + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & (\cos \alpha - \sin \alpha)^2 \\ & = \cos^2 \alpha - 2 \sin \alpha \cos \alpha + \sin^2 \alpha \\ & = \cos^2 \alpha + \sin^2 \alpha - 2 \sin \alpha \cos \alpha \\ & = 1 - \sin 2\alpha \end{aligned}$$

REVIEW SET 13E

$$\begin{aligned} 1 \quad \mathbf{a} \quad & \frac{1 - \cos^2 \theta}{1 + \cos \theta} \\ & = \frac{1(1 + \cos \theta)(1 - \cos \theta)}{1 + \cos \theta} \\ & = 1 - \cos \theta \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{\sin \alpha - \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\ & = \frac{\sin \alpha - \cos \alpha}{(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)} \\ & = \frac{1}{\sin \alpha + \cos \alpha} \end{aligned}$$

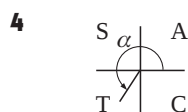
$$\begin{aligned} \mathbf{c} \quad & \frac{4 \sin^2 \alpha - 4}{8 \cos \alpha} \\ & = \frac{-4(1 - \sin^2 \alpha)}{8 \cos \alpha} \\ & = \frac{-4 \cos^2 \alpha}{8 \cos \alpha} \\ & = -\frac{1}{2} \cos \alpha \end{aligned}$$

$$\begin{aligned} 2 \quad \mathbf{a} \quad & \frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} \\ & = \frac{\cos^2 \theta + (1 + \sin \theta)^2}{(1 + \sin \theta) \cos \theta} \\ & = \frac{\cos^2 \theta + 1 + 2 \sin \theta + \sin^2 \theta}{(1 + \sin \theta) \cos \theta} \\ & = \frac{2 + 2 \sin \theta}{(1 + \sin \theta) \cos \theta} \quad \{\cos^2 \theta + \sin^2 \theta = 1\} \\ & = \frac{2(1 + \sin \theta)}{(1 + \sin \theta) \cos \theta} \\ & = \frac{2}{\cos \theta} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \left(1 + \frac{1}{\cos \theta}\right) (\cos \theta - \cos^2 \theta) \\ & = \cos \theta - \cos^2 \theta + 1 - \cos \theta \\ & = 1 - \cos^2 \theta \\ & = \sin^2 \theta \end{aligned}$$

$$\begin{aligned} 3 \quad \mathbf{a} \quad \sin 2A &= 2 \sin A \cos A \\ &= 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) \\ &= \frac{120}{169} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cos 2A &= \cos^2 A - \sin^2 A \\ &= \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \\ &= \frac{144 - 25}{169} \\ &= \frac{119}{169} \end{aligned}$$



$$\begin{aligned} \cos^2 \alpha + \sin^2 \alpha &= 1 \\ \therefore \cos^2 \alpha + \frac{9}{16} &= 1 \\ \therefore \cos^2 \alpha &= \frac{7}{16} \\ \therefore \cos \alpha &= \pm \frac{\sqrt{7}}{4} \end{aligned}$$

But in Q3, $\cos \alpha < 0$

$$\therefore \cos \alpha = -\frac{\sqrt{7}}{4}$$

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= 2\left(-\frac{3}{4}\right)\left(-\frac{\sqrt{7}}{4}\right) \\ &= \frac{3\sqrt{7}}{8} \end{aligned}$$

5

$$\cos 2A = 1 - 2\sin^2 A$$

$$\therefore \cos x = 1 - 2\sin^2\left(\frac{x}{2}\right) \quad \{\text{letting } 2A = x, A = \frac{x}{2}\}$$

$$\therefore -\frac{3}{4} = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

$$\therefore 2\sin^2\left(\frac{x}{2}\right) = \frac{7}{4}$$

$$\therefore \sin^2\left(\frac{x}{2}\right) = \frac{7}{8}$$

$$\therefore \sin\left(\frac{x}{2}\right) = \pm \frac{\sqrt{7}}{2\sqrt{2}}$$

$$\text{But } \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \quad (\text{in Q2})$$

$$\therefore \sin\left(\frac{x}{2}\right) = \frac{\sqrt{7}}{2\sqrt{2}}$$

6

a i $\tan x = 4$

$$\therefore x \doteq 1.326 + k\pi$$

$$\text{i.e., } x \doteq 1.33 + k\pi$$

ii $\tan\left(\frac{x}{4}\right) = 4$

$$\therefore \frac{x}{4} \doteq 1.326 + k\pi$$

$$\therefore x \doteq 5.30 + k4\pi$$

iii $\tan(x - 1.5) = 4$

$$\therefore x - 1.5 \doteq 1.326 + k2\pi$$

$$\therefore x \doteq 2.83 + k\pi$$

b i $\tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}$

$$\therefore x + \frac{\pi}{6} = \frac{2\pi}{3} + k\pi$$

$$\therefore x = \frac{\pi}{2} + k\pi$$

ii $\tan 2x = -\sqrt{3}$

$$\therefore 2x = \frac{2\pi}{3} + k\pi$$

$$\therefore x = \frac{\pi}{3} + \frac{k\pi}{2}$$

iii $\tan^2 x - 3 = 0$

$$\therefore \tan x = \pm\sqrt{3}$$

$$\therefore x = \left\{ \frac{\pi}{3}, \frac{2\pi}{3} \right\} + k\pi$$

c $3\tan(x - 1.2) = -2$

$$\therefore \tan(x - 1.2) = -\frac{2}{3}$$

$$\therefore x - 1.2 \doteq -0.588 + k\pi$$

$$\therefore x \doteq 0.612 + k\pi$$

7

$$\tan \theta = -\frac{2}{3}, \quad \frac{\pi}{2} < \theta < \pi$$

$$\therefore \frac{\sin \theta}{\cos \theta} = -\frac{2}{3}$$

$$\therefore \sin \theta = -2k, \quad \cos \theta = 3k$$

$$\text{but } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore 9k^2 + 4k^2 = 1$$

$$\therefore 13k^2 = 1$$

$$\therefore k = \pm \frac{1}{\sqrt{13}}$$



But in Q2,

$$\sin \theta > 0, \quad \cos \theta < 0$$

$$\therefore k = -\frac{1}{\sqrt{13}}$$

$$\therefore \sin \theta = \frac{2}{\sqrt{13}}, \quad \cos \theta = -\frac{3}{\sqrt{13}}$$

8

$$\frac{\sin 2\alpha - \sin \alpha}{\cos 2\alpha - \cos \alpha + 1} = \frac{2\sin \alpha \cos \alpha - \sin \alpha}{2\cos^2 \alpha - 1 - \cos \alpha + 1}$$

$$= \frac{\sin \alpha (2\cos \alpha - 1)}{\cos \alpha (2\cos \alpha - 1)}$$

$$= \frac{\sin \alpha}{\cos \alpha}$$

$$= \tan \alpha$$

Chapter 14

MATRICES

EXERCISE 14A

1 a 1 row and 4 columns $\therefore 1 \times 4$

c 2 rows and 2 columns $\therefore 2 \times 2$

b 2 rows and 1 column $\therefore 2 \times 1$

d 3 rows and 3 columns $\therefore 3 \times 3$

2 $\begin{bmatrix} 2 & 1 & 6 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1.95 \\ 2.35 \\ 0.15 \\ 0.95 \end{bmatrix}$

c $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.15) + (1 \times 0.95)$
represents the total cost of the groceries.

3 $\begin{matrix} 200 \text{ g} & 300 \text{ g} & 500 \text{ g} \\ \begin{bmatrix} 1000 & 1500 & 1250 \\ 1500 & 1000 & 1000 \\ 800 & 2300 & 1300 \\ 1200 & 1200 & 1200 \end{bmatrix} & \begin{matrix} \text{week 1} \\ \text{week 2} \\ \text{week 3} \\ \text{week 4} \end{matrix} \end{matrix}$

4 $\begin{matrix} \text{pies} & \text{pasties} & \text{rolls} & \text{buns} \\ \begin{bmatrix} 40 & 50 & 55 & 40 \\ 25 & 65 & 44 & 30 \\ 35 & 40 & 40 & 35 \\ 35 & 40 & 35 & 50 \end{bmatrix} & \begin{matrix} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \\ \text{Monday} \end{matrix} \end{matrix}$

EXERCISE 14B

1 a $A + B$

$$= \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix}$$

c $B + C$

$$= \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ -6 & -1 \end{bmatrix}$$

b $A + B + C$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ -1 & 1 \end{bmatrix}$$

d $C + B - A$

$$= \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -11 & -3 \end{bmatrix}$$

2 a $P + Q$

$$= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} + \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{bmatrix}$$

b $P - Q$

$$= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} - \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{bmatrix}$$

c $Q - P = \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{bmatrix}$

3 a $\begin{matrix} \text{Friday} & \text{Saturday} \\ \begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix} & \begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix} \end{matrix}$

b Total for Friday and Saturday $= \begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix} + \begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix} = \begin{bmatrix} 187 \\ 229 \\ 101 \end{bmatrix}$

4 a i Cost price

$$\begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix}$$

ii Selling price

$$\begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix}$$

b In order to find David's profit/loss matrix we subtract the cost price matrix from the selling price matrix.

$$\text{c Profit/Loss matrix} = \begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix} - \begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.90 \\ 0.41 \\ -0.28 \\ -0.05 \end{bmatrix}$$

5 a i Lou Rose **ii** Lou Rose **iii** Total sales for November and December

$$\begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix} + \begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix} = \begin{bmatrix} 41 & 44 \\ 24 & 42 \\ 67 & 43 \end{bmatrix}$$

b If **F** represents the furniture in one flat, **12F** represents the furniture in all 12 flats.

$$\text{6 a } \begin{bmatrix} x & x^2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} y & 4 \\ 3 & y+1 \end{bmatrix}$$

Equating corresponding elements:

$$x = y, \quad x^2 = 4 \quad \text{and} \quad -1 = y + 1$$

$$\therefore y = -2 \quad \text{and} \quad x = \pm 2$$

$$\text{But } x = y \quad \therefore x = y = -2$$

$$\text{b } \begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} -y & x \\ x & -y \end{bmatrix}$$

Equating corresponding elements:

$$\left. \begin{array}{l} x = -y \\ y = x \end{array} \right\} \therefore y = 0, x = 0$$

$$\text{7 a } \mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-1) & 1+2 \\ 3+2 & -1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

$$\mathbf{B} + \mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & 2+1 \\ 2+3 & 3+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

b $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for all 2×2 matrices **A** and **B** because addition of numbers is commutative.

$$\text{8 a } (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \left(\begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} \right) + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \left(\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix}$$

b Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$\begin{aligned} \therefore (\mathbf{A} + \mathbf{B}) + \mathbf{C} &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right) + \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A} + (\mathbf{B} + \mathbf{C}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix} \\ &= \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix} \\ &= (\mathbf{A} + \mathbf{B}) + \mathbf{C} \end{aligned}$$

EXERCISE 14C

1 a $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

$$\begin{aligned} 2\mathbf{B} &= 2 \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 48 & 12 \end{bmatrix} \\ \frac{1}{3}\mathbf{B} &= \frac{1}{3} \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix} \\ \frac{1}{12}\mathbf{B} &= \frac{1}{12} \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{bmatrix} \\ -\frac{1}{2}\mathbf{B} &= -\frac{1}{2} \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -12 & -3 \end{bmatrix} \end{aligned}$$

2 a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2+1 & 3+2 & 5+1 \\ 1+1 & 6+2 & 4+3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{bmatrix} \end{aligned}$$

c $2\mathbf{A} + \mathbf{B}$ **d** $3\mathbf{A} - \mathbf{B}$

$$\begin{aligned} 2\mathbf{A} + \mathbf{B} &= 2 \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 & 10 \\ 2 & 12 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4+1 & 6+2 & 10+1 \\ 2+1 & 12+2 & 8+3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2-1 & 3-2 & 5-1 \\ 1-1 & 6-2 & 4-3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 3\mathbf{A} - \mathbf{B} &= 3 \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 9 & 15 \\ 3 & 18 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6-1 & 9-2 & 15-1 \\ 3-1 & 18-2 & 12-3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{bmatrix} \end{aligned}$$

3 a $2\mathbf{H}$ **b** $\frac{1}{2}\mathbf{H}$ **c** 150% of $\mathbf{H}$

$$\begin{aligned} 2\mathbf{H} &= 2 \begin{bmatrix} 6 \\ 12 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \\ 120 \\ 60 \end{bmatrix} \\ \frac{1}{2}\mathbf{H} &= \frac{1}{2} \begin{bmatrix} 6 \\ 12 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 30 \\ 15 \end{bmatrix} \\ 150\% \text{ of } \mathbf{H} &= 1.5 \begin{bmatrix} 6 \\ 12 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ 90 \\ 45 \end{bmatrix} \end{aligned}$$

- 4 a** Increase of 15% = 1.15
- $$\begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 35 & 46 & 46 & 69 \\ 58 & 46 & 35 & 86 \\ 46 & 46 & 58 & 58 \\ 12 & 23 & 23 & 17 \end{bmatrix} \text{ rounded to the nearest whole number.}$$
- b** Decrease of 15% = 0.85
- $$\begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 26 & 34 & 34 & 51 \\ 43 & 34 & 26 & 64 \\ 34 & 34 & 43 & 43 \\ 9 & 17 & 17 & 13 \end{bmatrix} \text{ rounded to the nearest whole number.}$$
- 5 a** Weekdays Weekends
- $$\begin{bmatrix} 75 \\ 27 \\ 102 \end{bmatrix} \quad \begin{bmatrix} 136 \\ 43 \\ 129 \end{bmatrix} \quad \begin{matrix} \text{VHS} \\ \text{DVD} \\ \text{games} \end{matrix}$$
- b**
- $$\begin{bmatrix} 75 \\ 27 \\ 102 \end{bmatrix} + \begin{bmatrix} 136 \\ 43 \\ 129 \end{bmatrix} = \begin{bmatrix} 211 \\ 70 \\ 231 \end{bmatrix}$$
- c** The sum matrix of **b** represents total weekly average hirings.
- 6** The matrix is $12\mathbf{F} = 12 \begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 48 \\ 24 \\ 12 \end{bmatrix}$

EXERCISE 14D

- 1 a** $\mathbf{A} + 2\mathbf{A} = 3\mathbf{A}$ **b** $3\mathbf{B} - 3\mathbf{B} = \mathbf{O}$ **c** $\mathbf{C} - 2\mathbf{C} = -\mathbf{C}$
- d** $-\mathbf{B} + \mathbf{B} = \mathbf{O}$ **e** $2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$ **f** $-(\mathbf{A} + \mathbf{B}) = -\mathbf{A} - \mathbf{B}$
- g** $-(2\mathbf{A} - \mathbf{C}) = -2\mathbf{A} + \mathbf{C}$ **h** $3\mathbf{A} - (\mathbf{B} - \mathbf{A}) = 3\mathbf{A} - \mathbf{B} + \mathbf{A} = 4\mathbf{A} - \mathbf{B}$ **i** $\mathbf{A} + 2\mathbf{B} - (\mathbf{A} - \mathbf{B}) = \mathbf{A} + 2\mathbf{B} - \mathbf{A} + \mathbf{B} = 3\mathbf{B}$
- 2 a** if $\mathbf{X} + \mathbf{B} = \mathbf{A}$
then $\mathbf{X} + \mathbf{B} + (-\mathbf{B}) = \mathbf{A} + (-\mathbf{B})$
 $\therefore \mathbf{X} + \mathbf{O} = \mathbf{A} - \mathbf{B}$
 $\therefore \mathbf{X} = \mathbf{A} - \mathbf{B}$
- c** if $4\mathbf{B} + \mathbf{X} = 2\mathbf{C}$
then $4\mathbf{B} + \mathbf{X} + (-4\mathbf{B}) = 2\mathbf{C} + (-4\mathbf{B})$
 $\therefore \mathbf{O} + \mathbf{X} = 2\mathbf{C} - 4\mathbf{B}$
 $\therefore \mathbf{X} = 2\mathbf{C} - 4\mathbf{B}$
- e** if $3\mathbf{X} = \mathbf{B}$
then $\frac{1}{3}(3\mathbf{X}) = \frac{1}{3}\mathbf{B}$
 $\therefore 1\mathbf{X} = \frac{1}{3}\mathbf{B}$ i.e., $\mathbf{X} = \frac{1}{3}\mathbf{B}$
- g** if $\frac{1}{2}\mathbf{X} = \mathbf{C}$
then $2(\frac{1}{2}\mathbf{X}) = 2\mathbf{C}$
 $\therefore 1\mathbf{X} = 2\mathbf{C}$ i.e., $\mathbf{X} = 2\mathbf{C}$
- i** if $\mathbf{A} - 4\mathbf{X} = \mathbf{C}$
then $\mathbf{A} - 4\mathbf{X} + 4\mathbf{X} = \mathbf{C} + 4\mathbf{X}$
 $\therefore \mathbf{A} + \mathbf{O} = \mathbf{C} + 4\mathbf{X}$
 $\therefore \mathbf{A} = \mathbf{C} + 4\mathbf{X}$
and $\mathbf{A} - \mathbf{C} = 4\mathbf{X}$
 $\therefore \frac{1}{4}(\mathbf{A} - \mathbf{C}) = \frac{1}{4}(4\mathbf{X})$
 $\therefore \mathbf{X} = \frac{1}{4}(\mathbf{A} - \mathbf{C})$
- b** if $\mathbf{B} + \mathbf{X} = \mathbf{C}$
then $\mathbf{B} + \mathbf{X} + (-\mathbf{B}) = \mathbf{C} + (-\mathbf{B})$
 $\therefore \mathbf{O} + \mathbf{X} = \mathbf{C} - \mathbf{B}$
 $\therefore \mathbf{X} = \mathbf{C} - \mathbf{B}$
- d** if $2\mathbf{X} = \mathbf{A}$
then $\frac{1}{2}(2\mathbf{X}) = \frac{1}{2}\mathbf{A}$
 $\therefore 1\mathbf{X} = \frac{1}{2}\mathbf{A}$ i.e., $\mathbf{X} = \frac{1}{2}\mathbf{A}$
- f** if $\mathbf{A} - \mathbf{X} = \mathbf{B}$
then $\mathbf{A} - \mathbf{X} + \mathbf{X} = \mathbf{B} + \mathbf{X}$
 $\therefore \mathbf{A} + \mathbf{O} = \mathbf{B} + \mathbf{X}$
 $\therefore \mathbf{A} = \mathbf{B} + \mathbf{X}$
and $\mathbf{A} + (-\mathbf{B}) = \mathbf{B} + \mathbf{X} + (-\mathbf{B})$
 $\therefore \mathbf{A} - \mathbf{B} = \mathbf{X} + \mathbf{O}$
 $\therefore \mathbf{A} - \mathbf{B} = \mathbf{X}$
 $\therefore \mathbf{X} = \mathbf{A} - \mathbf{B}$
- h** if $2(\mathbf{X} + \mathbf{A}) = \mathbf{B}$
then $\frac{1}{2}[2(\mathbf{X} + \mathbf{A})] = \frac{1}{2}\mathbf{B}$
 $\therefore 1(\mathbf{X} + \mathbf{A}) = \frac{1}{2}\mathbf{B}$
 $\therefore \mathbf{X} + \mathbf{A} = \frac{1}{2}\mathbf{B}$
 $\therefore \mathbf{X} + \mathbf{A} + (-\mathbf{A}) = \frac{1}{2}\mathbf{B} + (-\mathbf{A})$
 $\therefore \mathbf{X} + \mathbf{O} = \frac{1}{2}\mathbf{B} - \mathbf{A}$
 $\therefore \mathbf{X} = \frac{1}{2}\mathbf{B} - \mathbf{A}$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & \text{if } \frac{1}{3}\mathbf{X} = \mathbf{M} \\ & \text{then } 3\left(\frac{1}{3}\mathbf{X}\right) = 3\mathbf{M} \\ & \therefore \mathbf{X} = 3\mathbf{M} \end{aligned}$$

$$\begin{aligned} &= 3 \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix} \end{aligned}$$

$$\mathbf{b} \quad \text{if } 4\mathbf{X} = \mathbf{N} \quad \text{then } \frac{1}{4}(4\mathbf{X}) = \frac{1}{4}\mathbf{N}$$

$$\therefore \mathbf{X} = \frac{1}{4}\mathbf{N}$$

$$\therefore \mathbf{X} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$$

$$\begin{aligned} \mathbf{c} \quad & \text{if } \mathbf{A} - 2\mathbf{X} = 3\mathbf{B} \\ & \text{then } \mathbf{A} - 2\mathbf{X} + 2\mathbf{X} = 3\mathbf{B} + 2\mathbf{X} \\ & \therefore \mathbf{A} = 3\mathbf{B} - 2\mathbf{X} \\ & \therefore \mathbf{A} + (-3\mathbf{B}) = 3\mathbf{B} - 2\mathbf{X} + (-3\mathbf{B}) \\ & \therefore \mathbf{A} - 3\mathbf{B} = 2\mathbf{X} \\ & \therefore \frac{1}{2}(\mathbf{A} - 3\mathbf{B}) = \frac{1}{2}(2\mathbf{X}) \\ & \therefore \frac{1}{2}(\mathbf{A} - 3\mathbf{B}) = \mathbf{X} \\ & \therefore \mathbf{X} = \frac{1}{2}(\mathbf{A} - 3\mathbf{B}) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{bmatrix} -2 & -12 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -6 \\ 1 & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

EXERCISE 14E.1

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ &= [3 \times 5 + (-1) \times 4] \\ &= [15 - 4] \\ &= [11] \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} \\ &= [1 \times 5 + 3 \times 1 + 2 \times 7] \\ &= [5 + 3 + 14] \\ &= [22] \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \begin{bmatrix} 6 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix} = [6 \times 1 + (-1) \times 0 + 2 \times (-1) + 3 \times 4] \\ &= [6 + 0 - 2 + 12] \\ &= [16] \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad & \begin{bmatrix} w & x & y & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = [w + x + y + z] \\ & \therefore \frac{1}{4}(w + x + y + z), \text{ which is the average} \\ & \text{of } w, x, y \text{ and } z, \text{ can be represented as} \end{aligned}$$

$$\begin{bmatrix} w & x & y & z \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \mathbf{Q} &= \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 27 & 35 & 39 \end{bmatrix} \quad \mathbf{b} \quad \text{total cost} = \mathbf{PQ} = \begin{bmatrix} 27 & 35 & 39 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \\ &= [27 \times 4 + 35 \times 3 + 39 \times 2] \\ &= [291] \quad \therefore \text{total cost is \$291} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \mathbf{P} &= \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix} \quad \mathbf{b} \quad \text{total points} = \mathbf{PN} = \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix} \\ &= [10 \times 3 + 6 \times 2 + 3 \times 4 + 1 \times 2] \\ &= [30 + 12 + 12 + 2] \\ &= [56] \quad \text{So, the number of points awarded is 56.} \end{aligned}$$

$$\mathbf{N} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

EXERCISE 14E.2

1 $\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$ which is 1 row $\times$ 3 columns

$\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ which is 2 rows $\times$ 3 columns

$\mathbf{AB}$ cannot be found because the number of columns in $\mathbf{A}$ does not equal the number of rows in $\mathbf{B}$.

2 $\mathbf{A}$ is $2 \times n$ and $\mathbf{B}$ is $m \times 3$.

a We can find $\mathbf{AB}$ if the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{B}$, i.e., $n = m$.

b If $\mathbf{AB}$ can be found its order is 2×3 .

c $\mathbf{BA}$ cannot be found because the number of columns in $\mathbf{B}$ does not equal the number of rows in $\mathbf{A}$.

3 a $\mathbf{B}$ is 1×2 and $\mathbf{A}$ is 2×2 $\mathbf{BA} = \begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$
 $\therefore \mathbf{BA}$ is 1×2 $= \begin{bmatrix} 5 \times 2 + 6 \times 3 & 5 \times 1 + 6 \times 4 \end{bmatrix}$
 $= \begin{bmatrix} 10 + 18 & 5 + 24 \end{bmatrix}$
 $= \begin{bmatrix} 28 & 29 \end{bmatrix}$

b i $\mathbf{A}$ is 1×3 and $\mathbf{B}$ is 3×1 , $\therefore \mathbf{AB}$ is 1×1 and

$$\mathbf{AB} = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = [2 \times 1 + 0 \times 4 + 3 \times 2] = [2 + 0 + 6] = [8]$$

ii $\mathbf{B}$ is 3×1 and $\mathbf{A}$ is 1×3 , $\therefore \mathbf{BA}$ is 3×3 and

$$\mathbf{BA} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 & 1 \times 0 & 1 \times 3 \\ 4 \times 2 & 4 \times 0 & 4 \times 3 \\ 2 \times 2 & 2 \times 0 & 2 \times 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 8 & 0 & 12 \\ 4 & 0 & 6 \end{bmatrix}$$

4 a $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ is 1×3 by 3×3 $\therefore$ resultant matrix is 1×3
 $= \begin{bmatrix} 1 \times 2 + 2 \times 0 + 1 \times 1 & 1 \times 3 + 2 \times 1 + 1 \times 0 & 1 \times 1 + 2 \times 0 + 1 \times 2 \end{bmatrix}$
 $= \begin{bmatrix} 2 + 0 + 1 & 3 + 2 + 0 & 1 + 0 + 2 \end{bmatrix}$
 $= \begin{bmatrix} 3 & 5 & 3 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is 3×3 by 3×1 $\therefore$ resultant matrix is 3×1
 $= \begin{bmatrix} 1 \times 2 + 0 \times 3 + (-1) \times 4 \\ (-1) \times 2 + 1 \times 3 + 0 \times 4 \\ 0 \times 2 + (-1) \times 3 + 1 \times 4 \end{bmatrix}$
 $= \begin{bmatrix} 2 + 0 - 4 \\ -2 + 3 + 0 \\ 0 - 3 + 4 \end{bmatrix}$
 $= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

5 a

$$\mathbf{C} = \begin{bmatrix} 12.5 \\ 9.5 \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} 2375 & 5156 \\ 2502 & 3612 \end{bmatrix} \quad \begin{array}{l} \text{adults} \quad \text{children} \\ \text{first day} \\ \text{second day} \end{array}$$

b

$\mathbf{N}$ is 2×2 and $\mathbf{C}$ is 2×1
 $\therefore \mathbf{NC}$ is 2×1

$$\begin{aligned} \mathbf{NC} &= \begin{bmatrix} 2375 & 5156 \\ 2502 & 3612 \end{bmatrix} \begin{bmatrix} 12.5 \\ 9.5 \end{bmatrix} \\ &= \begin{bmatrix} 2375 \times 12.5 + 5156 \times 9.5 \\ 2502 \times 12.5 + 3612 \times 9.5 \end{bmatrix} \\ &= \begin{bmatrix} 29\,687.5 + 48\,982 \\ 31\,275 + 34\,314 \end{bmatrix} \\ &= \begin{bmatrix} 78\,669.5 \\ 65\,589 \end{bmatrix} \quad \begin{array}{l} \text{income from day 1} \\ \text{income from day 2} \end{array} \end{aligned}$$

c Total income = \$78 669.50 + \$65 589 = \$144 258.50

6 a

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \begin{array}{l} \text{me} \quad \text{friend} \\ \text{hammers} \\ \text{screwdrivers} \\ \text{cans of paint} \end{array}$$

b

$$\mathbf{P} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix} \quad \begin{array}{l} \text{store A} \\ \text{store B} \end{array}$$

c $\mathbf{P}$ is 2×3 and $\mathbf{R}$ is 3×2 , $\therefore \mathbf{PR}$ is 2×2

$$\mathbf{PR} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 \times 1 + 3 \times 1 + 19 \times 2 & 7 \times 1 + 3 \times 2 + 19 \times 3 \\ 6 \times 1 + 2 \times 1 + 22 \times 2 & 6 \times 1 + 2 \times 2 + 22 \times 3 \end{bmatrix}$$

$$\therefore \mathbf{PR} = \begin{bmatrix} 7 + 3 + 38 & 7 + 6 + 57 \\ 6 + 2 + 44 & 6 + 4 + 66 \end{bmatrix} = \begin{bmatrix} 48 & 70 \\ 52 & 76 \end{bmatrix}$$

d My costs at Store A are \$48; my friend's costs at Store B are \$76.

e My costs at Store B are \$52. Therefore I should shop at Store A, which is cheaper.

7

$$\begin{array}{ccc} \text{apples} & \text{bananas} & \text{oranges} \\ \downarrow & \downarrow & \downarrow \\ \text{Fruit matrix } \mathbf{F} = \begin{bmatrix} 6 & 7 & 9 \\ 5 & 8 & 4 \\ 4 & 7 & 2 \end{bmatrix} & \begin{array}{l} \text{first day} \\ \text{second day} \\ \text{third day} \end{array} & \text{cost matrix } \mathbf{C} = \begin{bmatrix} 18 \\ 15 \\ 13 \end{bmatrix} \quad \begin{array}{l} \text{apples} \\ \text{bananas} \\ \text{oranges} \end{array} \end{array}$$

The total cost of the three types of fruit over the three-day period is represented by $\mathbf{FC}$.

$$\begin{array}{l} \mathbf{F} \text{ is } 3 \times 3 \text{ and} \\ \mathbf{C} \text{ is } 3 \times 1, \\ \therefore \mathbf{FC} \text{ is } 3 \times 1. \end{array} \quad \mathbf{FC} = \begin{bmatrix} 6 & 7 & 9 \\ 5 & 8 & 4 \\ 4 & 7 & 2 \end{bmatrix} \begin{bmatrix} 18 \\ 15 \\ 13 \end{bmatrix} = \begin{bmatrix} 6 \times 18 + 7 \times 15 + 9 \times 13 \\ 5 \times 18 + 8 \times 15 + 4 \times 13 \\ 4 \times 18 + 7 \times 15 + 2 \times 13 \end{bmatrix}$$

$$\therefore \mathbf{FC} = \begin{bmatrix} 108 + 105 + 117 \\ 90 + 120 + 52 \\ 72 + 105 + 26 \end{bmatrix} = \begin{bmatrix} 330 \\ 262 \\ 203 \end{bmatrix}$$

$$\therefore \text{total cost} = \$330 + \$262 + \$203 = \$795$$

EXERCISE 14F**1 a**

$$\begin{bmatrix} 16 & 18 & 15 \\ 13 & 21 & 16 \\ 10 & 22 & 24 \end{bmatrix}$$

b

$$\begin{bmatrix} 10 & 6 & -7 \\ 9 & 3 & 0 \\ 4 & -4 & -10 \end{bmatrix}$$

$$\mathbf{c} \quad \begin{bmatrix} 22 & 0 & 132 & 176 & 198 \\ 44 & 154 & 88 & 110 & 0 \\ 176 & 44 & 88 & 88 & 132 \end{bmatrix} \quad \mathbf{d} \quad \begin{bmatrix} 115 \\ 136 \\ 46 \\ 106 \end{bmatrix}$$

2 a

nights breakfasts dinners
 ↘ ↓ ↙
 Numbers matrix $\mathbf{N} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix}$

b

	Bay View	Terrace	Staunton Star	
Prices matrix $\mathbf{P} =$	$\begin{bmatrix} 125 \\ 44 \\ 75 \end{bmatrix}$	$\begin{bmatrix} 150 \\ 40 \\ 80 \end{bmatrix}$	$\begin{bmatrix} 140 \\ 40 \\ 65 \end{bmatrix}$	room breakfast dinner

c Total prices for each venue = numbers matrix $\times$ prices matrix = $\mathbf{NP}$

$\mathbf{N}$ is 1×3 and $\mathbf{P}$ is 3×3 , $\therefore \mathbf{NP}$ is 1×3

$$\mathbf{NP} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ \$657 for Bay View, \$730 for Terrace, \$670 for Staunton Star.

d

$$\text{Total prices} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 369 & 420 & 385 \end{bmatrix}, \quad \{\text{using technology}\}$$

i.e., \$369 for Bay View, \$420 for Terrace, \$385 for Staunton Star.

e To include both scenarios we calculate

$$\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \\ 369 & 420 & 385 \end{bmatrix}, \quad \text{using technology}$$

3 To find total income we calculate:

Prices matrix = $\begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix}$

prices matrix $\times$ numbers matrix.

$$= \begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix} \begin{bmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{bmatrix}$$

$$= \begin{bmatrix} 51\,145 \\ 60\,655 \\ 61\,575 \\ 51\,285 \end{bmatrix} \quad \text{using technology}$$

$\therefore$ total income = \$51 145 + \$60 655 + \$61 575 + \$51 285 = \$224 660

4

$$\text{Numbers matrix } \mathbf{N} = \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix}$$

$$\text{Cost price matrix } \mathbf{C} = \begin{bmatrix} 1.95 & 2.10 & 1.45 & 0.95 \end{bmatrix}$$

$$\text{Selling price matrix } \mathbf{S} = \begin{bmatrix} 2.55 & 4.40 & 3.50 & 1.80 \end{bmatrix}$$

$$\begin{aligned}
\text{Profit per day} &= (\text{selling price per drink}) \times (\text{number of drinks sold}) \\
&\quad - (\text{cost price per drink}) \times (\text{number of drinks sold}) \\
&= \mathbf{SN} - \mathbf{CN} \\
&= \begin{bmatrix} 2.55 & 4.40 & 3.50 & 1.80 \end{bmatrix} \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix} \\
&\quad - \begin{bmatrix} 1.95 & 2.10 & 1.45 & 0.95 \end{bmatrix} \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix} \\
&= \begin{bmatrix} 515.35 & 421.30 & 414.55 & 575.45 & 702.05 & 769.40 \end{bmatrix} \text{ using technology}
\end{aligned}$$

$\therefore$ the profit for the week = \$515.35 + + \$769.40 = \$3398.10

A quicker way to solve this problem is to create a ‘profit/loss per drink’ matrix,

i.e., a selling price matrix – cost price matrix

$$\begin{aligned}
&= \mathbf{S} - \mathbf{C} \\
&= \begin{bmatrix} 2.55 & 4.40 & 3.50 & 1.80 \end{bmatrix} - \begin{bmatrix} 1.95 & 2.10 & 1.45 & 0.95 \end{bmatrix} \\
&= \begin{bmatrix} 0.60 & 2.30 & 2.05 & 0.85 \end{bmatrix}
\end{aligned}$$

$$\text{We then calculate } \begin{bmatrix} 0.60 & 2.30 & 2.05 & 0.85 \end{bmatrix} \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix}$$

Using technology the result checks with the result obtained above.

5 a

$$\text{Income matrix } \mathbf{I} = \begin{bmatrix} 125 & 195 & 225 \end{bmatrix}$$

$$\text{Cost matrix } \mathbf{C} = \begin{bmatrix} 85 & 120 & 130 \end{bmatrix}$$

$$\text{Numbers (bookings) matrix } \mathbf{N} = \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$\begin{aligned}
\text{profit per day} &= (\text{income from room}) \times (\text{bookings per day}) \\
&\quad - (\text{maintenance cost per room}) \times (\text{bookings per day})
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{IN} - \mathbf{CN} \\
&= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1185 & 800 & 1350 & 970 & 845 & 1130 & 845 \end{bmatrix}, \text{ using technology}
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ profit for the week} &= \$1185 + \$800 + \$1350 + \$970 + \$845 + \$1130 + \$845 \\
&= \$7125
\end{aligned}$$

- b** If the hotel maintained every room every day we would need to calculate
 (income from room) $\times$ (bookings per day) – (maintenance costs per room) $\times$ (number of rooms)

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \times \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \times \begin{bmatrix} 20 & 20 & 20 & 20 & 20 & 20 & 20 \\ 15 & 15 & 15 & 15 & 15 & 15 & 15 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$$\therefore \text{ using technology, profit per day} = \begin{bmatrix} -820 & -1840 & -455 & -1485 & -1725 & -920 & -1785 \end{bmatrix}$$

$$\therefore \text{ the profit per week would be } (-\$820) + \dots + (-\$1785) = -\$9030, \text{ i.e., a loss of } \$9030.$$

c Profit per room matrix = income per room matrix – cost per room matrix

$$= \mathbf{I} - \mathbf{C}$$

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} - \begin{bmatrix} 85 & 120 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 75 & 95 \end{bmatrix}$$

$\therefore$ the result in **a** can be obtained if we calculate:

$$\begin{bmatrix} 40 & 75 & 95 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \quad \begin{array}{l} \text{This checks} \\ \text{using technology.} \end{array}$$

EXERCISE 14G

1 $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1+0 & 1+0 \\ -1+0 & 1+6 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 7 \end{bmatrix}$

$$\mathbf{BA} = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 0+2 \\ 0+3 & 0+6 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 6 \end{bmatrix}$$

$\mathbf{AB} \neq \mathbf{BA} \quad \therefore$ in the general case $\mathbf{AB}$ does not necessarily equal $\mathbf{BA}$.

2 $\mathbf{AO} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

$$\mathbf{OA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O} \quad \therefore \mathbf{AO} = \mathbf{OA} = \mathbf{O}$$

3 a Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$, say

$$\begin{aligned} & \mathbf{A}(\mathbf{B} + \mathbf{C}) \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3+4 & 4+4 \\ 9+8 & 12+8 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} & \mathbf{AB} + \mathbf{AC} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 1+2 \\ 3+4 & 3+4 \end{bmatrix} + \begin{bmatrix} 2+2 & 3+2 \\ 6+4 & 9+4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 10 & 13 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix} \\ &= \mathbf{A}(\mathbf{B} + \mathbf{C}) \end{aligned}$$

$$\mathbf{b} \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix} = \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} + \mathbf{AC} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} + \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} \\ &= \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix} = \mathbf{A}(\mathbf{B} + \mathbf{C}) \end{aligned}$$

c Using the matrices in **a**,

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} & \mathbf{A}(\mathbf{BC}) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 6+3 & 9+3 \\ 14+7 & 21+7 \end{bmatrix} & &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2+1 & 3+1 \\ 2+1 & 3+1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix} & &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \\ & & &= \begin{bmatrix} 3+6 & 4+8 \\ 9+12 & 12+16 \end{bmatrix} \\ & & &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix} = (\mathbf{AB})\mathbf{C} \end{aligned}$$

d Using the matrices in **b**,

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix} \\ \mathbf{A}(\mathbf{BC}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left[\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right] \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{bmatrix} \\ &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix} = (\mathbf{AB})\mathbf{C} \end{aligned}$$

$$4 \quad \mathbf{a} \quad \text{If } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore$ equating coefficients of corresponding elements: $w = 1, \quad y = 0, \quad x = 0, \quad z = 1$

which checks with the coefficients in the second line. $\therefore w = z = 1, \quad \text{i.e., } \mathbf{X} \text{ is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 $x = y = 0$

b In **a** we showed that $\mathbf{AX} = \mathbf{A}$ where $\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore \mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \mathbf{A}^2 &= \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} & \mathbf{b} \quad \mathbf{A}^3 &= \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 4+3 & 2+(-2) \\ 6+(-6) & 3+4 \end{bmatrix} & & = \begin{bmatrix} 25+(-2) & -5+(-4) \\ 10+8 & -2+16 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} & & = \begin{bmatrix} 23 & -9 \\ 18 & 14 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \\
 & & & = \begin{bmatrix} 115+(-18) & -23+(-36) \\ 90+28 & -18+56 \end{bmatrix} \\
 & & & = \begin{bmatrix} 97 & -59 \\ 118 & 38 \end{bmatrix}
 \end{aligned}$$

$$\mathbf{6} \quad \mathbf{a} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \therefore \mathbf{A}^2 \text{ is } 3 \times 2 \text{ by } 3 \times 2$$

$2 \neq 3$ so $\mathbf{A}^2$ does not exist.

b We can square a matrix when the number of columns equals the number of rows, i.e., if it is a square matrix.

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{I}^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \\
 \mathbf{I}^3 &= \mathbf{I}\mathbf{I}^2 = \mathbf{I}\mathbf{I} = \mathbf{I}^2 = \mathbf{I}
 \end{aligned}$$

$$\begin{array}{lll}
 \mathbf{8} \quad \mathbf{a} & \mathbf{A}(\mathbf{A} + \mathbf{I}) & \mathbf{b} \quad (\mathbf{B} + 2\mathbf{I})\mathbf{B} & \mathbf{c} \quad \mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}) \\
 & = \mathbf{A}^2 + \mathbf{AI} & & = \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{AI} \\
 & = \mathbf{A}^2 + \mathbf{A} & & = \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} \\
 \mathbf{d} & \mathbf{A}(\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I}) & \mathbf{e} \quad (\mathbf{A} + \mathbf{B})(\mathbf{C} + \mathbf{D}) & \mathbf{f} \quad (\mathbf{A} + \mathbf{B})^2 \\
 & = \mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{AI} & & = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) \\
 & = \mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{A} & & = (\mathbf{A} + \mathbf{B})\mathbf{A} + (\mathbf{A} + \mathbf{B})\mathbf{B} \\
 & & & = \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2 \\
 \mathbf{g} & (\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) & \mathbf{h} \quad (\mathbf{A} + \mathbf{I})^2 & \mathbf{i} \quad (3\mathbf{I} - \mathbf{B})^2 \\
 & = (\mathbf{A} + \mathbf{B})\mathbf{A} - (\mathbf{A} + \mathbf{B})\mathbf{B} & & = (3\mathbf{I} - \mathbf{B})(3\mathbf{I} - \mathbf{B}) \\
 & = \mathbf{A}^2 + \mathbf{BA} - \mathbf{AB} - \mathbf{B}^2 & & = (3\mathbf{I} - \mathbf{B})3\mathbf{I} - (3\mathbf{I} - \mathbf{B})\mathbf{B} \\
 & & & = 9\mathbf{I}^2 - 3\mathbf{BI} - 3\mathbf{IB} + \mathbf{B}^2 \\
 & & & = 9\mathbf{I} - 3\mathbf{B} - 3\mathbf{B} + \mathbf{B}^2 \\
 & & & = 9\mathbf{I} - 6\mathbf{B} + \mathbf{B}^2
 \end{array}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad \mathbf{A}^2 &= 2\mathbf{A} - \mathbf{I} & \therefore \mathbf{A}^3 &= \mathbf{A} \times \mathbf{A}^2 & \text{and} \quad \mathbf{A}^4 &= \mathbf{A} \times \mathbf{A}^3 \\
 & & & = \mathbf{A}(2\mathbf{A} - \mathbf{I}) & & = \mathbf{A}(3\mathbf{A} - 2\mathbf{I}) \\
 & & & = 2\mathbf{A}^2 - \mathbf{AI} & & = 3\mathbf{A}^2 - 2\mathbf{AI} \\
 & & & = 2(2\mathbf{A} - \mathbf{I}) - \mathbf{A} & & = 3(2\mathbf{A} - \mathbf{I}) - 2\mathbf{A} \\
 & & & = 4\mathbf{A} - 2\mathbf{I} - \mathbf{A} & & = 6\mathbf{A} - 3\mathbf{I} - 2\mathbf{A} \\
 & & & = 3\mathbf{A} - 2\mathbf{I} & & = 4\mathbf{A} - 3\mathbf{I}
 \end{aligned}$$

b $B^2 = 2I - B$

$$\begin{array}{lll}
 \therefore B^3 = B \times B^2 & \text{and } B^4 = B \times B^3 & \text{and } B^5 = B \times B^4 \\
 = B(2I - B) & = B(3B - 2I) & = B(6I - 5B) \\
 = 2BI - B^2 & = 3B^2 - 2BI & = 6BI - 5B^2 \\
 = 2B - (2I - B) & = 3(2I - B) - 2B & = 6B - 5(2I - B) \\
 = 2B - 2I + B & = 6I - 3B - 2B & = 6B - 10I + 5B \\
 = 3B - 2I & = 6I - 5B & = 11B - 10I
 \end{array}$$

c $C^2 = 4C - 3I$ $C^3 = C \times C^2$ $C^5 = C^2 \times C^3$

$$\begin{array}{ll}
 = C(4C - 3I) & = (4C - 3I)(13C - 12I) \\
 = 4C^2 - 3CI & = (4C - 3I)13C - (4C - 3I)12I \\
 = 4(4C - 3I) - 3C & = 52C^2 - 39IC - 48CI + 36I^2 \\
 = 16C - 12I - 3C & = 52(4C - 3I) - 39C - 48C + 36I \\
 = 13C - 12I & = 208C - 156I - 87C + 36I \\
 & = 121C - 120I
 \end{array}$$

10 a If $A^2 = I$:

i $A(A + 2I)$
 $= A^2 + 2AI$
 $= I + 2A$

ii $(A - I)^2$
 $= (A - I)(A - I)$
 $= (A - I)A - (A - I)I$
 $= A^2 - IA - AI + I^2$
 $= I - A - A + I$
 $= 2I - 2A$

iii $A(A + 3I)^2$
 $= A(A + 3I)(A + 3I)$
 $= A[(A + 3I)A + (A + 3I)3I]$
 $= A[A^2 + 3IA + 3AI + 9I^2]$
 $= A[I + 3A + 3A + 9I]$
 $= A[10I + 6A]$
 $= 10AI + 6A^2$
 $= 10A + 6I$

b If $A^3 = I$, $A^2(A + I)^2 = A^2(A^2 + 2A + I)$
 $= A^4 + 2A^3 + A^2I$
 $= A(A^3) + 2A^3 + A^2I$
 $= AI + 2I + A^2$
 $= A^2 + A + 2I$

c If $A^2 = O$:

i $A(2A - 3I)$
 $= 2A^2 - 3AI$
 $= 2O - 3A$
 $= -3A$

iii $A(A + I)^3$
 $= A(A + I)(A + I)^2$
 $= A(A + I)(A^2 + 2A + I)$
 $= (A^2 + AI)(O + 2A + I)$
 $= (O + A)(2A + I)$
 $= 2A^2 + AI$
 $= 2O + A$
 $= A$

ii $A(A + 2I)(A - I)$
 $= A[(A + 2I)A - (A + 2I)I]$
 $= A(A^2 + 2IA - AI - 2I^2)$
 $= A(O + A - 2I)$
 $= A^2 - 2AI$
 $= O - 2A$
 $= -2A$

$$\begin{aligned}
 \mathbf{11} \quad \mathbf{a} \quad \mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
 &= \mathbf{O}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{A}^2 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{4} + \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \\ \frac{1}{4} + \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
 &= \mathbf{A}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \mathbf{A}^2 &= \mathbf{A} \\
 \therefore \mathbf{A}^2 - \mathbf{A} &= \mathbf{O} \\
 \therefore \mathbf{A}(\mathbf{A} - \mathbf{I}) &= \mathbf{O} \\
 \therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} &= \mathbf{O} \\
 \therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{I}
 \end{aligned}$$

The argument contains a false step. As the example in **a** illustrates, $\mathbf{AB} = \mathbf{O}$ does not imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.

This is a property of real numbers that does not hold for matrices. Therefore it is false to say that if $\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} - \mathbf{I}$ equals $\mathbf{O}$.

$$\begin{aligned}
 \mathbf{d} \quad \text{Let } \mathbf{A} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \text{If } \mathbf{A}^2 = \mathbf{A}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
 &\quad \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
 \end{aligned}$$

Equating corresponding elements:

$$a^2 + bc = a \quad \therefore \quad bc = a(1 - a) \quad \dots (1)$$

$$ab + bd = b \quad \therefore \quad b(a + d - 1) = 0 \quad \dots (2)$$

$$ac + cd = c \quad \therefore \quad c(a + d - 1) = 0 \quad \dots (3)$$

$$bc + d^2 = d \quad \therefore \quad bc = d(1 - d) \quad \dots (4)$$

If $a + d - 1 \neq 0$ then from (2) and (3), $b = c = 0$.

$\therefore$ from (1) and (4), $a = 0$ or 1 and $d = 0$ or 1

$\therefore a = 0, d = 0$ or $a = 0, d = 1$ or $a = 1, d = 0$ or $a = 1, d = 1$

where the last two cases are not possible as $a + d \neq 1$.

$$\text{So, if } a = 0, d = 0 \text{ then } \mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and if } a = 1, d = 1 \text{ then } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{If } a + d - 1 = 0 \text{ then } d = 1 - a \text{ and } c = \frac{a - a^2}{b}$$

$$\text{So } \mathbf{A} \text{ is } \begin{bmatrix} a & b \\ \frac{a - a^2}{b} & 1 - a \end{bmatrix} \text{ provided } b \neq 0$$

$$\begin{aligned}
 \mathbf{12} \quad \text{Choose } \mathbf{A} &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \text{ then } \mathbf{A}^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 + (-1) & -1 + 1 \\ 1 + (-1) & -1 + 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

$\therefore \mathbf{A}^2 = \mathbf{O}$, but $\mathbf{A} \neq \mathbf{O}$ so “if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$ ” is a false statement.

13 a Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = a \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 + (-2) & 2 + 4 \\ -1 + (-2) & -2 + 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ -a & 2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1 & 6 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} a + b & 2a \\ -a & 2a + b \end{bmatrix}$$

$$\therefore a + b = -1 \quad \text{and} \quad 2a = 6$$

$$\text{i.e., } a = 3 \quad \text{and} \quad b = -4$$

Checking for consistency: $-a = -3$, $2a + b = 6 + (-4) = 2 \quad \checkmark$

$$\therefore \mathbf{A}^2 = 3\mathbf{A} - 4\mathbf{I}$$

b Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} = a \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 9 + 2 & 3 + (-2) \\ 6 + (-4) & 2 + 4 \end{bmatrix} = \begin{bmatrix} 3a & a \\ 2a & -2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} 11 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3a + b & a \\ 2a & -2a + b \end{bmatrix}$$

$$\therefore 3a + b = 11 \quad \text{and} \quad a = 1$$

$$\therefore a = 1 \quad \text{and} \quad b = 8$$

Checking for consistency $2a = 2(1) = 2$, $-2a + b = -2(1) + 8 = 6 \quad \checkmark$

$$\therefore \mathbf{A}^2 = \mathbf{A} + 8\mathbf{I}$$

14 If $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = p \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} + q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 + (-2) & 2 + (-6) \\ -1 + 3 & -2 + 9 \end{bmatrix} = \begin{bmatrix} p & 2p \\ -p & -3p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\begin{bmatrix} -1 & -4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} p + q & 2p \\ -p & -3p + q \end{bmatrix}$$

$$\therefore p + q = -1 \quad \text{and} \quad 2p = -4$$

$$\therefore p = -2 \quad \text{and} \quad q = 1$$

Checking for consistency $-p = -(-2) = 2$, $-3p + q = -3(-2) + 1 = 7 \quad \checkmark$

$$\therefore \mathbf{A}^2 = -2\mathbf{A} + \mathbf{I}$$

a $\mathbf{A}^3 = \mathbf{A} \times \mathbf{A}^2$

$$\begin{aligned} &= \mathbf{A}(-2\mathbf{A} + \mathbf{I}) \\ &= -2\mathbf{A}^2 + \mathbf{A}\mathbf{I} \\ &= -2(-2\mathbf{A} + \mathbf{I}) + \mathbf{A} \\ &= 4\mathbf{A} - 2\mathbf{I} + \mathbf{A} \\ &= 5\mathbf{A} - 2\mathbf{I} \end{aligned}$$

b $\mathbf{A}^4 = \mathbf{A} \times \mathbf{A}^3$

$$\begin{aligned} &= \mathbf{A}(5\mathbf{A} - 2\mathbf{I}) \\ &= 5\mathbf{A}^2 - 2\mathbf{A}\mathbf{I} \\ &= 5(-2\mathbf{A} + \mathbf{I}) - 2\mathbf{A} \\ &= -10\mathbf{A} + 5\mathbf{I} - 2\mathbf{A} \\ &= -12\mathbf{A} + 5\mathbf{I} \end{aligned}$$

EXERCISE 14H

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\
 &= 3\mathbf{I} \\
 \therefore & \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \times \frac{1}{3} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} = \mathbf{I} \\
 \therefore & \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\
 &= 10\mathbf{I} \\
 \therefore & \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \times \frac{1}{10} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = \mathbf{I} \\
 \therefore & \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{10} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 0.2 & 0.4 \\ -0.1 & 0.3 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\mathbf{I} \\
 \therefore & \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix} = \mathbf{I} \\
 \text{and so} \quad & \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{bmatrix}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad |\mathbf{A}| = 12 - 14 = -2 \quad \mathbf{b} \quad |\mathbf{A}| = 2 - 3 = -1 \quad \mathbf{c} \quad |\mathbf{A}| = 0 - 0 = 0 \quad \mathbf{d} \quad |\mathbf{A}| = 1 - 0 = 1$$

$$\mathbf{3} \quad \mathbf{a} \quad \det \mathbf{B} = 12 - (-14) = 26 \quad \mathbf{b} \quad \det \mathbf{B} = 6 - 0 = 6 \quad \mathbf{c} \quad \det \mathbf{B} = 0 - 1 = -1 \quad \mathbf{d} \quad \det \mathbf{B} = a^2 - (-a) = a^2 + a$$

$$\mathbf{4} \quad \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{a} \quad |\mathbf{A}| = 2(-1) - (-1)(-1) = -3$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{A}^2 &= \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 4+1 & -2+1 \\ -2+1 & 1+1 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \\
 \therefore |\mathbf{A}^2| &= 5(2) - (-1)(-1) = 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad 2\mathbf{A} &= 2 \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & -2 \\ -2 & -2 \end{bmatrix} \\
 \therefore |2\mathbf{A}| &= 4(-2) - (-2)(-2) = -12
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \text{Let } \mathbf{A} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then } k\mathbf{A} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \quad \text{and } |k\mathbf{A}| = ka(kd) - kb(kc) \\
 &= k^2(ad - bc) \\
 &= k^2 |\mathbf{A}|
 \end{aligned}$$

$$6 \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$a \quad |\mathbf{A}| = ad - bc \\ \text{and } |\mathbf{B}| = wz - xy$$

$$b \quad \mathbf{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix} \\ \therefore |\mathbf{AB}| = (aw + by)(cx + dz) - (ax + bz)(cw + dy)$$

c Expanding brackets,

$$|\mathbf{AB}| = awcx + awdz + bycx + bydz - axcw - axdy - bzcw - bzdy \\ = wz(ad - bc) - xy(ad - bc) \\ = (ad - bc)(wz - xy) \\ = |\mathbf{A}| |\mathbf{B}|$$

$$7 \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$a \quad i \quad |\mathbf{A}| = 1(4) - 2(3) = 4 - 6 = -2 \quad ii \quad |2\mathbf{A}| = 2^2 |\mathbf{A}| = 4(-2) = -8 \quad iii \quad |-\mathbf{A}| = (-1)^2 |\mathbf{A}| = 1(-2) = -2$$

$$iv \quad |\mathbf{B}| = (-1)(1) - 2(0) = -1 \quad \therefore |-3\mathbf{B}| = (-3)^2 |\mathbf{B}| = 9(-1) = -9 \quad v \quad |\mathbf{AB}| = |\mathbf{A}| |\mathbf{B}| = (-2)(-1) = 2$$

b Checking:

$$ii \quad 2\mathbf{A} = 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad iii \quad |-\mathbf{A}| = \begin{vmatrix} -1 & -2 \\ -3 & -4 \end{vmatrix} \\ |2\mathbf{A}| = 2(8) - 4(6) = -8 \quad \checkmark \quad = (-1)(-4) - (-2)(-3) = -2 \quad \checkmark$$

$$iv \quad -3\mathbf{B} = -3 \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 0 & -3 \end{bmatrix} \quad v \quad \mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} -1+0 & 2+2 \\ -3+0 & 6+4 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -3 & 10 \end{bmatrix} \\ |-3\mathbf{B}| = (3)(-3) - (-6)(0) = -9 \quad \checkmark \quad |\mathbf{AB}| = (-1)(10) - 4(-3) = 2 \quad \checkmark$$

$$8 \quad a \quad \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{2(5) - 4(-1)} \begin{bmatrix} 5 & -4 \\ -(-1) & 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 5 & -4 \\ 1 & 2 \end{bmatrix}$$

$$b \quad \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{1(-1) - 0(1)} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = - \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$c \quad \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}^{-1} \text{ does not exist, since } ad - bc = 2(2) - 4(1) = 0$$

$$d \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{1(1) - 0(0)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e \quad \begin{bmatrix} 3 & 5 \\ -6 & -10 \end{bmatrix}^{-1} \text{ does not exist, since } ad - bc = 3(-10) - 5(-6) = 0$$

$$\mathbf{f} \quad \begin{bmatrix} -1 & 2 \\ 4 & 7 \end{bmatrix}^{-1} = \frac{1}{(-1)(7) - 2(4)} \begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix} = -\frac{1}{15} \begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix}$$

$$\mathbf{g} \quad \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{3(2) - (-1)(4)} \begin{bmatrix} 2 & -4 \\ -(-1) & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}$$

$$\mathbf{h} \quad \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{(-1)(3) - (-1)(2)} \begin{bmatrix} 3 & -(-1) \\ -2 & -1 \end{bmatrix} = - \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

EXERCISE 14I

$$\mathbf{1} \quad \mathbf{a} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix} \quad \mathbf{b} \quad \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a + 3b \\ a - 4b \end{bmatrix}$$

$$\mathbf{2} \quad \mathbf{a} \quad \left. \begin{array}{l} 3x - y = 8 \\ 2x + 3y = 6 \end{array} \right\} \text{ can be written as } \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$\mathbf{b} \quad \left. \begin{array}{l} 4x - 3y = 11 \\ 3x + 2y = -5 \end{array} \right\} \text{ can be written as } \begin{bmatrix} 4 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

$$\mathbf{c} \quad \left. \begin{array}{l} 3a - b = 6 \\ 2a + 7b = -4 \end{array} \right\} \text{ can be written as } \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$\mathbf{3} \quad \mathbf{a} \quad \begin{array}{l} 2x - y = 6 \\ x + 3y = 14 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 18 + 14 \\ -6 + 28 \end{bmatrix} = \begin{bmatrix} \frac{32}{7} \\ \frac{22}{7} \end{bmatrix} \quad \text{and so } x = \frac{32}{7}, \quad y = \frac{22}{7}$$

$$\mathbf{b} \quad \begin{array}{l} 5x - 4y = 5 \\ 2x + 3y = -13 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -13 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ -13 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 15 + (-52) \\ -10 + (-65) \end{bmatrix} = \begin{bmatrix} -\frac{37}{23} \\ -\frac{75}{23} \end{bmatrix} \quad \text{and so } x = -\frac{37}{23}, \quad y = -\frac{75}{23}$$

$$\mathbf{c} \quad \begin{array}{l} x - 2y = 7 \\ 5x + 3y = -2 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 21 + (-4) \\ -35 + (-2) \end{bmatrix} = \begin{bmatrix} \frac{17}{13} \\ -\frac{37}{13} \end{bmatrix} \quad \text{and so } x = \frac{17}{13}, \quad y = -\frac{37}{13}$$

$$\mathbf{d} \quad \begin{array}{l} 3x + 5y = 4 \\ 2x - y = 11 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \frac{1}{-13} \begin{bmatrix} -1 & -5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} -4 + (-55) \\ -8 + 33 \end{bmatrix} = \begin{bmatrix} \frac{59}{13} \\ -\frac{25}{13} \end{bmatrix} \quad \text{and so } x = \frac{59}{13}, \quad y = -\frac{25}{13}$$

e $4x - 7y = 8$ In matrix form, the system is: $\begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$
 $3x - 5y = 0$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -5 & 7 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -40 + 0 \\ -24 + 0 \end{bmatrix} \quad \text{and so } x = -40, \quad y = -24$$

f $7x + 11y = 18$ In matrix form, the system is: $\begin{bmatrix} 7 & 11 \\ 11 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ -11 \end{bmatrix}$
 $11x - 7y = -11$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 11 & -7 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ -11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-170} \begin{bmatrix} -7 & -11 \\ -11 & 7 \end{bmatrix} \begin{bmatrix} 18 \\ -11 \end{bmatrix} = -\frac{1}{170} \begin{bmatrix} -126 + 121 \\ -198 - 77 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{170} \begin{bmatrix} -5 \\ -275 \end{bmatrix} = \begin{bmatrix} \frac{1}{34} \\ \frac{55}{34} \end{bmatrix} \quad \text{and so } x = \frac{1}{34}, \quad y = \frac{55}{34}$$

4 a If $\mathbf{AX} = \mathbf{B}$

then $\mathbf{A}^{-1}\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ {pre-mult by $\mathbf{A}^{-1}$ }

$$\therefore \mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

If $\mathbf{XA} = \mathbf{B}$

then $\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$ {post-mult by $\mathbf{A}^{-1}$ }

$$\therefore \mathbf{XI} = \mathbf{BA}^{-1}$$

$$\therefore \mathbf{X} = \mathbf{BA}^{-1}$$

b i $\mathbf{X} \begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$

$$\therefore \mathbf{X} = \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix} \frac{1}{-11} \begin{bmatrix} -1 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= -\frac{1}{11} \begin{bmatrix} -14 + 25 & -28 - 5 \\ -22 + 0 & -44 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= \frac{1}{-7} \begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= -\frac{1}{7} \begin{bmatrix} -1 + (-12) & 3 + (-6) \\ -2 + 4 & 6 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13}{7} & \frac{3}{7} \\ -\frac{2}{7} & -\frac{8}{7} \end{bmatrix}$$

5 a $\mathbf{A} = \begin{bmatrix} k & 1 \\ -6 & 2 \end{bmatrix}$ $\therefore \mathbf{A}^{-1} = \frac{1}{2k+6} \begin{bmatrix} 2 & -1 \\ 6 & k \end{bmatrix}$, provided that $2k+6 \neq 0$,
i.e., $k \neq -3$

b $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 0 & k \end{bmatrix}$ $\therefore \mathbf{A}^{-1} = \frac{1}{3k} \begin{bmatrix} k & 1 \\ 0 & 3 \end{bmatrix}$, provided that $k \neq 0$

$$\begin{aligned} \mathbf{c} \quad \mathbf{A} &= \begin{bmatrix} k+1 & 2 \\ 1 & k \end{bmatrix} \quad \therefore \mathbf{A}^{-1} = \frac{1}{k(k+1)-2} \begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix}, \\ &= \frac{1}{(k+2)(k-1)} \begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix}, \quad \text{provided that } k \neq -2 \text{ or } 1 \end{aligned}$$

$$6 \quad \mathbf{a} \quad \mathbf{A} \text{ is } 2 \times 3 \text{ and } \mathbf{B} \text{ is } 3 \times 2 \quad \therefore \mathbf{AB} \text{ is } 2 \times 2$$

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -4 & 6 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+0+2 & 2+0+(-2) \\ 1+(-4)+3 & -2+6+(-3) \end{bmatrix} \\ \therefore \mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \end{aligned}$$

b $\mathbf{B}$ is 3×2 and $\mathbf{A}$ is 2×3 $\therefore \mathbf{BA}$ is 3×3 whereas $\mathbf{AB}$ is 2×2 . Hence $\mathbf{BA} \neq \mathbf{AB}$, and $\mathbf{A}$ and $\mathbf{B}$ are not inverses. The inverse of a matrix $\mathbf{A}$ satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1}$. This requires that $\mathbf{A}$ has the same number of rows as columns, i.e., that $\mathbf{A}$ is a square matrix.

$$7 \quad \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

Since $\mathbf{AXB} = \mathbf{C}$,

and

$$\text{then } \mathbf{A}^{-1}\mathbf{AXB} = \mathbf{A}^{-1}\mathbf{C} \quad \{\text{premult. by } \mathbf{A}^{-1}\} \quad \mathbf{XBB}^{-1} = \mathbf{A}^{-1}\mathbf{CB}^{-1} \quad \{\text{postmult. by } \mathbf{B}^{-1}\}$$

$$\therefore \mathbf{IXB} = \mathbf{A}^{-1}\mathbf{C}$$

$$\therefore \mathbf{XI} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$$

$$\therefore \mathbf{XB} = \mathbf{A}^{-1}\mathbf{C}$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$$

$$\therefore \mathbf{X} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0+(-1) & 3+(-2) \\ 0+2 & 0+4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0+1 & 2+1 \\ 0+4 & -4+4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 1 & 0 \end{bmatrix}$$

$$8 \quad \mathbf{a} \quad \mathbf{i} \quad \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$$\text{and } |\mathbf{A}| = -2 - (-12)$$

$$= -2 + 12$$

$$= 10$$

ii As $|\mathbf{A}| \neq 0$, the system has a unique solution

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 25 \\ -10 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{2} \\ -1 \end{bmatrix} \end{aligned}$$

$$\therefore x = \frac{5}{2}, \quad y = -1$$

$$\mathbf{b} \quad \mathbf{i} \quad \begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$$\text{and } |\mathbf{A}| = -2 - 4k$$

ii The system has a unique solution if

$$-2 - 4k \neq 0 \quad \text{i.e., } k \neq -\frac{1}{2}$$

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{-2-4k} \begin{bmatrix} -1 & -k \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{-2-4k} \begin{bmatrix} -8-11k \\ -10 \end{bmatrix} \\ \therefore x &= \frac{8+11k}{2+4k}, \quad y = \frac{5}{1+2k}, \quad k \neq -\frac{1}{2} \end{aligned}$$

is the unique solution

iii When $k = -\frac{1}{2}$, the equations are

$$\begin{cases} 2x - \frac{1}{2}y = 8 \\ 4x - y = 11 \end{cases} \quad \text{i.e., } \begin{cases} 4x - y = 16 \\ 4x - y = 11 \end{cases}$$

So, we have no solutions (as the lines are parallel and so do not meet).

9 a If $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

b If $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is its own inverse, then $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & b^2 + a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{lll} \therefore a^2 + b^2 = 1 & \text{then } a = 0 & \text{or } b = 0 \\ \text{and } 2ab = 0 & \text{and } b^2 = 1 & \text{and } a^2 = 1 \\ \text{If } 2ab = 0, & \therefore a = 0 & \text{and } b = \pm 1 \quad \therefore a = \pm 1 \quad \text{and } b = 0 \end{array}$$

This gives four possible combinations: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

10 a $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \therefore \mathbf{A}^{-1} = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$\therefore (\mathbf{A}^{-1})^{-1} = \frac{1}{\frac{1}{2}} \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \mathbf{A}$$

b If $\mathbf{A}^{-1} = \mathbf{B}$

then $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1}) = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$

and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1} = \mathbf{B}\mathbf{B}^{-1} = \mathbf{I}$

c We can deduce from **b** that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
and $(\mathbf{A}^{-1})^{-1}$ is the inverse of $\mathbf{A}^{-1}$.

11 a $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$

i $\mathbf{A}^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ **ii** $\mathbf{B}^{-1} = \frac{1}{-2} \begin{bmatrix} -3 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

iii $(\mathbf{AB})^{-1}$

$$\begin{aligned} &= \left(\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 0+2 & 1+(-3) \\ 0+(-2) & 2+3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{i.e., } = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

iv $(\mathbf{BA})^{-1}$

$$\begin{aligned} &= \left(\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 0+2 & 0+(-1) \\ 2+(-6) & 2+3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{i.e., } = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

v $\mathbf{A}^{-1}\mathbf{B}^{-1}$

$$\begin{aligned} &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{6} + \frac{1}{3} & \frac{1}{6} + 0 \\ \frac{6}{6} + (-\frac{1}{3}) & \frac{2}{6} + 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

vi $\mathbf{B}^{-1}\mathbf{A}^{-1}$

$$\begin{aligned} &= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{6} + \frac{2}{6} & \frac{3}{6} + (-\frac{1}{6}) \\ \frac{1}{3} + 0 & \frac{1}{3} + 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

b Choose appropriate vectors and repeat question **a**.

c The results of **a** and **b** suggest that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

$$\begin{array}{ll} \mathbf{d} & (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) \quad \text{and} \quad (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) \\ & = (\mathbf{AB})(\mathbf{AB})^{-1} \quad \{\text{from c}\} \quad = (\mathbf{AB})^{-1}(\mathbf{AB}) \quad \{\text{from c}\} \\ & = \mathbf{I} \quad \quad \quad = \mathbf{I} \end{array}$$

$\therefore (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$ i.e., $\mathbf{AB}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$ are inverses.

$$\mathbf{12} \quad (k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = k \times \frac{1}{k}(\mathbf{AA}^{-1}) = \mathbf{I} \quad \text{also} \quad \left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \frac{1}{k} \times k(\mathbf{A}^{-1}\mathbf{A}) = \mathbf{I}$$

$\therefore (k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = \left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \mathbf{I}$ i.e., $k\mathbf{A}$ and $\frac{1}{k}\mathbf{A}^{-1}$ are inverses.

13 $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$

a $\mathbf{X} = \mathbf{AY} = \mathbf{A}(\mathbf{BZ}) = \mathbf{ABZ}$

b $(\mathbf{AB})^{-1}\mathbf{X} = (\mathbf{AB})^{-1}\mathbf{ABZ} \quad \{\text{premult by } (\mathbf{AB})^{-1}\}$

$(\mathbf{AB})^{-1}\mathbf{X} = \mathbf{IZ}$

$\therefore \mathbf{Z} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X} \quad \{\text{as } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}\}$

14 In each example we premultiply by $\mathbf{A}^{-1}$.

a $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$

$\therefore \mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}(4\mathbf{A} - \mathbf{I})$

$\therefore \mathbf{A}^{-1}\mathbf{AA} = 4\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I}$

$\therefore \mathbf{IA} = 4\mathbf{I} - \mathbf{A}^{-1}$

$\therefore \mathbf{A} - 4\mathbf{I} = -\mathbf{A}^{-1}$

$\therefore \mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$

b $5\mathbf{A} = \mathbf{I} - \mathbf{A}^2$

$\therefore \mathbf{A}^{-1}5\mathbf{A} = \mathbf{A}^{-1}(\mathbf{I} - \mathbf{A}^2)$

$\therefore 5\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{-1}\mathbf{I} - \mathbf{A}^{-1}\mathbf{AA}$

$\therefore 5\mathbf{I} = \mathbf{A}^{-1} - \mathbf{IA}$

$\therefore 5\mathbf{I} = \mathbf{A}^{-1} - \mathbf{A}$

$\therefore \mathbf{A}^{-1} = 5\mathbf{I} + \mathbf{A}$

c $2\mathbf{I} = 3\mathbf{A}^2 - 4\mathbf{A}$

$\therefore \mathbf{A}^{-1}2\mathbf{I} = \mathbf{A}^{-1}3\mathbf{A}^2 - \mathbf{A}^{-1}4\mathbf{A}$

$\therefore 2\mathbf{A}^{-1} = 3\mathbf{A}^{-1}\mathbf{AA} - 4\mathbf{A}^{-1}\mathbf{A}$

$\therefore 2\mathbf{A}^{-1} = 3\mathbf{IA} - 4\mathbf{I}$

$\therefore 2\mathbf{A}^{-1} = 3\mathbf{A} - 4\mathbf{I}$

$\therefore \mathbf{A}^{-1} = \frac{3}{2}\mathbf{A} - 2\mathbf{I}$

15 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$, let $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

$$\therefore \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} = p \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} + q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 9 + (-4) & 6 + (-2) \\ -6 + 2 & -4 + 1 \end{bmatrix} = \begin{bmatrix} 3p & 2p \\ -2p & -p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 3p + q & 2p \\ -2p & -p + q \end{bmatrix}$$

$\therefore 5 = 3p + q \quad \text{and} \quad 4 = 2p$

$\therefore p = 2 \quad \text{and} \quad q = -1 \quad \text{and} \quad \mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$

Checking for consistency:

$-2p = -2(2) = -4 \quad \checkmark$

$-p + q = -2 + (-1) = -3 \quad \checkmark$

$$\text{Now } \mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$$

$$\therefore \mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}2\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \quad \{\text{premultiplying by } \mathbf{A}^{-1}\}$$

$$\therefore \mathbf{A}^{-1}\mathbf{A}\mathbf{A} = 2\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}$$

$$\therefore \mathbf{IA} = 2\mathbf{I} - \mathbf{A}^{-1}$$

$$\therefore \mathbf{A} = 2\mathbf{I} - \mathbf{A}^{-1}$$

$$\therefore \mathbf{A}^{-1} = 2\mathbf{I} - \mathbf{A}$$

16 If $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$,

$$\text{then } \mathbf{A}^2 = \mathbf{AA}$$

$$= (\mathbf{AB})\mathbf{A}$$

$$= \mathbf{A}(\mathbf{BA}) \quad \{\text{associative rule}\}$$

$$= \mathbf{AB}$$

$$\therefore \mathbf{A}^2 = \mathbf{A}$$

$ab = ac$ implies that $b = c$ for real numbers, but this property does not hold for matrices.

Thus from $\mathbf{AB} = \mathbf{AI} = \mathbf{A}$ it does not follow that $\mathbf{B} = \mathbf{I}$.

17 If $\mathbf{AB} = \mathbf{AC}$

$$\text{then } \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \quad \{\text{premultiplying by } \mathbf{A}^{-1}\}$$

$$\therefore \mathbf{IB} = \mathbf{IC}$$

$$\therefore \mathbf{B} = \mathbf{C}$$

i.e., if $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$ if $\mathbf{A}^{-1}$ exists, i.e., if $ad - bc \neq 0$.

18 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$

$$\text{then } \mathbf{X}^3 = (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})$$

$$= (\mathbf{P}^{-1}\mathbf{A})(\mathbf{PP}^{-1})\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} \quad \{\text{associative rule}\}$$

$$= \mathbf{P}^{-1}\mathbf{AIAIAP}$$

$$= \mathbf{P}^{-1}\mathbf{AAAP}$$

$$= \mathbf{P}^{-1}\mathbf{A}^3\mathbf{P}$$

$$= \mathbf{P}^{-1}\mathbf{IP}$$

$$= \mathbf{P}^{-1}\mathbf{P} \quad \text{and so } \mathbf{X}^3 = \mathbf{I}$$

19 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$

$$\text{then } a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = a(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}$$

$$= a\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}$$

$$= a\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}$$

$$= \mathbf{P}^{-1}(a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I})\mathbf{P}$$

$$= \mathbf{P}^{-1}\mathbf{OP}$$

$$= \mathbf{O}$$

EXERCISE 14J

1 a

$$\begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 0 & 5 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} + 0 \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} \\ = 2(10 - 0) + 3(2 - 5) + 0 \\ = 41$$

b

$$\begin{vmatrix} -1 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix} = -1 \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + (-3) \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} \\ = -1(0 - 0) + 2(0 - 1) - 3(2 - 0) \\ = -8$$

$$\begin{aligned}
 \mathbf{c} \quad \begin{vmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \\
 &= 2(3-2) + 1(4-3) + 3(-1-2) \\
 &= 0
 \end{aligned}$$

$$\mathbf{d} \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} + 0 \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 1(6-0) = 6$$

$$\mathbf{e} \quad \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = 2(0-3) = -6$$

$$\begin{aligned}
 \mathbf{f} \quad \begin{vmatrix} 4 & 1 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix} &= 4 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} \\
 &= 4(0-2) + 1(-2-1) + 3(-1-0) \\
 &= -12
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad \begin{vmatrix} 3 & -1 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & 3 \end{vmatrix} &= 3 \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} + (-2) \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \\
 &= 3(3-1) - 1(-1-0) - 2(0-1) \\
 &= 11
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad \begin{vmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 2 & 6 & 4 \end{vmatrix} &= 1 \begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 \\ 2 & 6 \end{vmatrix} \\
 &= 1(8-6) + 3(2-4) + 2(-6-4) \\
 &= 0
 \end{aligned}$$

$$\mathbf{i} \quad \begin{vmatrix} 0 & 3 & 0 \\ 1 & 2 & 5 \\ 6 & 0 & 1 \end{vmatrix} = 0 \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 5 & 1 \\ 1 & 6 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix} = 3(30-1) = 87$$

$$\mathbf{2} \quad \mathbf{a} \quad \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = a \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ c & 0 \end{vmatrix} + 0 \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} = a(bc-0) = abc$$

$$\begin{aligned}
 \mathbf{b} \quad \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} &= 0 \begin{vmatrix} 0 & z \\ -z & 0 \end{vmatrix} + x \begin{vmatrix} z & -x \\ 0 & -y \end{vmatrix} + y \begin{vmatrix} -x & 0 \\ -y & -z \end{vmatrix} \\
 &= x(-zy-0) + y(xz-0) \\
 &= -xyz + xyz \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} &= a \begin{vmatrix} c & a \\ a & b \end{vmatrix} + b \begin{vmatrix} a & b \\ b & c \end{vmatrix} + c \begin{vmatrix} b & c \\ c & a \end{vmatrix} \\
 &= a(cb-a^2) + b(ac-b^2) + c(ba-c^2) \\
 &= abc - a^3 + abc - b^3 + abc - c^3 \\
 &= 3abc - a^3 - b^3 - c^3
 \end{aligned}$$

$$\mathbf{3} \quad \begin{cases} x + 2y - 3z = 5 \\ 2x - y - z = 8 \\ kx + y + 2z = 14 \end{cases} \quad \text{has matrix equation} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 14 \end{bmatrix}$$

$\mathbf{A} \qquad \qquad \mathbf{X} \qquad \qquad \mathbf{B}$

$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ has a unique solution if $|\mathbf{A}| \neq 0$.

$$\begin{aligned} \text{Now } \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{vmatrix} &= 1 \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 \\ 2 & k \end{vmatrix} + (-3) \begin{vmatrix} 2 & -1 \\ k & 1 \end{vmatrix} \\ &= 1(-2 - -1) + 2(-k - 4) - 3(2 - -k) \\ &= -1 - 2k - 8 - 6 - 3k \\ &= -15 - 5k \quad \text{i.e., a unique solution for all } k \text{ provided } k \neq -3. \end{aligned}$$

$$\mathbf{4} \quad \begin{cases} 2x - y - 4z = 8 \\ 3x - ky + z = 1 \\ 5x - y + kz = -2 \end{cases} \quad \text{has matrix equation} \quad \begin{bmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix}$$

$\mathbf{A} \qquad \qquad \mathbf{X} \qquad \qquad \mathbf{B}$

This has a unique solution if $|\mathbf{A}| \neq 0$.

$$\begin{aligned} \text{Now } \begin{vmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{vmatrix} &= 2 \begin{vmatrix} -k & 1 \\ -1 & k \end{vmatrix} + (-1) \begin{vmatrix} 1 & 3 \\ k & 5 \end{vmatrix} + (-4) \begin{vmatrix} 3 & -k \\ 5 & -1 \end{vmatrix} \\ &= 2(-k^2 - -1) - 1(5 - 3k) - 4(-3 - -5k) \\ &= -2k^2 + 2 - 5 + 3k + 12 - 20k \\ &= -2k^2 - 17k + 9 \\ &= -(2k^2 + 17k - 9) \\ &= -(2k - 1)(k + 9) \end{aligned}$$

$\therefore$ there is a unique solution for all k provided $k \neq \frac{1}{2}$ and $k \neq -9$.

5 a

$$\begin{vmatrix} 1 & k & 3 \\ k & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 7$$

$$\therefore 1 \begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} + k \begin{vmatrix} -1 & k \\ 2 & 3 \end{vmatrix} + 3 \begin{vmatrix} k & 1 \\ 3 & 4 \end{vmatrix} = 7$$

$$\therefore 1(2 - -4) + k(-3 - 2k) + 3(4k - 3) = 7$$

$$\therefore 6 - 3k - 2k^2 + 12k - 9 = 7$$

$$\therefore 2k^2 - 9k + 10 = 0$$

$$\therefore (2k - 5)(k - 2) = 0 \quad \text{and so } k = \frac{5}{2} \text{ or } 2$$

b

$$\begin{vmatrix} k & 2 & 1 \\ 2 & k & 2 \\ 1 & 2 & k \end{vmatrix} = 0$$

$$\therefore k \begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ k & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & k \\ 1 & 2 \end{vmatrix} = 0$$

$$\therefore k(k^2 - 4) + 2(2 - 2k) + (4 - k) = 0$$

$$\therefore k^3 - 4k + 4 - 4k + 4 - k = 0$$

$$\therefore k^3 - 9k + 8 = 0$$

Using technology there is one rational zero, $k = 1$

$$\therefore (k - 1)(k^2 + k - 8) = 0$$

$$\therefore k = 1 \quad \text{or} \quad k = \frac{-1 \pm \sqrt{1 - 4(1)(-8)}}{2} = \frac{-1 \pm \sqrt{33}}{2}$$

6 Using technology,

$$\mathbf{a} \quad \begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{vmatrix} = 16$$

$$\mathbf{b} \quad \begin{vmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{vmatrix} = -34$$

7 a Using Excel, $\det = -17$ and

$$\text{inverse} = \begin{bmatrix} -\frac{5}{17} & \frac{2}{17} & \frac{14}{17} & -\frac{1}{17} \\ -\frac{11}{17} & \frac{1}{17} & \frac{7}{17} & \frac{8}{17} \\ -\frac{3}{17} & \frac{8}{17} & \frac{5}{17} & -\frac{4}{17} \\ \frac{10}{17} & -\frac{4}{17} & -\frac{11}{17} & \frac{2}{17} \end{bmatrix}$$

b Using Excel, $\det = 2$ and

$$\text{inverse} = \begin{bmatrix} \frac{7}{2} & -4 & -\frac{5}{2} & \frac{11}{2} & 0 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 2 & 0 \\ -\frac{1}{2} & 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ -3 & 3 & 2 & -4 & 1 \end{bmatrix}$$

8 a Let o, a, p, c, l represent (respectively) the cost (in dollars) of each orange, apple, pear, cabbage and lettuce. The system, in matrix form, becomes:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} o \\ a \\ p \\ c \\ l \end{bmatrix} = \begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 10.9 \end{bmatrix}$$

$\mathbf{A} \qquad \qquad \mathbf{X} \qquad \qquad \mathbf{B}$

b $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

Using technology, $|\mathbf{A}| = 0$,
 $\therefore \mathbf{A}^{-1}$ does not exist and $\mathbf{X}$
 cannot be found using this
 information.

c If the last line is amended, the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \det \mathbf{A} = 6$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 9.2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.7 \\ 2 \\ 1.5 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ oranges cost 50 cents, apples cost 80 cents, pears cost 70 cents, cabbages cost \$2.00, and
 lettuces cost \$1.50

EXERCISE 14K

$$\mathbf{1} \quad \mathbf{a} \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{bmatrix}$$

$$\mathbf{b} \quad \mathbf{A}^{-1} = \begin{bmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{bmatrix}$$

$$\mathbf{2} \quad \mathbf{a} \quad \mathbf{B}^{-1} = \begin{bmatrix} 0.050\,23 & -0.011\,48 & -0.066\,34 \\ 4.212 \times 10^{-4} & 0.013\,53 & 0.027\,75 \\ -0.029\,90 & 0.039\,33 & 0.030\,06 \end{bmatrix} \div \begin{bmatrix} 0.050 & -0.011 & -0.066 \\ 0.000 & 0.014 & 0.028 \\ -0.030 & 0.039 & 0.030 \end{bmatrix}$$

$$\mathbf{b} \quad \mathbf{B}^{-1} = \begin{bmatrix} 1.596 & -0.9964 & -0.1686 \\ -3.224 & 1.925 & 0.6291 \\ 2.000 & -1.086 & -0.3958 \end{bmatrix} \div \begin{bmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2.000 & -1.086 & -0.396 \end{bmatrix}$$

EXERCISE 14L

$$1 \quad a \quad \left. \begin{array}{l} x - y - z = 2 \\ x + y + 3z = 7 \\ 9x - y - 3z = -1 \end{array} \right\} \text{ has matrix equation } \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$$

$$b \quad \left. \begin{array}{l} 2x + y - z = 3 \\ y + 2z = 6 \\ x - y + z = 13 \end{array} \right\} \text{ has matrix equation } \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 13 \end{bmatrix}$$

$$c \quad \left. \begin{array}{l} a + b - c = 7 \\ a - b + c = 6 \\ 2a + b - 3c = -2 \end{array} \right\} \text{ has matrix equation } \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -2 \end{bmatrix}$$

$$2 \quad \mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-1+0 & 1-1+0 & 1-1+0 \\ -2+0+2 & -1+0+2 & -1+0+1 \\ 0+2-2 & 0+2-2 & 0+2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and so } \mathbf{AB} = \mathbf{I}$$

$$\mathbf{BA} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2-1+0 & -2+0+2 & 0+1-1 \\ 1-1+0 & -1+0+2 & 0+1-1 \\ 2-2+0 & -2+0+2 & 0+2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \mathbf{BA} = \mathbf{I}$$

i.e., $\mathbf{AB} = \mathbf{BA} = \mathbf{I} \quad \therefore \mathbf{A}$ and $\mathbf{B}$ are inverses of each other.

$$3 \quad \mathbf{AB} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 8-1-6 & 14-2-12 & -6+1+5 \\ -4-2+6 & -7-4+12 & 3+2-5 \\ 0-6+6 & 0-12+12 & 0+6-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \mathbf{AB} = \mathbf{I} \quad \text{and so } \mathbf{A} = \mathbf{B}^{-1}$$

$$\left. \begin{array}{l} 4a + 7b - 3c = -8 \\ -a - 2b + c = 3 \\ 6a + 12b - 5c = -15 \end{array} \right\} \text{ has matrix equation } \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -16+3+15 \\ 8+6-15 \\ 0+18-15 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \therefore a = 2, \quad b = -1, \quad c = 3$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{MN} &= \begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 15+3-14 & 10-3-7 & 15+6-21 \\ -3-3+6 & -2+3+3 & -3-6+9 \\ -9-1+10 & -6+1+5 & -9-2+15 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \therefore \mathbf{MN} = 4\mathbf{I} \\
 &\quad \therefore \left(\frac{1}{4}\mathbf{M}\right)\mathbf{N} = \mathbf{I} \\
 &\quad \therefore \frac{1}{4}\mathbf{M} = \mathbf{N}^{-1}
 \end{aligned}$$

$$\text{Now } \left. \begin{aligned} 3u + 2v + 3w &= 18 \\ u - v + 2w &= 6 \\ 2u + v + 3w &= 16 \end{aligned} \right\} \text{ has matrix equation } \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 90+18-112 \\ -18-18+48 \\ -54-6+80 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 12 \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \quad \therefore u = -1, \quad v = 3, \quad w = 5$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix} \\
 \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix}
 \end{aligned}$$

$$\text{Using technology, } x = \frac{23}{10}, \quad y = \frac{13}{10}, \quad z = -\frac{9}{2}$$

$$\begin{aligned}
 \mathbf{b} \quad \begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix} \\
 \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}
 \end{aligned}$$

$$\text{Using technology, } x = -\frac{1}{3}, \quad y = -\frac{95}{21}, \quad z = \frac{2}{21}$$

$$\begin{aligned}
 \mathbf{c} \quad \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix} \\
 \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix}
 \end{aligned}$$

$$\text{Using technology, } x = 2, \quad y = 4, \quad z = -1$$

- 6** **a** $x = 2, y = -1, z = 5$ **b** $x = 4, y = -2, z = 1$
c $x = 4, y = -3, z = 2$ **d** $x = 4, y = 6, z = -7$
e $x = 3, y = 11, z = -7$ **f** $x = 0.326, y = 7.652, z = 4.156$

- 7** **a** Using technology, $x = 14, y = 11, z = 17$.

b Let x be the cost of a cricket ball in dollars,
 y be the cost of softball in dollars, and
 z be the cost of a netball in dollars.

- c** Cost of 4 cricket balls and 5 softballs is $4x + 5y = 4(14) + 5(11) = 111$, i.e., \$111
 $\therefore$ amount left for netballs is $\$315 - \$111 = \$204$

$$\text{Number of netballs bought} = \frac{204}{17} = 12 \quad \text{i.e., 12 netballs are bought.}$$

- 8** **a** System of equations is: $2x + 3y + 8z = 352$
 $x + 5y + 4z = 274$
 $x + 2y + 11z = 351$

b Using technology, $x = 42, y = 28, z = 23$,
i.e., the salaries are: manager \$42 000, clerk \$28 000 and labourer \$23 000.

- c** Salary bill is $3x + 8y + 37z$
 $= 3(42) + 8(28) + 37(23)$
 $= 1201$ (thousands of dollars)
i.e., \$1 201 000

- 9** Let the quadratic function have form $f(x) = ax^2 + bx + c$.

$(-2, -15)$ is a point on the curve $\therefore 4a - 2b + c = -15$

$(1, -3)$ is a point on the curve $\therefore a + b + c = -3$

$(3, -5)$ is a point on the curve $\therefore 9a + 3b + c = -5$

Using technology to solve these equations, $a = -1, b = 3$ and $c = -5$.

$$\therefore f(x) = -x^2 + 3x - 5$$

$$\text{Now } f(-3) = -(-3)^2 + 3(-3) - 5 = -23$$

$$f(-1) = -(-1)^2 + 3(-1) - 5 = -9$$

$$f(0) = -5$$

$$f(2) = -2^2 + 3(2) - 5 = -3$$

The table is

x	-3	-2	-1	0	1	2	3
y	-23	-15	-9	-5	-3	-3	-5

- 10** Let x be the cost in dollars of 1 kg of cashews,
 y be the cost in dollars of 1 kg of macadamias, and
 z be the cost in dollars of 1 kg of Brazil nuts.

The cost of 1 kg of mix A is $0.5x + 0.3y + 0.2z = 12.5$,

the cost of 1 kg of mix B is $0.2x + 0.4y + 0.4z = 12.4$,

the cost of 1 kg of mix C is $0.6x + 0.1y + 0.3z = 11.7$.

Using technology, $x = 12, y = 15$ and $z = 10$

i.e., the cost of 1 kg of cashews is \$12, the cost of 1 kg of macadamias is \$15 and the cost of 1 kg of Brazil nuts is \$10.

$$\begin{aligned}
 &\text{Cost per kg of 400 g cashews, 200 g macadamias and 400 g Brazil nuts} \\
 &= 0.4 \times 12 + 0.2 \times 15 + 0.4 \times 10 \quad \text{dollars} \\
 &= \$11.80
 \end{aligned}$$

11 a Number of students who study Chemistry is $\frac{1}{3}p + \frac{1}{3}q + \frac{2}{5}r = 27 \quad \dots (1)$

number of students who study Maths is $\frac{1}{2}p + \frac{2}{3}q + \frac{1}{5}r = 35 \quad \dots (2)$

number of students who study Geography is $\frac{1}{4}p + \frac{1}{3}q + \frac{3}{5}r = 30 \quad \dots (3)$

The required system of equations is $5p + 5q + 6r = 405 \quad \{(1) \times 15\}$

$$15p + 20q + 6r = 1050 \quad \{(2) \times 30\}$$

$$15p + 20q + 36r = 1800 \quad \{(3) \times 60\}$$

b Using technology, $p = 24$, $q = 27$, $r = 25$.

12 a As t is the number of years after 2000, then

profit in year 2000 is $P(0) = b + \frac{c}{4} = 160\,000$

profit in year 2001 is $P(1) = a + b + \frac{c}{5} = 198\,000$

profit in year 2002 is $P(2) = 2a + b + \frac{c}{6} = 240\,000$

Using technology, $a = 50\,000$, $b = 100\,000$ and $c = 240\,000$.

b Using the model given, the profit in 1999 would be

$$P(-1) = -a + b + \frac{c}{3} = 50\,000 + 100\,000 + 80\,000 = 130\,000$$

i.e., the profit would be \$130 000, which fits the model.

c Predicted profit in 2003 is $P(3) = 3a + b + \frac{c}{7} = 3(50\,000) + 100\,000 + \frac{240\,000}{7}$
 $\div \$284\,000$

Predicted profit in 2005 is $P(5) = 5a + b + \frac{c}{9} = 5(50\,000) + 100\,000 + \frac{240\,000}{9}$
 $\div \$377\,000$

REVIEW SET 14A

1 a $A + B$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

b $3A$

$$= 3 \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 \\ 0 & -3 \end{bmatrix}$$

c $-2B$

$$= -2 \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 4 & -8 \end{bmatrix}$$

d $A - B$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$$

e $B - 2A$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 \\ -2 & 6 \end{bmatrix}$$

f $3A - 2B$

$$= 3 \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 4 & -11 \end{bmatrix}$$

g AB

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix}$$

h BA

$$= \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -6 & -8 \end{bmatrix}$$

i A^{-1}

$$= \frac{1}{-3} \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{bmatrix}$$

j A^2

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 4 \\ 0 & 1 \end{bmatrix}$$

k ABA

$$= (AB)A$$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -10 \\ 6 & 8 \end{bmatrix}$$

l $(AB)^{-1}$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix}^{-1}$$

$$= \frac{1}{4 - 16} \begin{bmatrix} -4 & -8 \\ -2 & -1 \end{bmatrix}$$

$$= \frac{1}{-12} \begin{bmatrix} -4 & -8 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{bmatrix}$$

2 a Equating corresponding elements,

$$\begin{aligned} a &= -a \\ b - 2 &= 3 \\ c &= 2 - c & \therefore a = 0, \quad b = 5 \\ d &= -4 & c = 1, \quad d = -4 \end{aligned}$$

b Equating corresponding elements,

$$\begin{aligned} 3 + b &= a & \therefore a = 2 \\ 2a - a &= 2 & b = -1 \\ b + c &= 2 & c = 3 \\ -2 + d &= 6 & d = 8 \end{aligned}$$

3 a $B - Y = A$

$$\begin{aligned} \therefore -Y &= A - B \\ \therefore Y &= -(A - B) \\ \therefore Y &= B - A \end{aligned}$$

b $2Y + C = D$

$$\begin{aligned} \therefore 2Y &= D - C \\ \therefore Y &= \frac{1}{2}(D - C) \end{aligned}$$

c $AY = B$

$$\begin{aligned} \therefore A^{-1}AY &= A^{-1}B \\ \therefore IY &= A^{-1}B \\ \therefore Y &= A^{-1}B \end{aligned}$$

d $YB = C$

$$\begin{aligned} \therefore YBB^{-1} &= CB^{-1} \\ \therefore YI &= CB^{-1} \\ \therefore Y &= CB^{-1} \end{aligned}$$

e $C - AY = B$

$$\begin{aligned} \therefore -AY &= B - C \\ \therefore AY &= C - B \\ \therefore A^{-1}AY &= A^{-1}(C - B) \\ \therefore Y &= A^{-1}(C - B) \end{aligned}$$

f $AY^{-1} = B$

$$\begin{aligned} \therefore A^{-1}AY^{-1} &= A^{-1}B \\ \therefore Y^{-1} &= A^{-1}B \\ \therefore (Y^{-1})^{-1} &= (A^{-1}B)^{-1} \\ \therefore Y &= B^{-1}(A^{-1})^{-1} \\ \therefore Y &= B^{-1}A \end{aligned}$$

4 a

$$\begin{aligned} 3x - 4y &= 2 \\ 5x + 2y &= -1 \end{aligned}$$

$$\therefore \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{26} \begin{bmatrix} 2 & 4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{26} \begin{bmatrix} 0 \\ -13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$$

$$\therefore x = 0, \quad y = -\frac{1}{2}$$

b

$$\begin{aligned} 4x - y &= 5 \\ 2x + 3y &= 9 \end{aligned}$$

$$\therefore \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 24 \\ 26 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{7} \\ \frac{13}{7} \end{bmatrix}$$

$$\therefore x = \frac{12}{7}, \quad y = \frac{13}{7}$$

$$\mathbf{c} \quad \mathbf{X} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} -1 & 8 \\ -2 & 6 \end{bmatrix}$$

$$\mathbf{d} \quad \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{2} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$\mathbf{e} \quad \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{-3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{-3} \begin{bmatrix} -14 \\ -1 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} \frac{14}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\mathbf{f} \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{X} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$\therefore \mathbf{X} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} 5 & -2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} -3 & -9 \\ -9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{5} \quad \mathbf{a} \quad 2\mathbf{B}$$

$$= 2 \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{bmatrix}$$

$$\mathbf{b} \quad \frac{1}{2}\mathbf{B}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{AB}$$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 12 \end{bmatrix}$$

$$\mathbf{d} \quad \mathbf{B} \text{ is } 3 \times 2 \quad \leftarrow$$

$$\text{and } \mathbf{A} \text{ is } 1 \times 3.$$

not equal $\longleftarrow$

$\therefore \mathbf{BA}$ does not exist.

$$\begin{array}{lll}
 \mathbf{6} \quad \mathbf{a} \quad \mathbf{P} + \mathbf{Q} & \mathbf{b} \quad \mathbf{Q} - \mathbf{P} & \mathbf{c} \quad \frac{3}{2}\mathbf{P} - \mathbf{Q} \\
 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} & = \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} & = \begin{bmatrix} \frac{3}{2} & 3 \\ \frac{3}{2} & 0 \\ 3 & \frac{9}{2} \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \\
 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{bmatrix} & = \begin{bmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{bmatrix} & = \begin{bmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{bmatrix}
 \end{array}$$

REVIEW SET 14B

$$\begin{array}{ll}
 \mathbf{1} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} & \therefore a = 1, \quad b = 0 \quad \text{i.e., } a = 1, \quad b = 0, \quad c = 0, \quad d = 1 \\
 \therefore \begin{bmatrix} a & b \\ a + c & b + d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} & \begin{array}{l} a + c = 1, \\ b + d = 1 \end{array} \quad \therefore \text{matrix is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{2} \quad \mathbf{a} \quad \mathbf{AB} & \mathbf{b} \quad \mathbf{BA} & \mathbf{c} \quad \mathbf{AC} \\
 = \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} & = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} & = \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\
 = [10] & = \begin{bmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 0 & 0 & 0 \end{bmatrix} & = \begin{bmatrix} 15 & 18 & 21 \end{bmatrix}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{d} \quad \mathbf{CA} \text{ does not exist as } \mathbf{C} & \mathbf{e} \quad \mathbf{CB} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix} \\
 \text{is } 3 \times 3 \text{ and } \mathbf{A} \text{ is } 1 \times 3 & \\
 \uparrow \text{not equal} \uparrow &
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{3} \quad \mathbf{a} \quad \mathbf{A}^{-1} = \frac{1}{42 - 40} \begin{bmatrix} 7 & -8 \\ -5 & 6 \end{bmatrix} & \mathbf{b} \quad \mathbf{A}^{-1} \text{ does not exist as } |\mathbf{A}| & \mathbf{c} \quad \mathbf{A}^{-1} = \frac{1}{-3} \begin{bmatrix} -3 & -5 \\ 6 & 11 \end{bmatrix} \\
 = \begin{bmatrix} \frac{7}{2} & -4 \\ -\frac{5}{2} & 3 \end{bmatrix} & = -24 - -24 & = \begin{bmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{bmatrix} \\
 & = 0 &
 \end{array}$$

$$\mathbf{4} \quad \text{Sales matrix is } \begin{bmatrix} 42 - 27 & 54 - 31 \\ 36 - 28 & 27 - 15 \\ 34 - 28 & 30 - 22 \end{bmatrix} = \begin{bmatrix} 15 & 23 \\ 8 & 12 \\ 6 & 8 \end{bmatrix} \quad \text{Totals matrix is } \begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix}$$

$$\therefore \text{total profit} = \begin{bmatrix} 0.75 & 0.55 & 1.20 \end{bmatrix} \begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix} = [56.3] \text{ dollars i.e., } \$56.30$$

$$\begin{array}{lll}
 \mathbf{5} \quad \mathbf{A} = 2\mathbf{A}^{-1} & \mathbf{a} \quad \mathbf{A}^2 = \mathbf{A} \times \mathbf{A} & \mathbf{b} \quad (\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I}) = (\mathbf{A} - \mathbf{I})\mathbf{A} + (\mathbf{A} - \mathbf{I})3\mathbf{I} \\
 & = \mathbf{A}(2\mathbf{A}^{-1}) & = \mathbf{A}^2 - \mathbf{IA} + 3\mathbf{AI} - 3\mathbf{I}^2 \\
 & = 2\mathbf{AA}^{-1} & = 2\mathbf{I} - \mathbf{A} + 3\mathbf{A} - 3\mathbf{I} \\
 & = 2\mathbf{I} & = 2\mathbf{A} - \mathbf{I}
 \end{array}$$

$$\begin{array}{l}
 \mathbf{6} \quad \mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 + -8 + 6 & -2 + 2 + 0 & -1 - 2 + 3 \\ 6 - 20 + 14 & -4 + 5 + 0 & -2 - 5 + 7 \\ -6 + 16 - 10 & 4 - 4 + 0 & 2 + 4 - 5 \end{bmatrix} \\
 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{array}$$

$$\begin{aligned}\mathbf{BA} &= \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 3-4+2 & 6-10+4 & 9-14+5 \\ -4+2+2 & -8+5+4 & -12+7+5 \\ 2+0-2 & 4+0-4 & 6+0-5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

$$\text{i.e., } \mathbf{AB} = \mathbf{BA} = \mathbf{I} \quad \therefore \mathbf{A}^{-1} = \mathbf{B}$$

REVIEW SET 14C

1 In matrix form $\begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$ has a unique solution if $\begin{vmatrix} k & 3 \\ 1 & k+2 \end{vmatrix} \neq 0$

i.e., $k^2 + 2k - 3 \neq 0$
 $(k-1)(k+3) \neq 0$
 $k \neq 1 \text{ or } -3$

$$\begin{aligned}\text{and if } k \neq 1 \text{ or } -3, \quad \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ &= \frac{1}{(k-1)(k+3)} \begin{bmatrix} k+2 & -3 \\ -1 & k \end{bmatrix} \begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ &= \frac{1}{(k-1)(k+3)} \begin{bmatrix} -6k-18 \\ 6+2k \end{bmatrix} \\ &= \frac{1}{(k-1)(k+3)} (k+3) \begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{-6}{k-1} \\ \frac{2}{k-1} \end{bmatrix} \quad \text{So the unique solution is} \\ &\quad x = \frac{-6}{k-1}, \quad y = \frac{2}{k-1}\end{aligned}$$

If $k = 1$, the equations are: $\left. \begin{array}{l} x+3y = -6 \\ x+3y = 2 \end{array} \right\}$ parallel lines $\therefore$ no solutions exist

If $k = -3$, the equations are: $\left. \begin{array}{l} -3x+3y = -6 \\ x-y = 2 \end{array} \right\}$ coincident lines $\therefore$ infinitely many solutions

2 $\begin{vmatrix} x & 2 & 0 \\ 2 & x+1 & -2 \\ 0 & -2 & x+2 \end{vmatrix} = x \begin{vmatrix} x+1 & -2 \\ -2 & x+2 \end{vmatrix} + 2 \begin{vmatrix} -2 & 2 \\ x+2 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & x+1 \\ 0 & -2 \end{vmatrix}$

$$\begin{aligned}&= x[(x+1)(x+2) - 4] + 2[0 - 2(x+2)] \\ &= x(x^2 + 3x + 2 - 4) - 4(x+2) \\ &= x(x^2 + 3x - 2) - 4x - 8 \\ &= x^3 + 3x^2 - 2x - 4x - 8 \\ &= x^3 + 3x^2 - 6x - 8 \\ &= (x+4)(x+1)(x-2) \quad \{\text{using technology}\}\end{aligned}$$

But $(x+4)(x+1)(x-2) = 0$ {given}
 $\therefore x = -4, -1 \text{ or } 2$

3 $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}$

a $2\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} -4 & 6 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} -14 & 18 \\ 18 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -12 \\ -10 & 4 \end{bmatrix}$

b $\mathbf{A}$ is 2×2 and $\mathbf{C}$ is 2×3 $\therefore \mathbf{AC}$ is 2×3

$$\mathbf{AC} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+6 & -6+3 \\ -4+0 & 0-2 & 12-1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{bmatrix}$$

c $\mathbf{C}$ is 2×3 and $\mathbf{B}$ is 2×2 $\therefore \mathbf{CB}$ is not possible.

d $\mathbf{DA} = \mathbf{B}$

$$\therefore \mathbf{DAA}^{-1} = \mathbf{BA}^{-1} \quad \{\text{post multiplying by } \mathbf{A}^{-1}\}$$

$$\therefore \mathbf{D} = \mathbf{BA}^{-1}$$

$$\therefore \mathbf{D} = \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix} = \frac{1}{-2(-1) - 3(4)} \begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} 7-36 & 21-18 \\ -9+12 & -27+6 \end{bmatrix}$$

$$\therefore \mathbf{D} = \begin{bmatrix} \frac{29}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{21}{10} \end{bmatrix}$$

$$4 \quad \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 11 & 33 & 22 \\ -11 & 11 & 33 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

$$5 \quad 5\mathbf{A}^2 - 6\mathbf{B} = 3\mathbf{I}$$

$$\therefore \mathbf{A}(5\mathbf{A} - 6\mathbf{I}) = 3\mathbf{I}$$

$$\therefore \mathbf{A} \times \frac{1}{3}(5\mathbf{A} - 6\mathbf{I}) = \mathbf{I}$$

$$\therefore \mathbf{A}^{-1} = \frac{5}{3}\mathbf{A} - 2\mathbf{I}$$

6 a i If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{ABB}^{-1} = \mathbf{BB}^{-1}$ provided $\mathbf{B}^{-1}$ exists.

$$\therefore \mathbf{A} = \mathbf{I} \quad \text{provided } \mathbf{B}^{-1} \text{ exists, i.e., provided that } |\mathbf{B}| \neq 0.$$

ii $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B})$

$$= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$$

$$= \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 \quad \text{provided that } \mathbf{AB} = \mathbf{BA}.$$

b $\mathbf{M} = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix} \begin{bmatrix} k-1 & -2 \\ -3 & -k \end{bmatrix}$ and $\mathbf{M}^{-1}$ exists provided that $|\mathbf{M}| \neq 0$.

$$\text{Now } |\mathbf{M}| = \begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} = \begin{vmatrix} k-1 & -2 \\ -3 & k \end{vmatrix}$$

$$= (k^2 - 4)(k^2 - k - 6)$$

$$= (k+2)(k-2)(k+2)(k-3)$$

$$\therefore \mathbf{M}^{-1} \text{ exists provided that } k \neq 3 \text{ or } \pm 2$$

REVIEW SET 14D

1 This system in matrix form is $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \text{So, } x = 1, \quad y = -1, \quad z = 2. \quad \{\text{using technology}\}$$

$$\begin{aligned} 2 \quad \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} &= (a+b) \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} + c \begin{vmatrix} a & a \\ c+a & b \end{vmatrix} + c \begin{vmatrix} a & b+c \\ b & b \end{vmatrix} \\ &= (a+b)[(b+c)(c+a) - ab] + c[ab - a(c+a)] + c[ab - (b+c)b] \\ &= (a+b)(bc + ab + c^2 + ac - ab) + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= abc + b^2c + ac^2 + bc^2 + a^2c + abc + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= 4abc \end{aligned}$$

$$3 \quad \text{If } \mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I},$$

$$\begin{array}{lll} \mathbf{A}^3 = \mathbf{A}(5\mathbf{A} + 2\mathbf{I}) & \mathbf{A}^4 = \mathbf{A}(27\mathbf{A} + 10\mathbf{I}) & \mathbf{A}^5 = \mathbf{A}(145\mathbf{A} + 54\mathbf{I}) \\ = 5\mathbf{A}^2 + 2\mathbf{A} & = 27\mathbf{A}^2 + 10\mathbf{A} & = 145\mathbf{A}^2 + 54\mathbf{A} \\ = 5(5\mathbf{A} + 2\mathbf{I}) + 2\mathbf{A} & = 27(5\mathbf{A} + 2\mathbf{I}) + 10\mathbf{A} & = 145(5\mathbf{A} + 2\mathbf{I}) + 54\mathbf{A} \\ = 25\mathbf{A} + 10\mathbf{I} + 2\mathbf{A} & = 135\mathbf{A} + 54\mathbf{I} + 10\mathbf{A} & = 725\mathbf{A} + 290\mathbf{I} + 54\mathbf{A} \\ = 27\mathbf{A} + 10\mathbf{I} & = 145\mathbf{A} + 54\mathbf{I} & = 779\mathbf{A} + 290\mathbf{I} \end{array}$$

$$\begin{aligned} \mathbf{A}^6 &= \mathbf{A}(779\mathbf{A} + 290\mathbf{I}) \\ &= 779\mathbf{A}^2 + 290\mathbf{A} \\ &= 779(5\mathbf{A} + 2\mathbf{I}) + 290\mathbf{A} \\ &= 4185\mathbf{A} + 1558\mathbf{I} \end{aligned}$$

$$4 \quad \mathbf{a} \quad C(0) = 80 \quad \therefore a(0) + b(0) + c(0) + d = 80 \quad \therefore d = 80$$

$$\begin{array}{ll} \mathbf{b} \quad C(1) = 100 & \therefore a + b + c + 80 = 100 \\ C(2) = 148 & 8a + 4b + 2c + 80 = 148 \\ C(4) = 376 & 64a + 16b + 4c + 80 = 376 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\therefore a = 2, \quad b = 8, \quad c = 10 \quad \{\text{technology}\}$$

$$\begin{array}{ll} 5 \quad \mathbf{AXB} = \mathbf{C} & \text{So } \mathbf{X} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -12 & -11 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}^{-1} \\ \therefore \mathbf{A}^{-1}\mathbf{AXB}\mathbf{B}^{-1} = \mathbf{A}^{-1}\mathbf{CB}^{-1} & \\ \therefore \mathbf{IXI} = \mathbf{A}^{-1}\mathbf{CB}^{-1} & \therefore \mathbf{X} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \quad \{\text{using technology}\} \\ \therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{CB}^{-1} & \end{array}$$

$$6 \quad \mathbf{a} \quad \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore x = 6, \quad y = -2, \quad z = 1$$

$$\mathbf{b} \quad \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{7}{6} \\ -\frac{7}{6} \end{bmatrix}$$

$$\therefore x = \frac{3}{2}, \quad y = -\frac{7}{6}, \quad z = -\frac{7}{6}$$

REVIEW SET 14E

- 1 a** Let \$x\$ be the cost of an opera ticket $3x + 2y + 5z = 267$
 \$y\$ be the cost of a play ticket $2x + 3y + z = 145$
 \$z\$ be the cost of a concert ticket $x + 5y + 4z = 230$

$$\mathbf{b} \text{ So, } \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 18 \\ 27 \end{bmatrix} \quad \text{using technology}$$

$\therefore$ cost of each ticket is \$32 for opera, \$18 for play, \$27 for concert.

$$\mathbf{c} \text{ Total cost} = 4 \times \$32 + 1 \times \$18 + 2 \times \$27$$

$$= \$128 + \$18 + \$54 = \$200$$

$$\mathbf{2} \quad \begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore x = 2, \quad y = 1, \quad z = 3 \quad \{\text{using technology}\}$

$$\mathbf{3} \quad \mathbf{a} \quad 3\mathbf{A} = 3 \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad \mathbf{b} \quad \mathbf{AB} = \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 6 & 6 \\ 3 & -3 & 0 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{BA} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 4 \\ 10 & -7 & -6 \\ -1 & 0 & 2 \end{bmatrix}$$

$\mathbf{d} \quad \mathbf{A}$ is 2×3 and $\mathbf{C}$ is 2×2
 $\therefore \mathbf{AC}$ does not exist.

$$\mathbf{e} \quad \mathbf{BC} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 22 \\ 7 & -12 \\ 0 & 11 \end{bmatrix}$$

$$\mathbf{4} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1+4+3 & 2-2+4 & -3+8+1 \\ -2+8+18 & 4-4+24 & -6+16+6 \\ -3+2+6 & 6-1+8 & -9+4+2 \end{bmatrix}$$

$$\therefore \mathbf{AB} = \begin{bmatrix} 6 & 4 & 6 \\ 24 & 24 & 16 \\ 5 & 13 & -3 \end{bmatrix}$$

$$\begin{aligned}
 \text{and } \det(\mathbf{AB}) &= 6 \begin{vmatrix} 24 & 16 \\ 13 & -3 \end{vmatrix} + 4 \begin{vmatrix} 16 & 24 \\ -3 & 5 \end{vmatrix} + 6 \begin{vmatrix} 24 & 24 \\ 5 & 13 \end{vmatrix} \\
 &= 6(-72 - 208) + 4(80 + 72) + 6(312 - 120) \\
 &= 80
 \end{aligned}$$

$$\begin{aligned}
 \det \mathbf{A} &= 1 \begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} \\
 &= 1(8 - 6) + 2(18 - 4) + 1(2 - 12) \\
 &= 2 + 28 - 10 \\
 &= 20
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \det \mathbf{B} &= -1 \begin{vmatrix} -1 & 4 \\ 4 & 1 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + (-3) \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \\
 &= -1(-1 - 16) + 2(12 - 2) - 3(8 - -3) \\
 &= 4 \quad \therefore \det \mathbf{A} \times \det \mathbf{B} = 20 \times 4 = 80 = \det(\mathbf{AB})
 \end{aligned}$$

5 a $s(t) = at^2 + bt + c$

$$\begin{aligned}
 \text{But } s(1) &= 63 & \therefore a + b + c &= 63 & \therefore \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 49 & 7 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 63 \\ 72 \\ 27 \end{bmatrix} \\
 s(2) &= 72 & 4a + 2b + c &= 72 \\
 s(7) &= 27 & 49a + 7b + c &= 27 \\
 & & \therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 49 & 7 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 63 \\ 72 \\ 27 \end{bmatrix} \\
 & & \therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} -3 \\ 18 \\ 48 \end{bmatrix}
 \end{aligned}$$

b $s(t) = -3t^2 + 18t + 48$ metres
 $s(0) = 48$ m, so, the cliff is 48 m high

c $s(t) = 0$ when $-3t^2 + 18t + 48 = 0$
 $\therefore t^2 - 6t - 16 = 0$
 $\therefore (t + 2)(t - 8) = 0$
 $\therefore t = -2 \text{ or } 8$
 $\therefore t = 8 \quad \{\text{as } t > 0\}$

$\therefore$ it will take 8 seconds to reach the water.

6 If $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$ then $\mathbf{A}^3 = \mathbf{A}(\mathbf{A} - \mathbf{I})$ $\mathbf{A}^4 = \mathbf{A}\mathbf{A}^3$ $\mathbf{A}^5 = \mathbf{A}\mathbf{A}^4$
 $= \mathbf{A}^2 - \mathbf{A}\mathbf{I}$ $= \mathbf{A}(-\mathbf{I})$ $= \mathbf{A}(-\mathbf{A})$
 $= \mathbf{A} - \mathbf{I} - \mathbf{A}$ $= -\mathbf{A}$ $= -\mathbf{A}^2$
 $= -\mathbf{I}$ $= -(\mathbf{A} - \mathbf{I})$
 $= \mathbf{I} - \mathbf{A}$

$\mathbf{A}^6 = \mathbf{A}\mathbf{A}^5$ $\mathbf{A}^7 = \mathbf{A}\mathbf{A}^6$ $\mathbf{A}^8 = \mathbf{A}\mathbf{A}^7$
 $= \mathbf{A}(\mathbf{I} - \mathbf{A})$ $= \mathbf{A}\mathbf{I}$ $= \mathbf{A}\mathbf{A}$
 $= \mathbf{A}\mathbf{I} - \mathbf{A}^2$ $= \mathbf{A}$ $= \mathbf{A}^2$
 $= \mathbf{A} - (\mathbf{A} - \mathbf{I})$ $= \mathbf{A} - \mathbf{I}$
 $= \mathbf{I}$

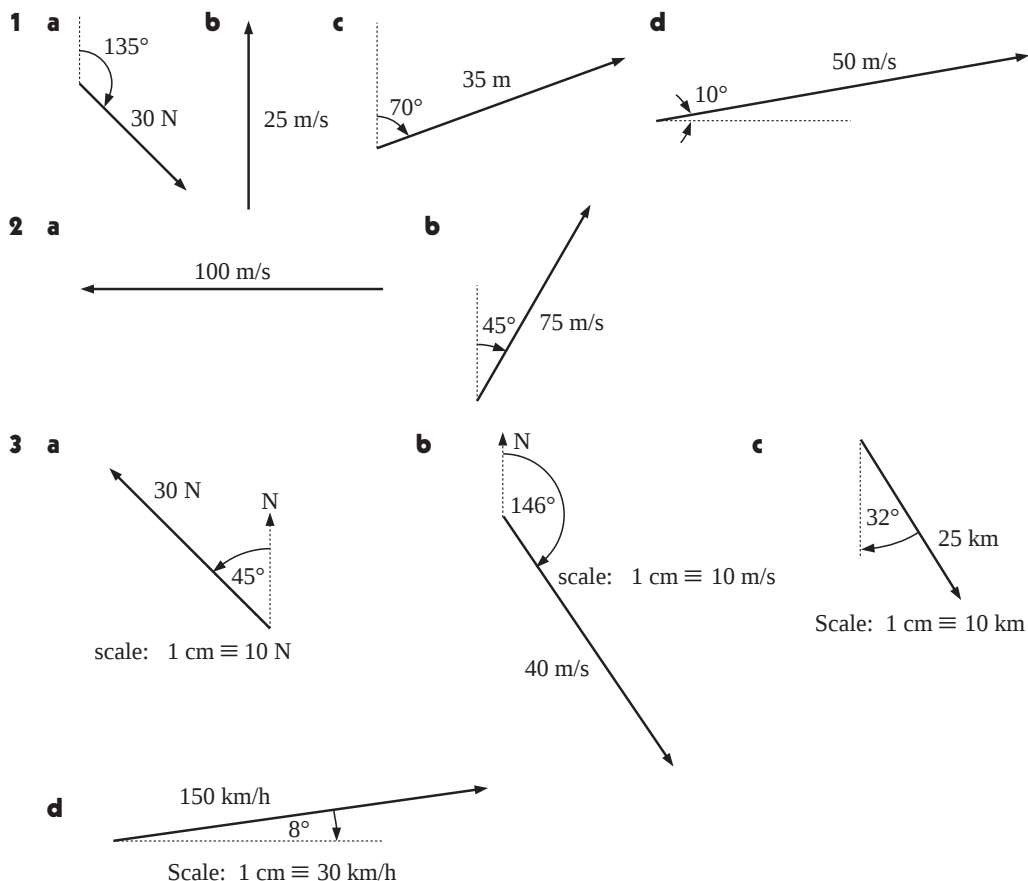
a $\mathbf{A}^{6n+3} = -\mathbf{I}$
 $\mathbf{A}^{6n+5} = \mathbf{I} - \mathbf{A}$

b Now $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$
 $\therefore \mathbf{A}^{-1}\mathbf{A}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I}$ {premultiplying by $\mathbf{A}^{-1}$ }
 $\therefore \mathbf{I}\mathbf{A} = \mathbf{I} - \mathbf{A}^{-1}$
 $\therefore \mathbf{A}^{-1} = \mathbf{I} - \mathbf{A}$

Chapter 15

VECTORS IN 2-DIMENSIONS

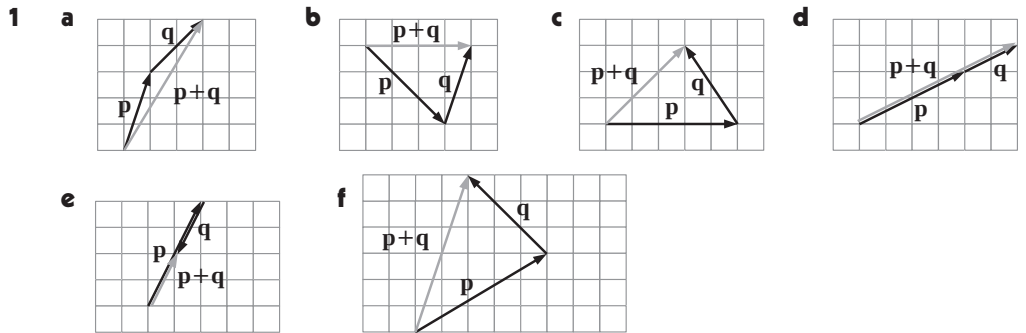
EXERCISE 15A.1



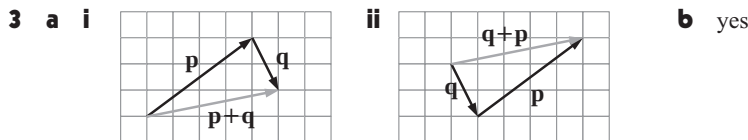
EXERCISE 15A.2

- 1
 - a If they are equal in magnitude, they have the same length. These are **p**, **q**, **s** and **t**.
 - b Those parallel are **p**, **q**, **r** and **t**.
 - c Those in the same direction are: **p** and **r**, **q** and **t**.
 - d To be equal they must have the same direction and be equal in length $\therefore$ **q** = **t**.
 - e **p** and **q** are negatives (equal length, but opposite direction). Likewise **p** and **t** are negatives. We write **p** = -**q** and **p** = -**t**.
- 2
 - a True, as they have the same length and are parallel.
 - b True, as they are sides of an equilateral triangle.
 - c False, as they do not have the same direction.
 - d False, as they have opposite directions.
 - e True, as they have the same length and direction.
 - f False, as they do not have the same direction.

EXERCISE 15B.1



2 a $\vec{AB} + \vec{BC} = \vec{AC}$ b $\vec{BC} + \vec{CD} = \vec{BD}$ c $\vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD}$ d $\vec{AC} + \vec{CB} + \vec{BD} = \vec{AB} + \vec{BD} = \vec{AD}$



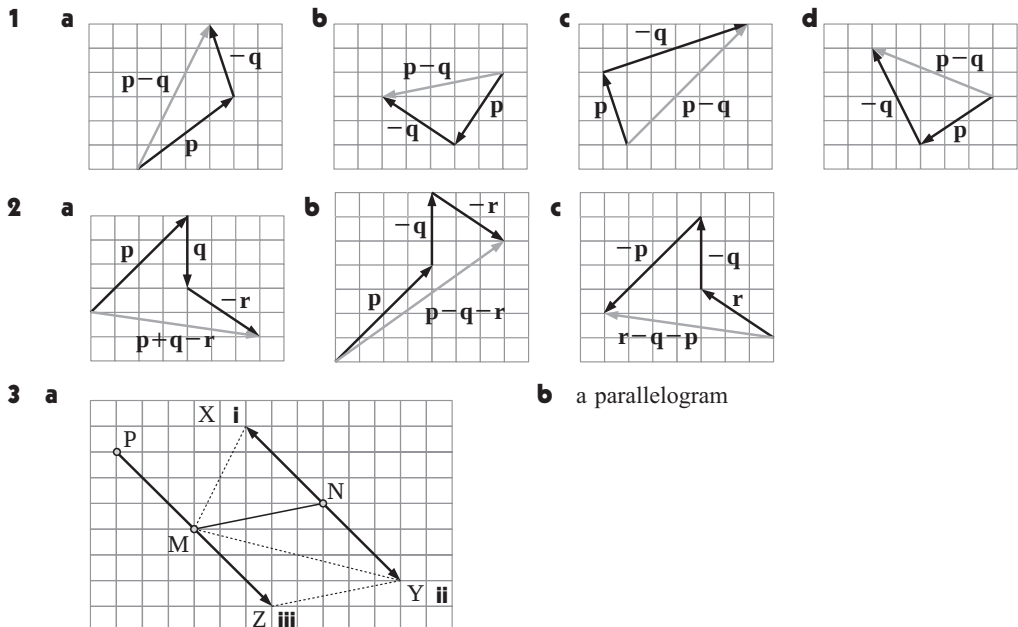
4 $\vec{PS} = \vec{PR} + \vec{RS} = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

But $\vec{PS} = \vec{PQ} + \vec{QS} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

$\therefore (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {as both are equal to $\vec{PS}$ }

Note: $\vec{PS} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ also

EXERCISE 15B.2



4 **a** $\vec{AC} + \vec{CB}$
 $= \vec{AB}$

b $\vec{AD} - \vec{BD}$
 $= \vec{AD} + \vec{DB}$
 $= \vec{AB}$

c $\vec{AC} + \vec{CA}$
 $= \vec{AA}$
 $= \mathbf{0}$

d $\vec{AB} + \vec{BC} + \vec{CD}$
 $= \vec{AC} + \vec{CD}$
 $= \vec{AD}$

e $\vec{BA} - \vec{CA} + \vec{CB}$
 $= \vec{BA} + \vec{AC} + \vec{CB}$
 $= \vec{BC} + \vec{CB}$
 $= \vec{BB}$
 $= \mathbf{0}$

f $\vec{AB} - \vec{CB} - \vec{DC}$
 $= \vec{AB} + \vec{BC} + \vec{CD}$
 $= \vec{AC} + \vec{CD}$
 $= \vec{AD}$

5 **a** $t = r + s$ **b** $r = -s - t$ **c** $r = -p - q - s$ **d** $r = q - p + s$
e $p = t + s + r - q$ **f** $p = -u + t + s - r - q$

6 **a** **i** $\vec{OB}$
 $= \vec{OA} + \vec{AB}$
 $= r + s$

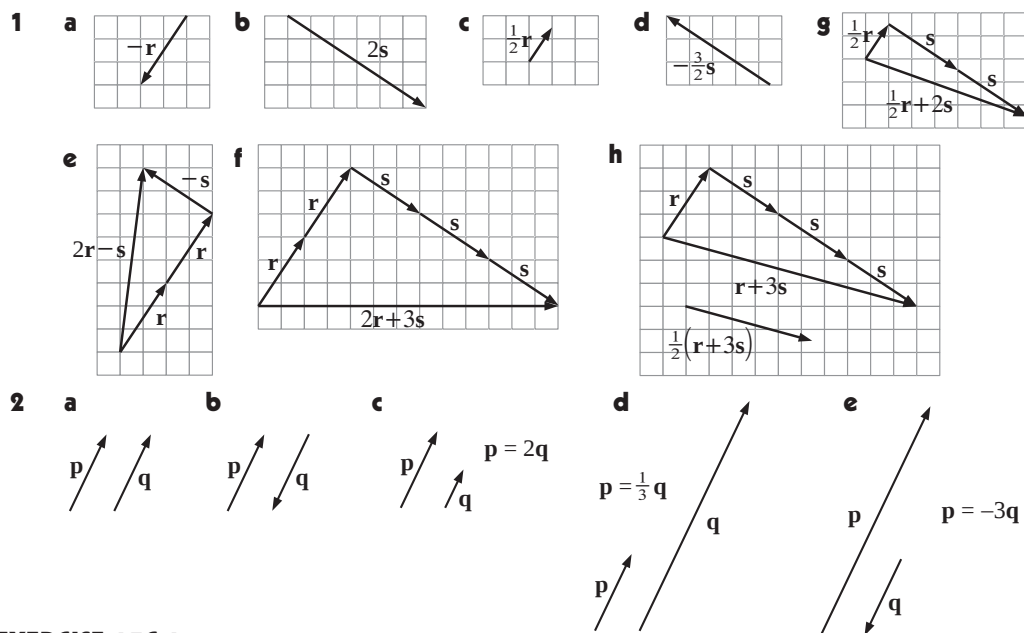
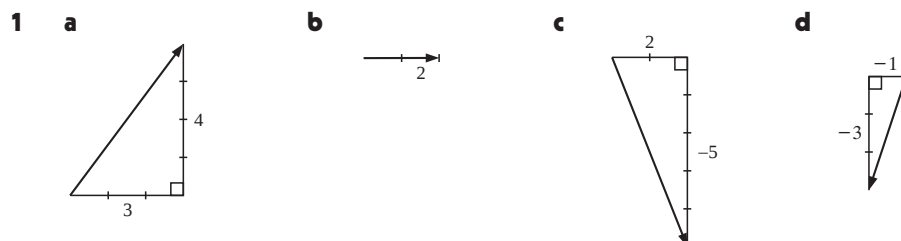
ii $\vec{CA}$
 $= \vec{CB} + \vec{BA}$
 $= -\vec{BC} - \vec{AB}$
 $= -t - s$

iii $\vec{OC} = \vec{OA} + \vec{AB} + \vec{BC}$
 $= r + s + t$

b **i** $\vec{AD}$
 $= \vec{AB} + \vec{BD}$
 $= p + q$

ii $\vec{BC}$
 $= \vec{BD} + \vec{DC}$
 $= q + r$

iii $\vec{AC} = \vec{AB} + \vec{BD} + \vec{DC}$
 $= p + q + r$

EXERCISE 15B.3

EXERCISE 15C.1


$$\mathbf{2} \quad \mathbf{a} \quad \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \mathbf{b} \quad \begin{bmatrix} -6 \\ 0 \end{bmatrix} \quad \mathbf{c} \quad \begin{bmatrix} 2 \\ -5 \end{bmatrix} \quad \mathbf{d} \quad \begin{bmatrix} 0 \\ 6 \end{bmatrix} \quad \mathbf{e} \quad \begin{bmatrix} -6 \\ 3 \end{bmatrix} \quad \mathbf{f} \quad \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

EXERCISE 15C.2

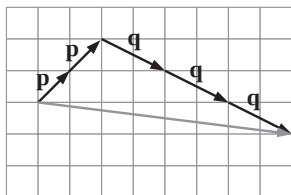
$$\begin{array}{llll} \mathbf{1} & \mathbf{a} & \mathbf{a} + \mathbf{b} & \mathbf{b} & \mathbf{b} + \mathbf{a} & \mathbf{c} & \mathbf{b} + \mathbf{c} & \mathbf{d} & \mathbf{c} + \mathbf{b} \\ & & = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} & & = \begin{bmatrix} -2 \\ -5 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ & & = \begin{bmatrix} -2 \\ 6 \end{bmatrix} & & = \begin{bmatrix} -2 \\ 6 \end{bmatrix} & & = \begin{bmatrix} -1 \\ -1 \end{bmatrix} & & = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ & \mathbf{e} & \mathbf{a} + \mathbf{c} & \mathbf{f} & \mathbf{c} + \mathbf{a} & \mathbf{g} & \mathbf{a} + \mathbf{a} & \mathbf{h} & \mathbf{b} + \mathbf{a} + \mathbf{c} \\ & & = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} & & = \begin{bmatrix} -2 \\ -5 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} & & = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \\ & & = \begin{bmatrix} -5 \\ -3 \end{bmatrix} & & = \begin{bmatrix} -5 \\ -3 \end{bmatrix} & & = \begin{bmatrix} -6 \\ 4 \end{bmatrix} & & = \begin{bmatrix} -2 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \\ & & & & & & & = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \end{array}$$

$$\begin{array}{llll} \mathbf{2} & \mathbf{a} & \mathbf{p} - \mathbf{q} & \mathbf{b} & \mathbf{q} - \mathbf{r} & \mathbf{c} & \mathbf{p} + \mathbf{q} - \mathbf{r} \\ & & = \begin{bmatrix} -4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \end{bmatrix} & & = \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} & & = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ & & = \begin{bmatrix} -3 \\ 7 \end{bmatrix} & & = \begin{bmatrix} -4 \\ -3 \end{bmatrix} & & = \begin{bmatrix} -8 \\ -1 \end{bmatrix} \\ & \mathbf{d} & \mathbf{p} - \mathbf{q} - \mathbf{r} & \mathbf{e} & \mathbf{q} - \mathbf{r} - \mathbf{p} & \mathbf{f} & \mathbf{r} + \mathbf{q} - \mathbf{p} \\ & & = \begin{bmatrix} -4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} & & = \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \end{bmatrix} & & = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \end{bmatrix} \\ & & = \begin{bmatrix} -6 \\ 9 \end{bmatrix} & & = \begin{bmatrix} 0 \\ -5 \end{bmatrix} & & = \begin{bmatrix} 6 \\ -9 \end{bmatrix} \end{array}$$

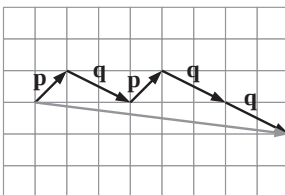
$$\begin{array}{llll} \mathbf{3} & \mathbf{a} & \overrightarrow{AC} & \mathbf{b} & \overrightarrow{CB} & \mathbf{c} & \overrightarrow{SP} \\ & & = \overrightarrow{AB} + \overrightarrow{BC} & & = \overrightarrow{CA} + \overrightarrow{AB} & & = \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP} \\ & & = -\overrightarrow{BA} + \overrightarrow{BC} & & = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} & & = -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ} \\ & & = -\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 2 \end{bmatrix} & & = -\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \end{bmatrix} \\ & & = \begin{bmatrix} -5 \\ 4 \end{bmatrix} & & & & = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \end{array}$$

EXERCISE 15C.3

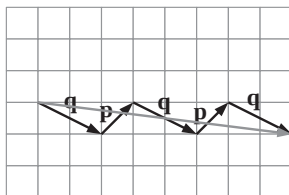
$$\begin{array}{llll} \mathbf{1} & \mathbf{a} & -3\mathbf{p} & \mathbf{b} & \frac{1}{2}\mathbf{q} & \mathbf{c} & 2\mathbf{p} + \mathbf{q} \\ & & = -3\begin{bmatrix} 1 \\ 5 \end{bmatrix} & & = \frac{1}{2}\begin{bmatrix} -2 \\ 4 \end{bmatrix} & & = 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ & & = \begin{bmatrix} -3 \\ -15 \end{bmatrix} & & = \begin{bmatrix} -1 \\ 2 \end{bmatrix} & & = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ & & & & & & = \begin{bmatrix} 0 \\ 14 \end{bmatrix} \\ & \mathbf{d} & \mathbf{p} - 2\mathbf{q} & \mathbf{e} & \mathbf{p} - \frac{1}{2}\mathbf{r} & \mathbf{f} & 2\mathbf{p} + 3\mathbf{r} \\ & & = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 4 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} & & = 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} -3 \\ -1 \end{bmatrix} \\ & & = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -4 \\ 8 \end{bmatrix} & & = \begin{bmatrix} 2\frac{1}{2} \\ 5\frac{1}{2} \end{bmatrix} & & = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} -9 \\ -3 \end{bmatrix} \\ & & = \begin{bmatrix} 5 \\ -3 \end{bmatrix} & & & & = \begin{bmatrix} -7 \\ 7 \end{bmatrix} \\ & \mathbf{g} & 2\mathbf{q} - 3\mathbf{r} & \mathbf{h} & 2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r} \\ & & = 2\begin{bmatrix} -2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} -3 \\ -1 \end{bmatrix} & & = \begin{bmatrix} 2 \\ 10 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{1}{3} \end{bmatrix} \\ & & = \begin{bmatrix} -4 \\ 8 \end{bmatrix} - \begin{bmatrix} -9 \\ -3 \end{bmatrix} & & = \begin{bmatrix} 3 \\ 5\frac{2}{3} \end{bmatrix} \\ & & = \begin{bmatrix} 5 \\ 11 \end{bmatrix} & & \end{array}$$

2 a

$$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

b

$$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

c

$$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

All vector expressions are equal {each consists of $2\mathbf{p}$ and $3\mathbf{q}$ }.

Each expression is equal to $2\mathbf{p} + 3\mathbf{q}$.

EXERCISE 15C.4**1**

$$\begin{aligned} \mathbf{a} \quad |\mathbf{r}| &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad |\mathbf{s}| &= \sqrt{(-1)^2 + 4^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

c

$$\begin{aligned} \mathbf{r} + \mathbf{s} &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{r} + \mathbf{s}| &= \sqrt{1^2 + 7^2} \\ &= \sqrt{50} \text{ units} \end{aligned}$$

d

$$\begin{aligned} \mathbf{r} - \mathbf{s} &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{r} - \mathbf{s}| &= \sqrt{3^2 + (-1)^2} \\ &= \sqrt{10} \text{ units} \end{aligned}$$

e

$$\begin{aligned} \mathbf{s} - 2\mathbf{r} &= \begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{s} - 2\mathbf{r}| &= \sqrt{(-5)^2 + (-2)^2} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

2

$$\begin{aligned} \mathbf{a} \quad |\mathbf{p}| &= \sqrt{1^2 + 3^2} \\ &= \sqrt{10} \text{ units} \end{aligned}$$

b

$$\begin{aligned} 2\mathbf{p} &= \begin{bmatrix} 2 \\ 6 \end{bmatrix} \\ \therefore |2\mathbf{p}| &= \sqrt{2^2 + 6^2} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \text{ units} \end{aligned}$$

c

$$\begin{aligned} -2\mathbf{p} &= \begin{bmatrix} -2 \\ -6 \end{bmatrix} \\ \therefore |-2\mathbf{p}| &= \sqrt{(-2)^2 + (-6)^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 3\mathbf{p} &= \begin{bmatrix} 3 \\ 9 \end{bmatrix} \\ \therefore |3\mathbf{p}| &= \sqrt{3^2 + 9^2} \\ &= \sqrt{9 + 81} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad -3\mathbf{p} &= \begin{bmatrix} -3 \\ -9 \end{bmatrix} \\ \therefore |-3\mathbf{p}| &= \sqrt{(-3)^2 + (-9)^2} \\ &= \sqrt{9 + 81} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad |\mathbf{q}| &= \sqrt{(-2)^2 + 4^2} \\ &= \sqrt{4 + 16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad 4\mathbf{q} &= \begin{bmatrix} -8 \\ 16 \end{bmatrix} \\ \therefore |4\mathbf{q}| &= \sqrt{(-8)^2 + 16^2} \\ &= \sqrt{64 + 256} \\ &= \sqrt{320} \\ &= 8\sqrt{5} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad -4\mathbf{q} &= \begin{bmatrix} 8 \\ -16 \end{bmatrix} \\ \therefore |-4\mathbf{q}| &= \sqrt{8^2 + (-16)^2} \\ &= \sqrt{64 + 256} \\ &= \sqrt{320} \\ &= 8\sqrt{5} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad \frac{1}{2}\mathbf{q} &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \therefore \left| \frac{1}{2}\mathbf{q} \right| &= \sqrt{(-1)^2 + 2^2} \\ &= \sqrt{5} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad -\frac{1}{2}\mathbf{q} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \therefore \left| -\frac{1}{2}\mathbf{q} \right| &= \sqrt{1^2 + (-2)^2} \\ &= \sqrt{5} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad k\mathbf{a} &= \begin{bmatrix} ka_1 \\ ka_2 \end{bmatrix} \quad \therefore |k\mathbf{a}| = \sqrt{(ka_1)^2 + (ka_2)^2} \\ &= \sqrt{k^2a_1^2 + k^2a_2^2} \\ &= \sqrt{k^2(a_1^2 + a_2^2)} \\ &= \sqrt{k^2} \sqrt{a_1^2 + a_2^2} \\ &= |k| \sqrt{a_1^2 + a_2^2} \\ &= |k| |\mathbf{a}| \end{aligned}$$

EXERCISE 15D

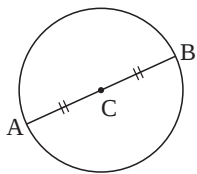
$$\begin{array}{lll}
 \mathbf{1} \quad \mathbf{a} & 2\mathbf{x} = \mathbf{q} & \mathbf{b} \quad \frac{1}{2}\mathbf{x} = \mathbf{n} & \mathbf{c} \quad -3\mathbf{x} = \mathbf{p} \\
 & \therefore \frac{1}{2}(2\mathbf{x}) = \frac{1}{2}\mathbf{q} & \therefore 2(\frac{1}{2}\mathbf{x}) = 2\mathbf{n} & \therefore 3\mathbf{x} = -\mathbf{p} \\
 & \therefore \mathbf{x} = \frac{1}{2}\mathbf{q} & \therefore \mathbf{x} = 2\mathbf{n} & \therefore \frac{1}{3}(3\mathbf{x}) = -\frac{1}{3}\mathbf{p} \\
 & & & \therefore \mathbf{x} = -\frac{1}{3}\mathbf{p} \\
 \\
 \mathbf{d} \quad \mathbf{q} + 2\mathbf{x} = \mathbf{r} & \mathbf{e} \quad 4\mathbf{s} - 5\mathbf{x} = \mathbf{t} & \mathbf{f} \quad 4\mathbf{m} - \frac{1}{3}\mathbf{x} = \mathbf{n} \\
 \therefore 2\mathbf{x} = \mathbf{r} - \mathbf{q} & \therefore -5\mathbf{x} = \mathbf{t} - 4\mathbf{s} & \therefore 4\mathbf{m} - \mathbf{n} = \frac{1}{3}\mathbf{x} \\
 \therefore \mathbf{x} = \frac{1}{2}(\mathbf{r} - \mathbf{q}) & \therefore 5\mathbf{x} = 4\mathbf{s} - \mathbf{t} & \therefore \mathbf{x} = 12\mathbf{m} - 3\mathbf{n} \\
 & \therefore \mathbf{x} = \frac{4}{5}\mathbf{s} - \frac{1}{5}\mathbf{t} &
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{2} \quad \mathbf{a} \quad 2\mathbf{y} = \mathbf{r} & \mathbf{b} \quad \frac{1}{2}\mathbf{y} = \mathbf{s} & \mathbf{c} \quad \mathbf{r} + 2\mathbf{y} = \mathbf{s} & \mathbf{d} \quad 3\mathbf{s} - 4\mathbf{y} = \mathbf{r} \\
 \therefore \mathbf{y} = \frac{1}{2}\mathbf{r} & \therefore \mathbf{y} = 2\mathbf{s} & \therefore 2\mathbf{y} = \mathbf{s} - \mathbf{r} & \therefore 3\mathbf{s} - \mathbf{r} = 4\mathbf{y} \\
 = \frac{1}{2} \begin{bmatrix} -2 \\ 3 \end{bmatrix} & = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \therefore \mathbf{y} = \frac{1}{2}\mathbf{s} - \frac{1}{2}\mathbf{r} & \therefore \mathbf{y} = \frac{3}{4}\mathbf{s} - \frac{1}{4}\mathbf{r} \\
 = \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix} & = \begin{bmatrix} 2 \\ 4 \end{bmatrix} & = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix} & = \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} -\frac{2}{4} \\ \frac{3}{4} \end{bmatrix} \\
 & & = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} & = \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \end{bmatrix} \\
 \\
 \mathbf{3} \quad k\mathbf{x} = \mathbf{a} & \therefore k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} & & \\
 & \therefore kx_1 = a_1 \quad \text{and} \quad kx_2 = a_2 & & \\
 & \therefore x_1 = \frac{1}{k}a_1 \quad \text{and} \quad x_2 = \frac{1}{k}a_2 & \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{bmatrix} = \frac{1}{k} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}
 \end{array}$$

EXERCISE 15E

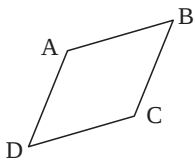
$$\begin{array}{lll}
 \mathbf{1} \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{bmatrix} 2-3 \\ 6--2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix} & \mathbf{b} \quad \overrightarrow{BC} = \begin{bmatrix} -1-2 \\ -4-6 \end{bmatrix} = \begin{bmatrix} -3 \\ -10 \end{bmatrix} & \mathbf{c} \quad \overrightarrow{CA} = \begin{bmatrix} 3--1 \\ -2--4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\
 \text{and } AB = |\overrightarrow{AB}| = \sqrt{1+64} & \text{and } BC = |\overrightarrow{BC}| & \text{and } CA = |\overrightarrow{CA}| \\
 \therefore AB = \sqrt{65} \text{ units} & = \sqrt{9+100} & = \sqrt{16+4} \\
 & = \sqrt{109} \text{ units} & = \sqrt{20} \text{ units}
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{2} \quad \mathbf{a} \quad \text{M is } \left(\frac{3+-1}{2}, \frac{6+2}{2} \right) & \mathbf{b} \quad \overrightarrow{CA} = \begin{bmatrix} 3--4 \\ 6-1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} & \mathbf{c} \quad \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB} \\
 \text{i.e., M is } (1, 4) & \overrightarrow{CM} = \begin{bmatrix} 1--4 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} & = \frac{1}{2} \begin{bmatrix} 7 \\ 5 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
 & \overrightarrow{CB} = \begin{bmatrix} -1--4 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} & = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \text{ which is } \overrightarrow{CM}
 \end{array}$$

3

$$\begin{array}{lll}
 \mathbf{a} \quad \overrightarrow{AC} = \begin{bmatrix} 1-3 \\ 4--2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} & \therefore \text{B is } (1-2, 4+6), \text{ i.e., } (-1, 10) \\
 \mathbf{b} \quad \overrightarrow{AC} = \begin{bmatrix} -1-0 \\ -2-5 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \end{bmatrix} & \therefore \text{B is } (-1-1, -2-7), \text{ i.e., } (-2, -9) \\
 \mathbf{c} \quad \overrightarrow{AC} = \begin{bmatrix} 3--1 \\ 0--4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} & \therefore \text{B is } (3+4, 0+4), \text{ i.e., } (7, 4)
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{4} \quad \overrightarrow{AB} = \begin{bmatrix} 2--1 \\ 3-5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} & \therefore \text{C is } (2+3, 3-2) \text{ i.e., } (5, 1) \\
 & \text{D is } (5+3, 1-2) \text{ i.e., } (8, -1) \\
 & \text{E is } (8+3, -1-2) \text{ i.e., } (11, -3)
 \end{array}$$

5

$$\mathbf{a} \quad \overrightarrow{AB} = \begin{bmatrix} 4-3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{DC} = \begin{bmatrix} -1-2 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\mathbf{b} \quad \overrightarrow{AB} = \begin{bmatrix} -1-5 \\ 2-0 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\overrightarrow{DC} = \begin{bmatrix} 4-10 \\ -3-6 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

$$\mathbf{c} \quad \overrightarrow{AB} = \begin{bmatrix} 1-2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{DC} = \begin{bmatrix} -2-1 \\ 6-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$\text{Now } \overrightarrow{AB} = \overrightarrow{DC}$$

$\therefore$ sides AB and DC are equal in length and parallel. This is sufficient to deduce that ABCD is a parallelogram.

$$\text{So, } \overrightarrow{AB} \neq \overrightarrow{DC}$$

$\therefore$ ABCD cannot be a parallelogram.

$$\text{Now } \overrightarrow{AB} = \overrightarrow{DC}$$

$\therefore$ sides AB and DC are equal in length and parallel. This is sufficient to deduce that ABCD is a parallelogram.

6

a Let D be (a, b)

$$\text{Now } \overrightarrow{CD} = \overrightarrow{BA}$$

$$\therefore \begin{bmatrix} a-8 \\ b-2 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 0-1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a-8 \\ b+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore a = 9, \quad b = -1$$

So, D is $(9, -1)$

b Let R be (a, b)

$$\text{Now } \overrightarrow{SR} = \overrightarrow{PQ}$$

$$\therefore \begin{bmatrix} a-4 \\ b-0 \end{bmatrix} = \begin{bmatrix} -2-1 \\ 5-4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a-4 \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore a = 3, \quad b = 1$$

So, R is $(3, 1)$

c Let X be (a, b)

$$\text{Now } \overrightarrow{WX} = \overrightarrow{ZY}$$

$$\therefore \begin{bmatrix} a-1 \\ b-5 \end{bmatrix} = \begin{bmatrix} 3-0 \\ -2-4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a+1 \\ b-5 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

$$\therefore a = 2, \quad b = -1$$

So, X is $(2, -1)$

7**a**

$$r \begin{bmatrix} 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$$

$$\therefore \begin{bmatrix} r+2s \\ -r+5s \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$$

$$\therefore r + 2s = -8 \quad \dots\dots (1)$$

$$-r + 5s = -27$$

$$\text{adding } 7s = -35$$

$$\therefore s = -5$$

$$\text{and in (1) } r + 2(-5) = -8$$

$$\therefore r - 10 = -8$$

$$\therefore r = 2$$

$$\text{So, } r = 2, \quad s = -5$$

b

$$r \begin{bmatrix} 2 \\ -3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2r+s \\ -3r+7s \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$$

$$\therefore \begin{array}{l} 2r + s = 7 \\ -3r + 7s = -19 \end{array} \left| \begin{array}{l} \times 3 \\ \times 2 \end{array} \right. \dots\dots (1)$$

$$\therefore 6r + 3s = 21$$

$$-6r + 14s = -38$$

$$\text{adding } 17s = -17$$

$$\therefore s = -1$$

$$\text{and in (1) } 2r - 1 = 7$$

$$\therefore 2r = 8$$

$$\therefore r = 4$$

$$\text{So, } r = 4, \quad s = -1$$

EXERCISE 15F**1**

$$\mathbf{d} = \begin{bmatrix} -20 \\ 50 \end{bmatrix} = 10 \begin{bmatrix} -2 \\ 5 \end{bmatrix} = 10\mathbf{c}, \quad \therefore \mathbf{d} \text{ is parallel to } \mathbf{c}$$

$$\mathbf{e} = \begin{bmatrix} -12 \\ 4 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = -4\mathbf{b} \quad \therefore \mathbf{e} \text{ is parallel to } \mathbf{b}$$

$$\mathbf{f} = \begin{bmatrix} 10 \\ 25 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 5\mathbf{a} \quad \therefore \mathbf{f} \text{ is parallel to } \mathbf{a}$$

2

$$\mathbf{a} \quad \begin{bmatrix} 9 \\ t \end{bmatrix} = k \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \therefore \begin{cases} 3k = 9 \\ t = 4k \end{cases} \quad \text{so, } k = 3 \quad \text{and } t = 12$$

$$\mathbf{b} \quad \begin{bmatrix} -10 \\ t \end{bmatrix} = k \begin{bmatrix} 5 \\ -4 \end{bmatrix} \quad \therefore \begin{cases} 5k = -10 \\ -4k = t \end{cases} \quad \text{so, } k = -2 \quad \text{and } t = 8$$

$$\mathbf{c} \quad \begin{bmatrix} 16 \\ t \end{bmatrix} = k \begin{bmatrix} 12 \\ 9 \end{bmatrix} \quad \therefore \begin{cases} 12k = 16 \\ 9k = t \end{cases} \quad \text{so, } k = \frac{4}{3} \quad \text{and } t = 9 \times \frac{4}{3} = 12$$

EXERCISE 15G**1 a** length

$$= \sqrt{0^2 + (-1)^2}$$

$$= \sqrt{0+1}$$

$$= 1 \text{ unit}$$

$\therefore$ is a unit vector

b length

$$= \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$= 1 \text{ unit}$$

$\therefore$ is a unit vector

c length

$$= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{5}{9}} \neq 1$$

$\therefore$ is not a unit vector

d length

$$= \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{25}{25}}$$

$$= 1 \text{ unit}$$

$\therefore$ is a unit vector

e length

$$= \sqrt{\left(\frac{2}{7}\right)^2 + \left(-\frac{5}{7}\right)^2}$$

$$= \sqrt{\frac{4}{49} + \frac{25}{49}}$$

$$= \sqrt{\frac{29}{49}} \neq 1$$

$\therefore$ is not a unit vector

2 a $\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\mathbf{i} - \mathbf{j}$

b $\begin{bmatrix} -3 \\ -4 \end{bmatrix} = -3\mathbf{i} - 4\mathbf{j}$

c $\begin{bmatrix} -3 \\ 0 \end{bmatrix} = -3\mathbf{i}$

d $\begin{bmatrix} 0 \\ 7 \end{bmatrix} = 7\mathbf{j}$

e $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$

3 a $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

b $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$

c $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$

d $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

e $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$

4 a length = 1

$$\therefore \sqrt{0^2 + k^2} = 1$$

$$\therefore k^2 = 1$$

$$\therefore k = \pm 1$$

b length = 1

$$\therefore \sqrt{k^2 + 0} = 1$$

$$\therefore k^2 = 1$$

$$\therefore k = \pm 1$$

c length = 1

$$\therefore \sqrt{k^2 + 1} = 1$$

$$\therefore k^2 + 1 = 1$$

$$\therefore k^2 = 0$$

$$\therefore k = 0$$

d length = 1

$$\therefore \sqrt{\left(-\frac{1}{2}\right)^2 + k^2} = 1$$

$$\therefore \frac{1}{4} + k^2 = 1$$

$$\therefore k^2 = \frac{3}{4}$$

$$\therefore k = \pm \frac{\sqrt{3}}{2}$$

e length = 1

$$\therefore \sqrt{k^2 + \left(\frac{2}{3}\right)^2} = 1$$

$$\therefore k^2 + \frac{4}{9} = 1$$

$$\therefore k^2 = \frac{5}{9} \text{ and so } k = \pm \frac{\sqrt{5}}{3}$$

5 a length

$$= \sqrt{3^2 + 4^2}$$

$$= 5 \text{ units}$$

b length

$$= \sqrt{3^2 + (-4)^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25}$$

$$= 5 \text{ units}$$

c length

$$= \sqrt{(-2)^2 + (-7)^2}$$

$$= \sqrt{4 + 49}$$

$$= \sqrt{53} \text{ units}$$

d length = $\sqrt{(-2.36)^2 + (5.65)^2} \doteq 6.12 \text{ units}$

6 a $\mathbf{i} + 2\mathbf{j}$ has length $\sqrt{1^2 + 2^2} = \sqrt{5} \text{ units}$ $\therefore$ unit vector = $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$

b $2\mathbf{i} - 2\mathbf{j}$ has length $\sqrt{2^2 + (-2)^2} = \sqrt{8}$ or $2\sqrt{2} \text{ units}$ $\therefore$ unit vector = $\frac{1}{2\sqrt{2}}(2\mathbf{i} - 2\mathbf{j})$
 $= \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

c $-2\mathbf{i} - 5\mathbf{j}$ has length $\sqrt{(-2)^2 + (-5)^2} = \sqrt{29} \text{ units}$ $\therefore$ unit vector is $\frac{1}{\sqrt{29}}(-2\mathbf{i} - 5\mathbf{j})$

EXERCISE 15H

$$\begin{array}{llll}
 \mathbf{1} & \mathbf{a} & \mathbf{q} \bullet \mathbf{p} & \mathbf{b} & \mathbf{q} \bullet \mathbf{r} & \mathbf{c} & \mathbf{q} \bullet (\mathbf{p} + \mathbf{r}) & \mathbf{d} & 3\mathbf{r} \bullet \mathbf{q} \\
 & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \end{bmatrix} & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 4 \end{bmatrix} & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 6 \end{bmatrix} & & = \begin{bmatrix} -6 \\ 12 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 5 \end{bmatrix} \\
 & & = -3 + 10 & & = 2 + 20 & & = -1 + 30 & & = 6 + 60 \\
 & & = 7 & & = 22 & & = 29 & & = 66
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{e} & 2\mathbf{p} \bullet 2\mathbf{p} & \mathbf{f} & \mathbf{i} \bullet \mathbf{p} & \mathbf{g} & \mathbf{q} \bullet \mathbf{j} & \mathbf{h} & \mathbf{i} \bullet \mathbf{i} \\
 & = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ 4 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \end{bmatrix} & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 & = 36 + 16 & & = 3 + 0 & & = 0 + 5 & & = 1 + 0 \\
 & = 52 & & = 3 & & = 5 & & = 1
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{2} & \mathbf{a} & \begin{bmatrix} 3 \\ t \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0 & \mathbf{b} & \begin{bmatrix} 3 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 10 \\ t \end{bmatrix} = 0 & \mathbf{c} & \begin{bmatrix} t \\ -2 \end{bmatrix} \bullet \begin{bmatrix} t \\ 6 \end{bmatrix} = 0 \\
 & \therefore & -6 + t = 0 & & \therefore & 30 + 5t = 0 & \therefore & t^2 - 12 = 0 \\
 & & \therefore & t = 6 & & \therefore & t = -6 & \therefore & t^2 = 12 \\
 & & & & & & & \therefore & t = \pm\sqrt{12} \\
 & \mathbf{d} & \begin{bmatrix} t \\ t+2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -4 \end{bmatrix} = 0 & \mathbf{e} & \begin{bmatrix} t \\ t+2 \end{bmatrix} \bullet \begin{bmatrix} 2-3t \\ t \end{bmatrix} = 0 \\
 & \therefore & 3t - 4(t+2) = 0 & & \therefore & 2t - 3t^2 + t^2 + 2t = 0 \\
 & \therefore & 3t - 4t - 8 = 0 & & \therefore & -2t^2 + 4t = 0 \\
 & & \therefore & -t = 8 & & \therefore & t^2 - 2t = 0 \\
 & & \therefore & t = -8 & & \therefore & t(t-2) = 0 \\
 & & & & & \therefore & t = 0 \text{ or } 2
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{3} & \mathbf{a} \quad \overrightarrow{AB} = \begin{bmatrix} -2-2 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\
 & \quad \overrightarrow{BC} = \begin{bmatrix} 3-2 \\ 1-5 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \\
 & \quad \overrightarrow{AC} = \begin{bmatrix} 3-2 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 & \text{and } \overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 & \quad = 0 + 0 \\
 & \quad = 0 \\
 & \therefore \angle BAC \text{ is a right angle} \\
 & \text{i.e., } \triangle ABC \text{ is right-angled at B.}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{b} & \overrightarrow{AB} = \begin{bmatrix} 1-4 \\ 2-7 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix} \\
 & \overrightarrow{BC} = \begin{bmatrix} -1-1 \\ 6-2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\
 & \overrightarrow{AC} = \begin{bmatrix} -1-4 \\ 6-7 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \\
 & \text{None of the possible dot products result in 0.} \\
 & \therefore \triangle ABC \text{ is not right-angled.}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{c} & \overrightarrow{AB} = \begin{bmatrix} 5-2 \\ 7-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\
 & \overrightarrow{BC} = \begin{bmatrix} -1-5 \\ -1-7 \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \end{bmatrix} \\
 & \overrightarrow{AC} = \begin{bmatrix} -1-2 \\ -1-2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \\
 & \text{and } \overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ -3 \end{bmatrix} \\
 & \quad = -9 + 9 \\
 & \quad = 0 \\
 & \therefore \angle BAC \text{ is a right angle} \\
 & \text{i.e., } \triangle ABC \text{ is right-angled at A.}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{d} & \overrightarrow{AB} = \begin{bmatrix} 5-10 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \\
 & \overrightarrow{BC} = \begin{bmatrix} 7-5 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
 & \overrightarrow{AC} = \begin{bmatrix} 7-10 \\ 4-1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \\
 & \text{and } \overrightarrow{BC} \bullet \overrightarrow{AC} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 3 \end{bmatrix} \\
 & \quad = -6 + 6 \\
 & \quad = 0 \\
 & \therefore \angle BCA \text{ is a right angle} \\
 & \text{i.e., } \triangle ABC \text{ is right-angled at C.}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{4} & \mathbf{a} \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 5 \end{bmatrix} = -10 + 10 = 0, \text{ so } \begin{bmatrix} -2 \\ 5 \end{bmatrix} \text{ is one such vector.} \\
 & \therefore \text{required vectors have form } k \begin{bmatrix} -2 \\ 5 \end{bmatrix} \text{ where } k \neq 0.
 \end{array}$$

Note: $k \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, $k \neq 0$ is also ok.

b $\begin{bmatrix} -1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2 - 2 = 0$, so $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $k \neq 0$.

c $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 - 3 = 0$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $k \neq 0$.

d $\begin{bmatrix} -4 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \end{bmatrix} = -12 + 12 = 0$, so $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $k \neq 0$.

e $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 + 0 = 0$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $k \neq 0$.

5 a $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos \theta$
 $= 2 \times 5 \times \cos 60^\circ$
 $= 5$

b $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos \theta$
 $= 6 \times 3 \times \cos 120^\circ$
 $= -9$

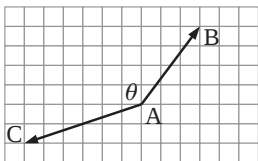
6 a i $\mathbf{a} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

ii $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$
 $\therefore \begin{bmatrix} 4 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \sqrt{4^2 + 4^2} \sqrt{7^2 + (-1)^2} \cos \theta$
 $\therefore 28 - 4 = \sqrt{32} \sqrt{50} \cos \theta$
 $\therefore \frac{24}{\sqrt{32} \sqrt{50}} = \cos \theta$
 $\therefore \theta = \cos^{-1} \left(\frac{24}{\sqrt{32} \sqrt{50}} \right)$
 $\therefore \theta \doteq 53.1^\circ$

b i $\mathbf{a} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$

ii $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$
 $\therefore \begin{bmatrix} -2 \\ 6 \end{bmatrix} \bullet \begin{bmatrix} 9 \\ -1 \end{bmatrix} = \sqrt{(-2)^2 + 6^2} \sqrt{9^2 + (-1)^2} \cos \theta$
 $\therefore -18 - 6 = \sqrt{40} \sqrt{82} \cos \theta$
 $\therefore \cos \theta = \frac{-24}{\sqrt{40} \sqrt{82}}$
 $\therefore \theta = \cos^{-1} \left(\frac{-24}{\sqrt{40} \sqrt{82}} \right)$
 $\therefore \theta \doteq 114.8^\circ$

7 a



b $\vec{AC} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$, $\vec{AB} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$
 $\vec{AC} \bullet \vec{AB} = |\vec{AC}| |\vec{AB}| \cos \theta$
 $\therefore \begin{bmatrix} -6 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \sqrt{(-6)^2 + (-2)^2} \sqrt{3^2 + 4^2} \cos \theta$
 $\therefore -18 - 8 = \sqrt{40} (5) \cos \theta$
 $\therefore \frac{-26}{5\sqrt{40}} = \cos \theta$
 $\therefore \theta = \cos^{-1} \left(\frac{-26}{5\sqrt{40}} \right)$
 $\therefore \theta \doteq 145.3^\circ$

$$\mathbf{8} \quad \mathbf{a} \quad \mathbf{r} \bullet \mathbf{s} = |\mathbf{r}| |\mathbf{s}| \cos \theta$$

$$\therefore \begin{bmatrix} 3 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \sqrt{3^2 + 3^2} \sqrt{(-1)^2 + 2^2} \cos \theta$$

$$\therefore -3 + 6 = \sqrt{18} \sqrt{5} \cos \theta$$

$$\therefore \frac{3}{\sqrt{90}} = \cos \theta$$

$$\therefore \theta = \cos^{-1} \left(\frac{3}{\sqrt{90}} \right)$$

$$\therefore \theta \doteq 71.6^\circ$$

$$\mathbf{c} \quad \mathbf{r} \bullet \mathbf{s} = |\mathbf{r}| |\mathbf{s}| \cos \theta$$

$$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \sqrt{1+1} \sqrt{4+1} \cos \theta$$

$$\therefore 2 + 1 = \sqrt{2} \sqrt{5} \cos \theta$$

$$\therefore \frac{3}{\sqrt{10}} = \cos \theta$$

$$\therefore \theta = \cos^{-1} \left(\frac{3}{\sqrt{10}} \right)$$

$$\therefore \theta \doteq 18.4^\circ$$

$$\mathbf{b} \quad \mathbf{r} \bullet \mathbf{s} = |\mathbf{r}| |\mathbf{s}| \cos \theta$$

$$\therefore \begin{bmatrix} -1 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \sqrt{(-1)^2 + (-3)^2} \sqrt{2^2 + 5^2} \cos \theta$$

$$\therefore -2 - 15 = \sqrt{10} \sqrt{29} \cos \theta$$

$$\therefore \frac{-17}{\sqrt{290}} = \cos \theta$$

$$\therefore \theta = \cos^{-1} \left(\frac{-17}{\sqrt{290}} \right)$$

$$\therefore \theta \doteq 176.6^\circ$$

$$\mathbf{d} \quad \mathbf{r} \bullet \mathbf{s} = |\mathbf{r}| |\mathbf{s}| \cos \theta$$

$$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sqrt{1+0} \sqrt{1+1} \cos \theta$$

$$\therefore 1 + 0 = \sqrt{1} \sqrt{2} \cos \theta$$

$$\therefore \frac{1}{\sqrt{2}} = \cos \theta$$

$$\therefore \theta = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right)$$

$$\therefore \theta = 45^\circ$$

$$\mathbf{9} \quad \mathbf{a} \quad \text{Let Y be } (a, b)$$

$$\text{Now } \overrightarrow{XY} = \overrightarrow{WZ}$$

$$\therefore \begin{bmatrix} a-7 \\ b-12 \end{bmatrix} = \begin{bmatrix} 20-4 \\ 5-3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a-7 \\ b-12 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \end{bmatrix}$$

$$\therefore a = 23, \quad b = 14$$

$$\therefore Y \text{ is } (23, 14).$$

$$\mathbf{c} \quad WX = \sqrt{90} \text{ units}, \quad WZ = \sqrt{260} \text{ units}$$

$$\text{and area} = 2 \left(\frac{1}{2} (WX)(WZ) \sin \theta \right)$$

$$= \sqrt{90} \sqrt{260} \sin 64.44^\circ$$

$$= 138 \text{ units}^2$$

$$\mathbf{b} \quad \overrightarrow{WX} = \begin{bmatrix} 7-4 \\ 12-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$\overrightarrow{WZ} = \begin{bmatrix} 20-4 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \end{bmatrix}$$

$$\overrightarrow{WX} \bullet \overrightarrow{WZ} = |\overrightarrow{WX}| |\overrightarrow{WZ}| \cos \theta$$

$$\therefore \begin{bmatrix} 3 \\ 9 \end{bmatrix} \bullet \begin{bmatrix} 16 \\ 2 \end{bmatrix} = \sqrt{9+81} \sqrt{256+4} \cos \theta$$

$$\therefore 48 + 18 = \sqrt{90} \sqrt{260} \cos \theta$$

$$\therefore \frac{66}{\sqrt{90} \sqrt{260}} = \cos \theta$$

$$\therefore \theta = \cos^{-1} \left(\frac{66}{\sqrt{90} \sqrt{260}} \right)$$

$$\therefore \theta \doteq 64.4^\circ$$

$$\mathbf{10} \quad \mathbf{a} \quad \overrightarrow{BA} = \begin{bmatrix} 3-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 4-1 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\overrightarrow{BA} \bullet \overrightarrow{BC} = |\overrightarrow{BA}| |\overrightarrow{BC}| \cos B$$

$$\therefore \begin{bmatrix} 2 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \sqrt{4+1} \sqrt{9+9} \cos B$$

$$\therefore 6 + 3 = \sqrt{5} \sqrt{18} \cos B$$

$$\therefore \frac{9}{\sqrt{90}} = \cos B$$

$$\therefore B = \cos^{-1} \left(\frac{9}{\sqrt{90}} \right) \doteq 18.4^\circ$$

$$\mathbf{b} \quad \overrightarrow{BA} = \begin{bmatrix} 5-1 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 2-1 \\ 8-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\overrightarrow{BA} \bullet \overrightarrow{BC} = |\overrightarrow{BA}| |\overrightarrow{BC}| \cos B$$

$$\therefore \begin{bmatrix} 4 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \sqrt{16+9} \sqrt{1+25} \cos B$$

$$\therefore 18 - 15 = \sqrt{25} \sqrt{26} \cos B$$

$$\therefore \frac{3}{\sqrt{45} \times 34} = \cos B$$

$$\therefore B \doteq 85.6^\circ$$

$$\mathbf{11} \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{bmatrix} -2-3 \\ 4-1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$\overrightarrow{AC} = \begin{bmatrix} 1-3 \\ 0-1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -5 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \sqrt{25+25} \sqrt{4+1} \cos A$$

$$\therefore 10 + 5 = \sqrt{50} \sqrt{5} \cos A$$

$$\therefore \frac{15}{\sqrt{250}} = \cos A$$

$$\therefore \angle A \doteq 18.4^\circ$$

$$\therefore \angle C \doteq 180^\circ - 18.43^\circ - 8.13^\circ \doteq 153.4^\circ$$

$$\overrightarrow{BC} = \begin{bmatrix} 1-2 \\ 0-4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$$\overrightarrow{BA} = -\begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \sqrt{9+16} \sqrt{25+25} \cos B$$

$$\therefore 15 + 20 = \sqrt{25} \sqrt{50} \cos B$$

$$\therefore 35 = 5\sqrt{50} \cos B$$

$$\therefore \frac{7}{\sqrt{50}} = \cos B$$

$$\therefore \angle B \doteq 8.1^\circ$$

$$\begin{aligned}
 \mathbf{b} \quad \vec{AB} &= \begin{bmatrix} 3-1 \\ -1-4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} & \vec{BC} &= \begin{bmatrix} -1-3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \\
 \vec{AC} &= \begin{bmatrix} -1-1 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} & \vec{BA} &= -\vec{AB} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \\
 \therefore \begin{bmatrix} 2 \\ -5 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ -2 \end{bmatrix} &= \sqrt{4+25}\sqrt{4+4}\cos A & \therefore \begin{bmatrix} -4 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 5 \end{bmatrix} &= \sqrt{16+9}\sqrt{4+25}\cos B \\
 \therefore -4+10 &= \sqrt{29}\sqrt{8}\cos A & \therefore 8+15 &= \sqrt{25}\sqrt{29}\cos B \\
 \therefore \frac{6}{\sqrt{29 \times 8}} &= \cos A & \therefore \frac{23}{5\sqrt{29}} &= \cos B \\
 \therefore \angle A &\doteq 66.8^\circ & \therefore \angle B &\doteq 31.3^\circ \\
 \therefore \angle C &\doteq 180^\circ - 66.80^\circ - 31.33^\circ \doteq 81.9^\circ
 \end{aligned}$$

12 a $x - y = 3$ has gradient $+\frac{1}{1}$ and so has direction vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
 $3x + 2y = 11$ has gradient $-\frac{3}{2}$ and so has direction vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

$$\begin{aligned}
 \therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -3 \end{bmatrix} &= \sqrt{1+1}\sqrt{4+9}\cos\theta \\
 \therefore 2-3 &= \sqrt{2}\sqrt{13}\cos\theta \\
 \therefore \frac{-1}{\sqrt{26}} &= \cos\theta \\
 \therefore \theta &\doteq 101.3 \quad \therefore \text{the angle is } 101.3^\circ \text{ or } 78.7^\circ
 \end{aligned}$$

b $y = x + 2$ has slope $1 = \frac{1}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
 $y = 1 - 3x$ has slope $-3 = \frac{-3}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

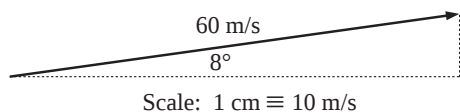
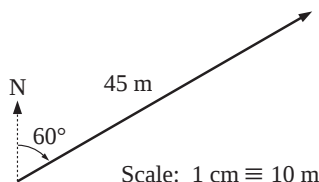
$$\begin{aligned}
 \therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -3 \end{bmatrix} &= \sqrt{1+1}\sqrt{1+9}\cos\theta \\
 \therefore 1-3 &= \sqrt{2}\sqrt{10}\cos\theta \\
 \therefore \frac{-2}{\sqrt{20}} &= \cos\theta \\
 \therefore \theta &\doteq 116.6 \quad \therefore \text{the angle is } 116.6^\circ \text{ or } 63.4^\circ
 \end{aligned}$$

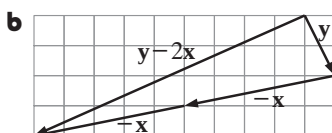
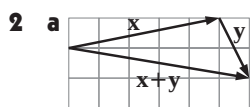
c $y + x = 7$ has slope $-1 = \frac{-1}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
 $x - 3y + 2 = 0$ has slope $\frac{1}{3}$ $\therefore$ direction vector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\begin{aligned}
 \therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \sqrt{1+1}\sqrt{9+1}\cos\theta \\
 \therefore 3-1 &= \sqrt{2}\sqrt{10}\cos\theta \\
 \therefore \frac{2}{\sqrt{20}} &= \cos\theta \\
 \therefore \theta &\doteq 63.4 \quad \therefore \text{the angle is } 63.4^\circ \text{ or } 116.6^\circ
 \end{aligned}$$

d $y = 2 - x$ has slope $-1 = \frac{-1}{1}$ $\therefore$ has direction vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
 $x - 2y = 7$ has slope $\frac{1}{2}$ $\therefore$ has direction vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{aligned}
 \therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \sqrt{1+1}\sqrt{4+1}\cos\theta \\
 \therefore 2-1 &= \sqrt{2}\sqrt{5}\cos\theta \\
 \therefore \cos\theta &= \frac{1}{\sqrt{10}} \\
 \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \doteq 71.6 \quad \therefore \text{the angle is } 71.6^\circ \text{ or } 108.4^\circ
 \end{aligned}$$

REVIEW SET 15A**1 a****b**



3 a $\vec{PR} + \vec{RQ}$
 $= \vec{PQ}$

b $\vec{PS} + \vec{SQ} + \vec{QR}$
 $= \vec{PQ} + \vec{QR}$
 $= \vec{PR}$

4 Dino's first displacement vector is $9 \begin{bmatrix} \cos 246^\circ \\ \sin 246^\circ \end{bmatrix}$, Dino's second displacement vector is $6 \begin{bmatrix} \cos 96^\circ \\ \sin 96^\circ \end{bmatrix}$

$\therefore$ Dino's resultant displacement vector is $\begin{bmatrix} 9 \cos 246^\circ \\ 9 \sin 246^\circ \end{bmatrix} + \begin{bmatrix} 6 \cos 96^\circ \\ 6 \sin 96^\circ \end{bmatrix} \doteq \begin{bmatrix} -4.288 \\ -2.255 \end{bmatrix}$

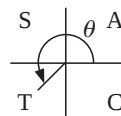
which has length $\sqrt{(-4.288)^2 + (-2.255)^2} \doteq 4.845$

$\therefore$ the resultant displacement vector $= 4.845 \begin{bmatrix} -0.8850 \\ -0.4654 \end{bmatrix} \begin{matrix} \leftarrow \cos \theta \\ \leftarrow \sin \theta \end{matrix}$

If $\cos \theta = -0.8850$ and $\sin \theta = -0.4654$

θ is in Quadrant 3 $\therefore \theta = 180^\circ + \cos^{-1}(0.8850) \doteq 207.7^\circ$

$\therefore$ Dino is 4.845 km from the start at bearing 208° .



5 a $\vec{AB} - \vec{CB}$
 $= \vec{AB} + \vec{BC}$
 $= \vec{AC}$

b $\vec{AB} + \vec{BC} - \vec{DC}$
 $= \vec{AC} + \vec{CD}$
 $= \vec{AD}$

6 a If $\vec{AB} = \frac{1}{2}\vec{CD}$ then
 $AB \parallel CD$ and $AB = \frac{1}{2}(CD)$

b If $\vec{AB} = 2\vec{AC}$ then
 $AB \parallel AC$ and $AB = 2(AC)$
 i.e., A, B and C are collinear and
 $AB = 2(AC)$.

So, C is the midpoint of AB.

7 a $\vec{p} + \vec{r} - \vec{q} = \vec{0}$
 $\therefore \vec{p} + \vec{r} = \vec{q}$

b $\vec{l} + \vec{m} - \vec{n} + \vec{j} - \vec{k} = \vec{0}$
 $\therefore \vec{l} + \vec{m} + \vec{j} = \vec{n} + \vec{k}$

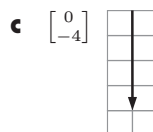
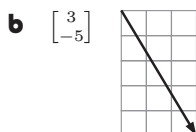
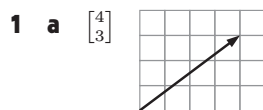
8 a $\vec{OQ} = \vec{OR} + \vec{RQ} = \vec{r} + \vec{q}$

b $\vec{PQ} = \vec{PO} + \vec{OR} + \vec{RQ} = -\vec{p} + \vec{r} + \vec{q}$

c $\vec{ON} = \vec{OR} + \vec{RN} = \vec{r} + \frac{1}{2}\vec{q}$

d $\vec{MN} = \vec{MQ} + \vec{QN}$
 $= \frac{1}{2}\vec{PQ} + \frac{1}{2}\vec{QR}$
 $= \frac{1}{2}(-\vec{p} + \vec{r} + \vec{q}) + \frac{1}{2}(-\vec{q})$
 $= -\frac{1}{2}\vec{p} + \frac{1}{2}\vec{r} + \frac{1}{2}\vec{q} - \frac{1}{2}\vec{q}$
 $= \frac{1}{2}\vec{r} - \frac{1}{2}\vec{p}$

REVIEW SET 15B



2 a $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$

3 a $2\vec{p} + \vec{q}$
 $= 2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$
 $= \begin{bmatrix} -6 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$
 $= \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

b $\vec{q} - 3\vec{r}$
 $= \begin{bmatrix} 2 \\ -4 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
 $= \begin{bmatrix} 2 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \end{bmatrix}$
 $= \begin{bmatrix} -1 \\ -13 \end{bmatrix}$

c $\vec{p} - \vec{q} + \vec{r}$
 $= \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
 $= \begin{bmatrix} -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
 $= \begin{bmatrix} -4 \\ 8 \end{bmatrix}$

4 a $\vec{p} + \vec{q} - \vec{r}$
 $= \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} - \begin{bmatrix 2 \\ -4 \end{bmatrix}$
 $= \begin{bmatrix} -2 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -4 \end{bmatrix}$
 $= \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

b $2\vec{q} - 3\vec{r}$
 $= 2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ -4 \end{bmatrix}$
 $= \begin{bmatrix} 2 \\ -6 \end{bmatrix} - \begin{bmatrix} 6 \\ -12 \end{bmatrix}$
 $= \begin{bmatrix} -4 \\ 6 \end{bmatrix}$

c $\vec{r} + 2\vec{p} - \vec{q}$
 $= \begin{bmatrix} 2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
 $= \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
 $= \begin{bmatrix} -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
 $= \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

$$\begin{aligned}
 \mathbf{5} \quad \overrightarrow{CB} &= \overrightarrow{CA} + \overrightarrow{AB} \\
 &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\
 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \overrightarrow{SP} &= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP} \\
 &= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ} \\
 &= -\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \\ 4 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad |\mathbf{r}| &= \sqrt{4^2 + 1^2} \\
 &= \sqrt{17} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad |\mathbf{s}| &= \sqrt{(-3)^2 + 2^2} \\
 &= \sqrt{13} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \mathbf{r} + \mathbf{s} &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\
 \therefore |\mathbf{r} + \mathbf{s}| &= \sqrt{1^2 + 3^2} \\
 &= \sqrt{10} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad 2\mathbf{s} - \mathbf{r} &= 2\begin{bmatrix} -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} -10 \\ 3 \end{bmatrix} \\
 \therefore |2\mathbf{s} - \mathbf{r}| &= \sqrt{100 + 9} \\
 &= \sqrt{109} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad \overrightarrow{BC} &= 2\overrightarrow{OA} = 2\mathbf{p} \\
 \text{Now } \overrightarrow{AC} &= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{BC} \\
 &= -\mathbf{p} + \mathbf{q} + 2\mathbf{p} \\
 &= \mathbf{p} + \mathbf{q}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\
 &= \mathbf{p} + \frac{1}{2}\overrightarrow{AC} \\
 &= \mathbf{p} + \frac{1}{2}(\mathbf{p} + \mathbf{q}) \\
 &= \frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}
 \end{aligned}$$

REVIEW SET 15C

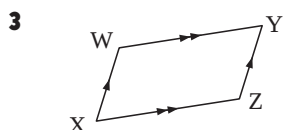
$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \mathbf{p} - 3\mathbf{x} &= \mathbf{0} \\
 \therefore \mathbf{p} &= 3\mathbf{x} \\
 \therefore \frac{1}{3}\mathbf{p} &= \mathbf{x} \\
 \therefore \mathbf{x} &= \begin{bmatrix} -1 \\ \frac{1}{3} \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad 2\mathbf{q} - \mathbf{x} &= \mathbf{r} \\
 \therefore 2\mathbf{q} - \mathbf{r} &= \mathbf{x} \\
 \therefore \mathbf{x} &= 2\begin{bmatrix} 2 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
 \therefore \mathbf{x} &= \begin{bmatrix} 1 \\ -10 \end{bmatrix}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \overrightarrow{PQ} = \begin{bmatrix} 2 & -3 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

$$\mathbf{b} \quad \overrightarrow{PR} = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{aligned}
 \mathbf{c} \quad |\overrightarrow{PR}| &= \sqrt{4^2 + (-2)^2} \\
 &= \sqrt{20} \text{ units}
 \end{aligned}$$

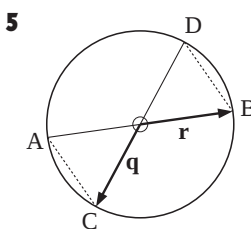


$$\begin{aligned}
 \overrightarrow{WY} &= \begin{bmatrix} 3 & -3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \\
 \overrightarrow{XZ} &= \begin{bmatrix} 4 & -2 \\ 10 & -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}
 \end{aligned}$$

So, $\overrightarrow{WY} = \overrightarrow{XZ}$
 $\therefore$ WY is parallel to XZ and they are equal in length. This is sufficient to deduce that WYXZ is a parallelogram.

$$\begin{aligned}
 \mathbf{4} \quad r \begin{bmatrix} -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -4 \end{bmatrix} &= \begin{bmatrix} 13 \\ -24 \end{bmatrix} \\
 \therefore \begin{bmatrix} -2r+3s \\ r-4s \end{bmatrix} &= \begin{bmatrix} 13 \\ -24 \end{bmatrix} \\
 \therefore \begin{cases} -2r+3s = 13 \\ r-4s = -24 \end{cases} &\quad \times 2 \dots (1) \\
 \therefore \begin{cases} -2r+3s = 13 \\ 2r-8s = -48 \end{cases} & \\
 \hline & -5s = -35 \\
 \therefore s &= 7
 \end{aligned}$$

$$\begin{aligned}
 \text{and in (1)} \quad r - 4(7) &= -24 \\
 \therefore r &= -24 + 28 \\
 \therefore r &= 4 \\
 \text{i.e., } r &= 4 \quad \text{and} \quad s = 7
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{a} \quad \overrightarrow{DB} &= \overrightarrow{DO} + \overrightarrow{OB} \\
 &= \overrightarrow{OC} + \overrightarrow{OB} \\
 &= \mathbf{q} + \mathbf{r} \\
 \mathbf{b} \quad \overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\
 &= \overrightarrow{OB} + \overrightarrow{OC} \\
 &= \mathbf{r} + \mathbf{q}
 \end{aligned}$$

We see that $\overrightarrow{DB} = \overrightarrow{AC}$

$\therefore$ DB is parallel to AC and equal in length.

This is sufficient to deduce that ACBD is a parallelogram.

6 a $\mathbf{p} = -2\mathbf{i} + \mathbf{j} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\therefore |\mathbf{p}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \text{ units}$$

$\therefore$ the required unit vector is

$$\frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{or} \quad \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$$

b The magnitude of $\mathbf{p}$ is $|\mathbf{p}| = \sqrt{5}$ units

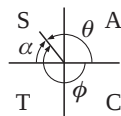
$$\cos \theta = \frac{-2}{\sqrt{5}} \quad \text{and} \quad \sin \theta = \frac{1}{\sqrt{5}}$$

$$\therefore \theta = 180 - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$\therefore \theta \doteq 153^\circ \quad \text{and} \quad \alpha = 27^\circ$$

i.e., the direction of $\mathbf{p}$

$$\text{is } \phi \doteq 270^\circ + 27^\circ = 297^\circ$$



7 a $3\mathbf{x} - \mathbf{y}$

$$= 3(-\mathbf{i} + 3\mathbf{j}) - (-\mathbf{i} - 2\mathbf{j})$$

$$= -3\mathbf{i} + 9\mathbf{j} + \mathbf{i} + 2\mathbf{j}$$

$$= -2\mathbf{i} + 11\mathbf{j}$$

b $|\mathbf{x}|$

$$= \sqrt{(1)^2 + 3^2}$$

$$= \sqrt{10} \text{ units}$$

c $\mathbf{y} - \mathbf{x}$

$$= -\mathbf{i} - 2\mathbf{j} - (-\mathbf{i} + 3\mathbf{j})$$

$$= -\mathbf{i} - 2\mathbf{j} + \mathbf{i} - 3\mathbf{j}$$

$$= -5\mathbf{j}$$

$\therefore$ the unit vector is $-\mathbf{j}$

8 a Is a unit vector if

$$\sqrt{\left(\frac{4}{7}\right)^2 + \left(\frac{1}{k}\right)^2} = 1$$

$$\therefore \frac{16}{49} + \frac{1}{k^2} = 1$$

$$\therefore \frac{1}{k^2} = \frac{33}{49}$$

$$\therefore k = \pm \frac{7}{\sqrt{33}}$$

b Is a unit vector if

$$\sqrt{k^2 + k^2} = 1$$

$$\therefore 2k^2 = 1$$

$$\therefore k^2 = \frac{1}{2}$$

$$\therefore k = \pm \frac{1}{\sqrt{2}}$$

REVIEW SET 15D

1 a $\mathbf{p} \bullet \mathbf{q}$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$= -3 + (-10)$$

$$= -13$$

b $\mathbf{p} - \mathbf{r}$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$\therefore \mathbf{q} \bullet (\mathbf{p} - \mathbf{r})$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$= -6 - 30$$

$$= -36$$

2 LHS = $\mathbf{p} \bullet (\mathbf{q} - \mathbf{r})$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \left(\begin{bmatrix} -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

$$= -9 - 16$$

$$= -25$$

RHS = $\mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= (-6 - 10) - (3 + 6)$$

$$= -16 - 9$$

$$= -25$$

$$\therefore \text{LHS} = \text{RHS} \quad \checkmark$$

3 Since they are perpendicular

$$\begin{bmatrix} 3 \\ 3-2t \end{bmatrix} \bullet \begin{bmatrix} t^2+t \\ -2 \end{bmatrix} = 0$$

$$\therefore 3(t^2+t) - 2(3-2t) = 0$$

$$\therefore 3t^2 + 3t - 6 + 4t = 0$$

$$\therefore 3t^2 + 7t - 6 = 0$$

$$\therefore (3t-2)(t+3) = 0$$

$$\therefore t = \frac{2}{3} \text{ or } -3$$

4

$$\overrightarrow{\text{AB}} = \begin{bmatrix} -1-2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\overrightarrow{\text{AC}} = \begin{bmatrix} 3-2 \\ k-3 \end{bmatrix} = \begin{bmatrix} 1 \\ k-3 \end{bmatrix}$$

Now $\overrightarrow{\text{AB}} \bullet \overrightarrow{\text{AC}} = 0$ {as $\angle \text{BAC} = 90^\circ$ }

$$\therefore \begin{bmatrix} -3 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ k-3 \end{bmatrix} = 0$$

$$\therefore -3 + k - 3 = 0$$

$$\therefore k = 6$$

5 One vector perpendicular to $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as dot product = $-20 + 20$

$$= 0$$

$$\therefore \text{all vectors have form } k \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad k \neq 0$$

6

$$\vec{KL} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{KM} = \begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\text{Now } \vec{KL} \bullet \vec{KM} = |\vec{KL}| |\vec{KM}| \cos K$$

$$\therefore \begin{bmatrix} 5 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \sqrt{25+1} \sqrt{9+16} \cos K$$

$$\therefore 15 - 4 = \sqrt{26} \sqrt{25} \cos K$$

$$\therefore \frac{11}{5\sqrt{26}} = \cos K$$

$$\therefore K = \cos^{-1} \left(\frac{11}{5\sqrt{26}} \right)$$

$$\therefore K = 64.4^\circ$$

$$\vec{LK} = -\vec{KL} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$\vec{LM} = \begin{bmatrix} 1 & -3 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$\text{Now } \vec{LK} \bullet \vec{LM} = |\vec{LK}| |\vec{LM}| \cos L$$

$$\therefore \begin{bmatrix} -5 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \sqrt{25+1} \sqrt{4+25} \cos L$$

$$\therefore 10 + 5 = \sqrt{26} \sqrt{29} \cos L$$

$$\therefore \cos L = \frac{15}{\sqrt{26 \times 29}}$$

$$\therefore L \doteq 56.9^\circ$$

$$\text{and } \angle M \doteq 180^\circ - 56.89^\circ - 64.44^\circ \doteq 58.7^\circ$$

$$7 \quad 4x - 5y = 11 \quad \text{has gradient } \frac{4}{5} \quad \therefore \text{direction vector } \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$2x + 3y = 7 \quad \text{has gradient } -\frac{2}{3} \quad \therefore \text{direction vector is } \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

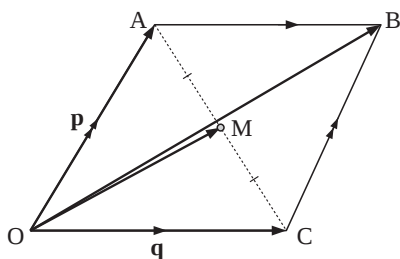
$$\text{If the angle is } \theta, \quad \begin{bmatrix} 5 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \sqrt{5^2 + 4^2} \sqrt{3^2 + (-2)^2} \cos \theta$$

$$\therefore 15 - 8 = \sqrt{41} \sqrt{13} \cos \theta$$

$$\therefore \frac{7}{\sqrt{41 \times 13}} = \cos \theta$$

$$\therefore \theta \doteq 72.3 \quad \therefore \text{the angle is } 72.3^\circ \text{ (or } 107.7^\circ)$$

8 a



i

$$(1) \quad \vec{OB}$$

$$= \vec{OA} + \vec{AB}$$

$$= \vec{OA} + \vec{OC}$$

$$= \mathbf{p} + \mathbf{q}$$

$$(2) \quad \vec{OM}$$

$$= \vec{OA} + \vec{AM}$$

$$= \vec{OA} + \frac{1}{2} \vec{AC}$$

$$= \mathbf{p} + \frac{1}{2} (\vec{AO} + \vec{OC})$$

$$= \mathbf{p} + \frac{1}{2} (-\mathbf{p} + \mathbf{q})$$

$$= \mathbf{p} - \frac{1}{2} \mathbf{p} + \frac{1}{2} \mathbf{q}$$

$$= \frac{1}{2} \mathbf{p} + \frac{1}{2} \mathbf{q}$$

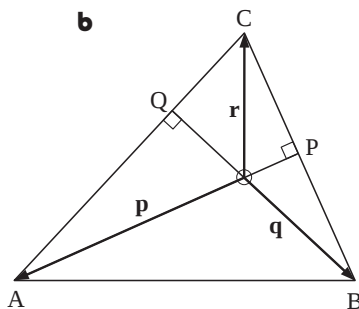
$$\text{ii} \quad \text{We notice that } \vec{OM} = \frac{1}{2} \vec{OB}$$

$$\therefore \vec{OM} \parallel \vec{OB} \quad \text{and} \quad OM = \frac{1}{2} (OB)$$

i.e., O, M and B are collinear (as O is common)

and $\therefore$ M is the midpoint of $\vec{OB}$.

b



$$\text{i} \quad \vec{AC} = \vec{AO} + \vec{OC} \quad \vec{BC} = \vec{BO} + \vec{OC}$$

$$= -\mathbf{p} + \mathbf{r}$$

$$= \mathbf{r} - \mathbf{p}$$

$$= -\mathbf{q} + \mathbf{r}$$

$$= \mathbf{r} - \mathbf{q}$$

ii

$$AP \perp BC$$

$$\therefore \mathbf{p} \perp \mathbf{r} - \mathbf{q}$$

$$\therefore \mathbf{p} \bullet (\mathbf{r} - \mathbf{q}) = 0$$

$$\therefore \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} = 0$$

$$\therefore \mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$$

and

$$BQ \perp AC$$

$$\therefore \mathbf{q} \perp (\mathbf{r} - \mathbf{p})$$

$$\therefore \mathbf{q} \bullet (\mathbf{r} - \mathbf{p}) = 0$$

$$\therefore \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} = 0$$

$$\therefore \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$$

$$\text{iii} \quad \mathbf{r} \bullet \vec{AB} = \mathbf{r} \bullet (-\mathbf{p} + \mathbf{q})$$

$$= -\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q}$$

$$= -\mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q} \quad \{\text{from ii}\}$$

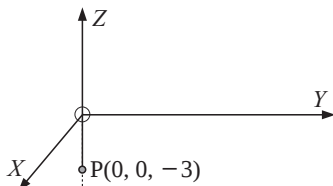
$$= 0 \quad \therefore \mathbf{r} \perp \vec{AB} \quad \text{i.e., } OC \perp AB$$

Chapter 16

VECTORS IN 3-DIMENSIONS

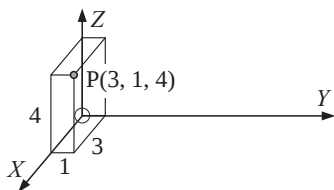
EXERCISE 16A

1 a



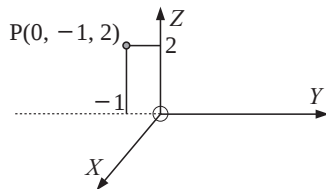
$$OP = \sqrt{0^2 + 0^2 + (-3)^2} = 3 \text{ units}$$

c



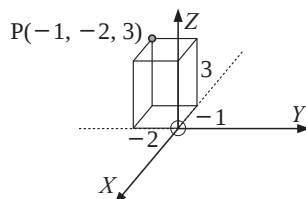
$$\begin{aligned} OP &= \sqrt{3^2 + 1^2 + 4^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

b



$$OP = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5} \text{ units}$$

d



$$\begin{aligned} OP &= \sqrt{(-1)^2 + (-2)^2 + 3^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

2 a i AB

$$\begin{aligned} &= \sqrt{(0 - (-1))^2 + (-1 - 2)^2 + (1 - 3)^2} \\ &= \sqrt{1 + 9 + 4} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{-1+0}{2}, \frac{2-1}{2}, \frac{3+1}{2}\right)$
i.e., $\left(-\frac{1}{2}, \frac{1}{2}, 2\right)$

c i AB

$$\begin{aligned} &= \sqrt{(-1 - 3)^2 + (0 - (-1))^2 + (1 - (-1))^2} \\ &= \sqrt{16 + 1 + 4} \\ &= \sqrt{21} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{3-1}{2}, \frac{-1+0}{2}, \frac{-1+1}{2}\right)$
i.e., $\left(1, -\frac{1}{2}, 0\right)$

b i AB

$$\begin{aligned} &= \sqrt{(2 - 0)^2 + (-1 - 0)^2 + (3 - 0)^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{0+2}{2}, \frac{0-1}{2}, \frac{0+3}{2}\right)$
i.e., $\left(1, -\frac{1}{2}, \frac{3}{2}\right)$

d i AB

$$\begin{aligned} &= \sqrt{(0 - 2)^2 + (1 - 0)^2 + (0 - (-3))^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{2+0}{2}, \frac{0+1}{2}, \frac{-3+0}{2}\right)$
i.e., $\left(1, \frac{1}{2}, -\frac{3}{2}\right)$

3 P(0, 4, 4) Q(2, 6, 5) R(1, 4, 3)

$$\begin{aligned} PQ &= \sqrt{(2-0)^2 + (6-4)^2 + (5-4)^2} \\ &= \sqrt{4+4+1} \\ &= 3 \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(1-2)^2 + (4-6)^2 + (3-5)^2} \\ &= \sqrt{1+4+4} \\ &= 3 \end{aligned}$$

$$\begin{aligned} PR &= \sqrt{(1-0)^2 + (4-4)^2 + (3-4)^2} \\ &= \sqrt{1+0+1} \\ &= \sqrt{2} \end{aligned}$$

$\therefore PQ = QR$ and so $\triangle PQR$ is isosceles.

4 a A(2, -1, 7) B(3, 1, 4) C(5, 4, 5)

$$\begin{aligned} AB &= \sqrt{(3-2)^2 + (1-(-1))^2 + (4-7)^2} & AC &= \sqrt{(5-2)^2 + (4-(-1))^2 + (5-7)^2} \\ &= \sqrt{1+4+9} & &= \sqrt{9+25+4} \\ &= \sqrt{14} & &= \sqrt{38} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(5-3)^2 + (4-1)^2 + (5-4)^2} \\ &= \sqrt{4+9+1} \\ &= \sqrt{14} \end{aligned}$$

 $\therefore$ as $AB = BC$, $\triangle ABC$ is isosceles.**b** A(0, 0, 3) B(2, 8, 1) C(-9, 6, 18)

$$\begin{aligned} AB &= \sqrt{(2-0)^2 + (8-0)^2 + (1-3)^2} & AC &= \sqrt{(-9-0)^2 + (6-0)^2 + (18-3)^2} \\ &= \sqrt{4+64+4} & &= \sqrt{81+36+225} \\ &= \sqrt{72} & &= \sqrt{342} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-9-2)^2 + (6-8)^2 + (18-1)^2} \\ &= \sqrt{121+4+289} \\ &= \sqrt{414} \end{aligned}$$

Since $BC^2 = AB^2 + AC^2$, $\triangle ABC$ is right-angled.**c** A(5, 6, -2) B(6, 12, 9) C(2, 4, 2)

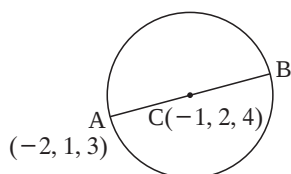
$$\begin{aligned} AB &= \sqrt{(6-5)^2 + (12-6)^2 + (9-(-2))^2} & AC &= \sqrt{(2-5)^2 + (4-6)^2 + (2-(-2))^2} \\ &= \sqrt{1+36+121} & &= \sqrt{9+4+16} \\ &= \sqrt{158} & &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(2-6)^2 + (4-12)^2 + (2-9)^2} \\ &= \sqrt{16+64+49} \\ &= \sqrt{129} \end{aligned}$$

Since $AB^2 = AC^2 + BC^2$, $\triangle ABC$ is right-angled.**d** A(1, 0, -3) B(2, 2, 0) C(4, 6, 6)

$$\begin{aligned} AB &= \sqrt{(2-1)^2 + (2-0)^2 + (0-(-3))^2} & AC &= \sqrt{(4-1)^2 + (6-0)^2 + (6-(-3))^2} \\ &= \sqrt{1^2+2^2+3^2} & &= \sqrt{3^2+6^2+9^2} \\ &= \sqrt{14} & &= \sqrt{126} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(4-2)^2 + (6-2)^2 + (6-0)^2} & &= 3\sqrt{14} \\ &= \sqrt{2^2+4^2+6^2} \\ &= \sqrt{56} \\ &= 2\sqrt{14} \end{aligned}$$

Since $AB + BC = AC$, the points A, B and C lie on a straight line, i.e., do not form a triangle.**5**If B is (a, b, c) then $\frac{a-2}{2} = -1$, $\frac{b+1}{2} = 2$, $\frac{c+3}{2} = 4$ $\therefore a = 0, b = 3, c = 5$ $\therefore$ B is (0, 3, 5)

$$\begin{aligned} r = AC &= \sqrt{(-1-(-2))^2 + (2-1)^2 + (4-3)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

6 a $(0, y, 0)$ for any y

b Distance between $(0, y, 0)$ and $B(-1, -1, 2)$ is $\sqrt{(-1)^2 + (-1 - y)^2 + 2^2}$

$$\therefore \sqrt{1 + (y + 1)^2 + 4} = \sqrt{14}$$

$$\therefore (y + 1)^2 = 9$$

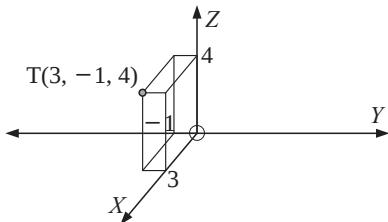
$$\therefore y + 1 = \pm 3$$

$$\therefore y = -1 \pm 3$$

$$\therefore y = -4 \text{ or } 2 \quad \therefore \text{the two points are } (0, -4, 0) \text{ and } (0, 2, 0)$$

EXERCISE 16B.1

1 a



b $\vec{OT} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$

c $OT = \sqrt{(3-0)^2 + (-1-0)^2 + (4-0)^2}$
 $= \sqrt{9 + 1 + 16}$
 $= \sqrt{26} \text{ units}$

2 a $\vec{AB} = \begin{bmatrix} 1 - (-3) \\ 0 - 1 \\ -1 - 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$

$$\vec{BA} = \begin{bmatrix} -3 - 1 \\ 1 - 0 \\ 2 - (-1) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

b $|\vec{AB}| = \sqrt{4^2 + (-1)^2 + (-3)^2}$
 $= \sqrt{26} \text{ units}$

$$|\vec{BA}| = \sqrt{(-4)^2 + 1^2 + 3^2}$$

$$= \sqrt{26} \text{ units}$$

3 $\vec{OA} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ $\vec{OB} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ $\vec{AB} = \begin{bmatrix} -1 - 3 \\ 1 - 1 \\ 2 - 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

4 a The position vector of M from N
 $= \vec{NM}$

$$= \begin{bmatrix} 4 - (-1) \\ -2 - 2 \\ -1 - 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}$$

b The position vector of N from M
 $= \vec{MN}$

$$= \begin{bmatrix} -1 - 4 \\ 2 - (-2) \\ 0 - (-1) \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

c $|\vec{MN}| = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{25 + 16 + 1} = \sqrt{42} \text{ units}$

5 a The position vector of A from O

$$= \vec{OA} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

$$|\vec{OA}| = \sqrt{(-1)^2 + 2^2 + 5^2}$$

$$= \sqrt{1 + 4 + 25}$$

$$= \sqrt{30} \text{ units}$$

b The position vector of C from A

$$= \vec{AC} = \begin{bmatrix} -3 - (-1) \\ 1 - 2 \\ 0 - 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix}$$

$$|\vec{AC}| = \sqrt{(-2)^2 + (-1)^2 + (-5)^2}$$

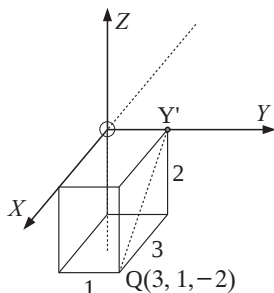
$$= \sqrt{4 + 1 + 25}$$

$$= \sqrt{30} \text{ units}$$

c The position vector of B from C

$$= \vec{CB} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

and $|\vec{CB}| = \sqrt{5^2 + (-1)^2 + 3^2}$
 $= \sqrt{25 + 1 + 9}$
 $= \sqrt{35} \text{ units}$

6

a The distance from Q to the Y-axis is the distance from Q to $Y'(0, 1, 0)$

$$\begin{aligned}\therefore QY' &= \sqrt{(3-0)^2 + (1-1)^2 + (-2-0)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \text{ units}\end{aligned}$$

b The distance from Q to the origin is

$$\begin{aligned}QO &= \sqrt{(3-0)^2 + (1-0)^2 + (-2-0)^2} \\ &= \sqrt{9+1+4} \\ &= \sqrt{14} \text{ units}\end{aligned}$$

c The distance from Q to the ZOY plane is the distance from Q to $(0, 1, -2)$, i.e., 3 units.

EXERCISE 16B.2**1**

a
$$\begin{bmatrix} a-4 \\ b-3 \\ c+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$$

$$\begin{aligned}\therefore a-4 &= 1, \\ b-3 &= 3, \\ c+2 &= -4\end{aligned}$$

$$\therefore a = 5, \quad b = 6, \quad c = -6$$

b

$$\begin{bmatrix} a-5 \\ b-2 \\ c+3 \end{bmatrix} = \begin{bmatrix} 3-a \\ 2-b \\ 5-c \end{bmatrix}$$

$$\begin{aligned}\therefore a-5 &= 3-a, \\ b-2 &= 2-b, \\ c+3 &= 5-c, \\ \therefore 2a &= 8, \quad 2b = 4, \quad 2c = 2 \\ \therefore a &= 4, \quad b = 2, \quad c = 1\end{aligned}$$

2

a
$$2 \begin{bmatrix} 1 \\ 0 \\ 3a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \\ 6a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$$

$$\therefore 6a = 2, \quad b = 2, \quad c-1 = 0 \quad \therefore a = \frac{1}{3}, \quad b = 2, \quad c = 1$$

c

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

$$\therefore a + 2b = -1 \dots\dots (1), \quad a + c = 3 \dots\dots (2) \quad \text{and} \quad -b + c = 3 \dots\dots (3)$$

$$(1) - (2) \text{ gives: } 2b - c = -4 \dots\dots (4)$$

$$\text{Adding (3) and (4) gives } b = -1$$

$$\therefore \text{ using (3), } c = 2$$

$$\text{and using (2), } a = 1$$

$$\therefore a = 1, \quad b = -1, \quad c = 2$$

3

$$A(-1, 3, 4) \quad B(2, 5, -1) \quad C(-1, 2, -2) \quad D(r, s, t)$$

a If $\overrightarrow{AC} = \overrightarrow{BD}$ then
$$\begin{bmatrix} -1 - (-1) \\ 2 - 3 \\ -2 - 4 \end{bmatrix} = \begin{bmatrix} r - 2 \\ s - 5 \\ t + 1 \end{bmatrix}$$

$$\therefore r - 2 = 0, \quad s - 5 = -1 \quad \text{and} \quad t + 1 = -6 \quad \therefore r = 2, \quad s = 4 \quad \text{and} \quad t = -7$$

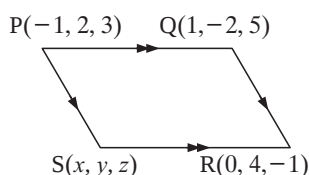
b If $\overrightarrow{AB} = \overrightarrow{DC}$ then
$$\begin{bmatrix} 2 - (-1) \\ 5 - 3 \\ -1 - 4 \end{bmatrix} = \begin{bmatrix} -1 - r \\ 2 - s \\ -2 - t \end{bmatrix}$$

$$\therefore -1 - r = 3, \quad 2 - s = 2 \quad \text{and} \quad -2 - t = -5 \quad \therefore r = -4, \quad s = 0 \quad \text{and} \quad t = 3$$

$$4 \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{bmatrix} 3-1 \\ -3-2 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \quad \text{and} \quad \overrightarrow{DC} = \begin{bmatrix} 7-5 \\ -4-1 \\ 5-6 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$$

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

5 a



Suppose S is at (x, y, z)

$$\overrightarrow{PQ} = \overrightarrow{SR} \quad \{\text{opposite sides are parallel and equal in length}\}$$

$$\therefore \begin{bmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 0-x \\ 4-y \\ -1-z \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -x \\ 4-y \\ -1-z \end{bmatrix}$$

$$\begin{aligned} \therefore -x &= 2 & 4-y &= -4 & -1-z &= 2 \\ \therefore x &= -2 & y &= 8 & z &= -3 \end{aligned}$$

$\therefore$ S is at $(-2, 8, -3)$

b The midpoint of PR is $\left(\frac{-1+0}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right)$ i.e., $(-\frac{1}{2}, 3, 1)$

The midpoint of QS is $\left(\frac{1+(-2)}{2}, \frac{-2+8}{2}, \frac{5+(-3)}{2}\right)$ i.e., $(-\frac{1}{2}, 3, 1)$

i.e., PR and QS have the same midpoint.

EXERCISE 16C

$$1 \quad \mathbf{a} \quad \mathbf{a} + \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \quad \mathbf{b} \quad \mathbf{a} - \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{b} + 2\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix}$$

$$\mathbf{d} \quad \mathbf{a} - 3\mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 10 \end{bmatrix}$$

$$\mathbf{e} \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$$\mathbf{f} \quad \mathbf{c} - \frac{1}{2}\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{7}{2} \end{bmatrix}$$

$$\mathbf{g} \quad \mathbf{a} - \mathbf{b} - \mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$$

$$\mathbf{h} \quad 2\mathbf{b} - \mathbf{c} + \mathbf{a} = 2 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad |\mathbf{a}| &= \sqrt{(-1)^2 + 1^2 + 3^2} \\ &= \sqrt{11} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad |\mathbf{b}| &= \sqrt{1^2 + (-3)^2 + 2^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad |\mathbf{b} + \mathbf{c}| &= \left| \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} \right| = \left| \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix} \right| \\ &= \sqrt{(-1)^2 + (-1)^2 + 6^2} \\ &= \sqrt{1 + 1 + 36} \\ &= \sqrt{38} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad |\mathbf{a} - \mathbf{c}| &= \left| \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} \right| = \left| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right| \\ &= \sqrt{1^2 + (-1)^2 + (-1)^2} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

$$\mathbf{e} \quad |\mathbf{a}|\mathbf{b} = \sqrt{11} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{bmatrix}$$

$$\mathbf{f} \quad \frac{1}{|\mathbf{a}|} \times \mathbf{a} = \frac{1}{\sqrt{11}} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{bmatrix}$$

$$\mathbf{3} \quad \mathbf{a} \quad 2\mathbf{x} + \mathbf{a} = \mathbf{b}$$

$$\therefore 2\mathbf{x} = \mathbf{b} - \mathbf{a}$$

$$\therefore \mathbf{x} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

$$\mathbf{b} \quad \mathbf{b} - 3\mathbf{x} = 2\mathbf{a}$$

$$\therefore \mathbf{b} - 2\mathbf{a} = 3\mathbf{x}$$

$$\therefore \mathbf{x} = \frac{1}{3}(\mathbf{b} - 2\mathbf{a})$$

$$\mathbf{c} \quad \mathbf{a} + 2\mathbf{x} = \mathbf{b} - \mathbf{x}$$

$$\therefore 3\mathbf{x} = \mathbf{b} - \mathbf{a}$$

$$\therefore \mathbf{x} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$$

$$\mathbf{4} \quad \mathbf{a} \quad 2\mathbf{a} + \mathbf{x} = \mathbf{b}$$

$$\therefore \mathbf{x} = \mathbf{b} - 2\mathbf{a} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -5 \end{bmatrix}$$

$$\mathbf{b} \quad 3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$$

$$\therefore 3\mathbf{x} = \mathbf{a} + 2\mathbf{b}$$

$$\therefore \mathbf{x} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b}) = \frac{1}{3} \left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right) = \frac{1}{3} \left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} \right)$$

$$\therefore \mathbf{x} = \frac{1}{3} \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\mathbf{c} \quad 2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$$

$$\therefore \mathbf{a} + 2\mathbf{b} = 2\mathbf{x}$$

$$\therefore \mathbf{x} = \frac{1}{2}(\mathbf{a} + 2\mathbf{b}) = \frac{1}{2} \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \quad \{\text{using } \mathbf{b}\} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{bmatrix}$$

$$\mathbf{5} \quad \overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{AO}} + \overrightarrow{\mathbf{OB}} = -\overrightarrow{\mathbf{OA}} + \overrightarrow{\mathbf{OB}} = - \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

$$\therefore |\overrightarrow{\mathbf{AB}}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9 + 16 + 4} = \sqrt{29} \text{ units}$$

$$\mathbf{6} \quad \overrightarrow{\mathbf{OA}} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \overrightarrow{\mathbf{OB}} = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}, \quad \overrightarrow{\mathbf{OC}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \overrightarrow{\mathbf{OD}} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$$

$$\therefore \overrightarrow{\mathbf{BD}} = \overrightarrow{\mathbf{BO}} + \overrightarrow{\mathbf{OD}} = -\overrightarrow{\mathbf{OB}} + \overrightarrow{\mathbf{OD}} = - \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix}$$

$$\text{and } \overrightarrow{\mathbf{AC}} = \overrightarrow{\mathbf{AO}} + \overrightarrow{\mathbf{OC}} = - \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \quad \therefore \overrightarrow{\mathbf{BD}} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = 2\overrightarrow{\mathbf{AC}}$$

$$\begin{array}{lll}
 \mathbf{7} \quad \mathbf{a} \quad \overrightarrow{BD} = \frac{1}{2}\overrightarrow{OA} & \mathbf{b} \quad \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} & \mathbf{c} \quad \overrightarrow{BA} = -\overrightarrow{AB} \\
 = \frac{1}{2}\mathbf{a} & = -\mathbf{a} + \mathbf{b} & = -(\mathbf{b} - \mathbf{a}) \\
 & = \mathbf{b} - \mathbf{a} & = -\mathbf{b} + \mathbf{a} \text{ or } \mathbf{a} - \mathbf{b} \\
 \mathbf{d} \quad \overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} & \mathbf{e} \quad \overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} & \mathbf{f} \quad \overrightarrow{DA} = -\overrightarrow{AD} \\
 = \mathbf{b} + \frac{1}{2}\mathbf{a} & = -\mathbf{a} + \mathbf{b} + \frac{1}{2}\mathbf{a} & = \frac{1}{2}\mathbf{a} - \mathbf{b} \\
 & = -\frac{1}{2}\mathbf{a} + \mathbf{b} &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{8} \quad \mathbf{a} \quad \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} \\
 \mathbf{b} \quad \overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = -\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix} \\
 \mathbf{c} \quad \overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \quad \{\text{using } \mathbf{b}\} = \begin{bmatrix} -3 \\ 6 \\ -5 \end{bmatrix}
 \end{array}$$

EXERCISE 16D

$$\begin{array}{ll}
 \mathbf{1} \quad \text{Since } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel, then } \mathbf{b} = k\mathbf{a} & \therefore \begin{bmatrix} -6 \\ r \\ s \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ 3k \end{bmatrix} \\
 \therefore 2k = -6, \quad r = -k, \quad s = 3k & \therefore k = -3, \quad r = 3, \quad s = -9
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{2} \quad \text{If } \begin{bmatrix} a \\ 2 \\ b \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \text{ are parallel, then } \begin{bmatrix} a \\ 2 \\ b \end{bmatrix} = k \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\
 \text{i.e., } 2 = -k, \quad a = 3k, \quad b = 2k & \therefore k = -2, \quad a = -6 \quad \text{and} \quad b = -4
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{3} \quad \mathbf{a} \quad \text{Let the vector parallel to } \mathbf{a} \text{ be } k\mathbf{a} & \mathbf{b} \quad \text{Let the vector parallel to } \mathbf{b} \text{ be } k\mathbf{b} \\
 \text{i.e., } k\mathbf{a} = k \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ -2k \end{bmatrix} & \text{i.e., } k\mathbf{b} = k \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ 2k \end{bmatrix} \\
 k\mathbf{a} \text{ has length} = 1, & k\mathbf{b} \text{ has length} = 2, \\
 \text{so } \sqrt{(2k)^2 + (-k)^2 + (-2k)^2} = 1 & \text{so } \sqrt{(-2k)^2 + (-k)^2 + (2k)^2} = 2 \\
 \therefore 4k^2 + k^2 + 4k^2 = 1 & \therefore 4k^2 + k^2 + 4k^2 = 4 \\
 \therefore 9k^2 = 1 & \therefore 9k^2 = 4 \\
 \therefore k^2 = \frac{1}{9} & \therefore k^2 = \frac{4}{9} \\
 \therefore k = \pm\frac{1}{3} & \therefore k = \pm\frac{2}{3}
 \end{array}$$

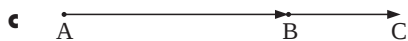
Choosing $k = \frac{1}{3}$,

the vector is $\begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$

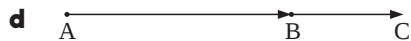
Choosing $k = \frac{2}{3}$,

the vector is $\begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$

$$\begin{array}{ll}
 \mathbf{4} \quad \mathbf{a} \quad \overrightarrow{AB} = 3\overrightarrow{CD} \text{ means that} & \mathbf{b} \quad \overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL} \text{ means that} \\
 \overrightarrow{AB} \text{ is parallel to } \overrightarrow{CD} \text{ and 3 times its} & \overrightarrow{RS} \text{ is parallel to } \overrightarrow{KL}, \text{ half its length} \\
 \text{length.} & \text{and in the opposite direction.}
 \end{array}$$



$\vec{AB} = 2\vec{BC}$ means that A, B and C are collinear and the length of $\vec{AB}$ is twice the length of $\vec{BC}$ (i.e., B cuts $\vec{AC}$ internally in the ratio 2 : 1).



$\vec{BC} = \frac{1}{3}\vec{AC}$ means that A, B and C are collinear and the length of $\vec{BC}$ is one third the length of $\vec{AC}$ (i.e., B cuts $\vec{AC}$ internally in the ratio 2 : 1).

5 $\vec{OP} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\vec{OQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\vec{OR} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\vec{OS} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$

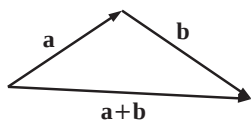
a $\vec{PR} = \vec{PO} + \vec{OR} = -\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$ and

$\vec{QS} = \vec{QO} + \vec{OS} = -\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix}$

$\therefore \vec{QS} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = 2\vec{PR}$ and so $QS \parallel PR$

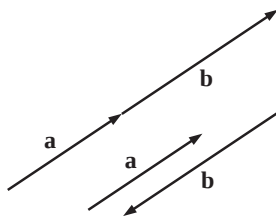
b As $\vec{QS} = 2\vec{PR}$ then $|\vec{QS}| = 2|\vec{PR}|$, i.e., QS is twice as long as PR.

6 a Consider **a** not parallel to **b**:



Clearly, $|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$

b Consider **a** parallel to **b**:



$|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

or

$|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$

c If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{b}$

$\therefore |\mathbf{a}| + |\mathbf{b}| = 0 + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

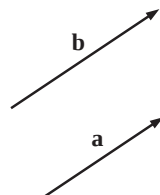
Similarly if $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{a}$

$\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}| + 0 = |\mathbf{a}| = |\mathbf{a} + \mathbf{b}|$

If $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{0}$

$\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

Combining all possibilities, $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$ i.e., $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$



EXERCISE 16E

1 a $\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore |\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{3}$ units

b $3\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$\therefore |3\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{9 + 1 + 1} = \sqrt{11}$ units

c $\mathbf{i} - 5\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$

$\therefore |\mathbf{i} - 5\mathbf{k}| = \sqrt{1 + 25} = \sqrt{26}$ units

d $\frac{1}{2}(\mathbf{j} + \mathbf{k}) = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\therefore \left| \frac{1}{2}(\mathbf{j} + \mathbf{k}) \right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ units

$$2 \quad \mathbf{a} \quad \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

$$\mathbf{c} \quad \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} = -3\mathbf{k}$$

$$3 \quad \mathbf{a} \quad \left| \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right| = \sqrt{2} \text{ units}$$

$$\mathbf{c} \quad |\mathbf{i} - \mathbf{j} + \mathbf{k}| = \left| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right| = \sqrt{3} \text{ units}$$

$$4 \quad \mathbf{a} \quad \begin{aligned} \mathbf{a} + \mathbf{b} \\ = (-\mathbf{i} + \mathbf{j} + \mathbf{k}) + (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ = 2\mathbf{i} + 3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2\mathbf{a} + 5\mathbf{b} \\ = 2(-\mathbf{i} + \mathbf{j} + \mathbf{k}) + 5(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + 15\mathbf{i} - 5\mathbf{j} + 10\mathbf{k} \\ = 13\mathbf{i} - 3\mathbf{j} + 12\mathbf{k} \end{aligned}$$

$$5 \quad \mathbf{a} \quad \mathbf{i} + 2\mathbf{j} + \mathbf{k} \text{ has length} \\ \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \text{ units} \\ \therefore \text{ the unit vector is } \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{b} \quad \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} = 2\mathbf{i} - 5\mathbf{k}$$

$$\mathbf{d} \quad \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = -3\mathbf{i} + \mathbf{k}$$

$$\begin{aligned} \mathbf{b} \quad \left| \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right| &= \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}} \\ &= \sqrt{1} \\ &= 1 \text{ unit} \end{aligned}$$

$$\mathbf{d} \quad |2\mathbf{i} - \mathbf{j}| = \left| \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right| = \sqrt{5} \text{ units}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{b} - \mathbf{a} \\ = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (-\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mathbf{i} - \mathbf{j} - \mathbf{k} \\ = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 3\mathbf{a} - 2\mathbf{b} \\ = 3(-\mathbf{i} + \mathbf{j} + \mathbf{k}) - 2(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ = -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} - 6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \\ = -9\mathbf{i} + 5\mathbf{j} - \mathbf{k} \end{aligned}$$

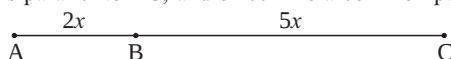
$$\begin{aligned} \mathbf{b} \quad 2\mathbf{i} - \mathbf{j} - 3\mathbf{k} \text{ has length} \\ \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{14} \text{ units} \\ \therefore \text{ the unit vector is } \frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \end{aligned}$$

EXERCISE 16F

$$1 \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{bmatrix} 4 - (-2) \\ 3 - 1 \\ 0 - 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \quad \text{and}$$

$$\overrightarrow{BC} = \begin{bmatrix} 19 - 4 \\ 8 - 3 \\ -10 - 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ -10 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$\therefore \overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, and since B is a common point, A, B and C are collinear.

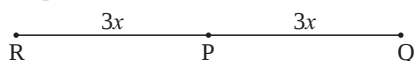


A divides CB in the ratio 7 : 2 externally.

$$\mathbf{b} \quad \overrightarrow{RP} = \begin{bmatrix} 2 - (-1) \\ 1 - 7 \\ 1 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \quad \text{and}$$

$$\overrightarrow{PQ} = \begin{bmatrix} 5 - 2 \\ -5 - 1 \\ -2 - 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore \overrightarrow{RP}$ is parallel to $\overrightarrow{PQ}$, and since P is a common point, P, Q and R are collinear.



$\therefore Q$ divides PR in the ratio 1 : 2 externally.

2 a

$$\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \\ C(-13, a, b) \quad A(2, -3, 4) \quad B(11, -9, 7) \end{array}$$

Since A, B and C are collinear, $\overrightarrow{CA}$ is parallel to $\overrightarrow{AB}$.

$$\therefore \begin{bmatrix} 15 \\ -3-a \\ 4-b \end{bmatrix} = k \begin{bmatrix} 9 \\ -6 \\ 3 \end{bmatrix}$$

$$\therefore 9k = 15,$$

$$-3-a = -6k$$

$$\text{and } 4-b = 3k$$

$$\therefore k = \frac{5}{3},$$

$$\text{and so } a = -3 + 6k$$

$$\therefore a = -3 + 10$$

$$\text{i.e., } a = 7$$

$$\text{and } b = 4 - 3k = 4 - 5 = -1$$

b

$$\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \\ L(4, -3, 7) \quad K(1, -1, 0) \quad M(a, 2, b) \end{array}$$

Since K, L and M are collinear, $\overrightarrow{LK}$ is parallel to $\overrightarrow{KM}$.

$$\therefore \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix} = k \begin{bmatrix} a-1 \\ 3 \\ b \end{bmatrix}$$

$$\therefore 3k = 2,$$

$$k(a-1) = -3$$

$$\text{and } kb = -7$$

$$\therefore k = \frac{2}{3},$$

$$a-1 = -\frac{3}{k} = -\frac{9}{2}$$

$$\text{i.e., } a = -\frac{7}{2}$$

$$\text{and } b = -\frac{7}{k} = -\frac{21}{2}$$

EXERCISE 16G**1 a**

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\ &= 2(-1) + 1(1) + 3(1) \\ &= -2 + 1 + 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad |\mathbf{a}|^2 &= (\sqrt{2^2 + 1^2 + 3^2})^2 \\ &= 14 \end{aligned}$$

e

$$\begin{aligned} \mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \\ &= 2(-1) + 1(0) + 3(2) \\ &= 4 \end{aligned}$$

2 a

$$\begin{aligned} (\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k}) &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \\ &= 1(0) + 1(2) - 1(1) \\ &= 0 + 2 - 1 \\ &= 1 \end{aligned}$$

b

$$\begin{aligned} \mathbf{b} \bullet \mathbf{a} &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= (-1)(2) + 1(1) + 1(3) \\ &= -2 + 1 + 3 \\ &= 2 \end{aligned}$$

d

$$\begin{aligned} \mathbf{a} \bullet \mathbf{a} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= 2(2) + 1(1) + 3(3) \\ &= 14 \end{aligned}$$

f

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} &= 2 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \{\text{using } \mathbf{a}\} \\ &= 2 + 2(0) + 1(-1) + 3(1) \\ &= 4 \end{aligned}$$

b

$$\begin{aligned} \mathbf{i} \bullet \mathbf{i} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= 1 \end{aligned}$$

c

$$\begin{aligned} \mathbf{i} \bullet \mathbf{j} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned}
 \mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \\
 &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix} \\
 &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
 &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\
 &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
 &= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) &= \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d} \quad \text{and if we let } \mathbf{p} = \mathbf{a} + \mathbf{b}, \\
 \text{then } (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) &= \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) \\
 &= \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d} \\
 &= (\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} + (\mathbf{a} + \mathbf{b}) \bullet \mathbf{d} \\
 &= \mathbf{c} \bullet (\mathbf{a} + \mathbf{b}) + \mathbf{d} \bullet (\mathbf{a} + \mathbf{b}) \\
 &= \mathbf{c} \bullet \mathbf{a} + \mathbf{c} \bullet \mathbf{b} + \mathbf{d} \bullet \mathbf{a} + \mathbf{d} \bullet \mathbf{b} \\
 &= \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{d}
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{Since the vectors are perpendicular, } \begin{bmatrix} 3 \\ -1 \\ t \end{bmatrix} \bullet \begin{bmatrix} 2t \\ -3 \\ -4 \end{bmatrix} &= 0 \\
 \therefore 3(2t) + (-1)(-3) + t(-4) &= 0 \\
 \therefore 6t + 3 - 4t &= 0 \\
 \therefore 2t + 3 &= 0 \\
 \therefore t &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{array}{lll}
 5 \quad \mathbf{a} \bullet \mathbf{b} & \mathbf{b} \bullet \mathbf{c} & \mathbf{a} \bullet \mathbf{c} \\
 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} & = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} & = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \\
 = 3(-1) + 1(1) + 2(1) & = (-1)(1) + 1(5) + 1(-4) & = (3)(1) + 1(5) + 2(-4) \\
 = 0 & = 0 & = 0
 \end{array}$$

$\therefore \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular.

$$\begin{array}{ll}
 6 \quad \mathbf{a} \quad \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} & \mathbf{b} \quad \begin{bmatrix} 3 \\ t \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1-t \\ -3 \\ 4 \end{bmatrix} = 0 \\
 = 1(2) + 1(3) + 5(-1) & \therefore 3(1-t) + t(-3) + (-2)4 = 0 \\
 = 0 & \therefore 3 - 3t - 3t - 8 = 0 \\
 & \therefore -6t = 5 \\
 \therefore \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ are perpendicular} & \therefore t = -\frac{5}{6}
 \end{array}$$

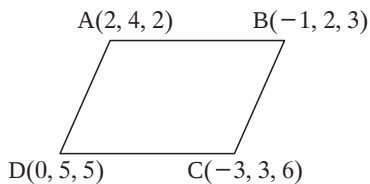
7 We have three points: A(5, 1, 2) B(6, -1, 0) C(3, 2, 0)

$$\text{Then } \overrightarrow{AB} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \quad \overrightarrow{AC} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BC} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}$$

$$\text{Now } \overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = (-2) + (-2) + 4 = 0$$

$\therefore \overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle ABC$ is right-angled.

8



$$\mathbf{a} \quad \overrightarrow{AB} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \quad \overrightarrow{BC} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \quad \therefore \overrightarrow{AB} \text{ is parallel to } \overrightarrow{DC} \text{ and } \overrightarrow{BC} \text{ is parallel to } \overrightarrow{AD}.$$

$\therefore$ ABCD is a parallelogram.

$$\overrightarrow{DC} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \quad \overrightarrow{AD} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$\mathbf{b} \quad |\overrightarrow{AB}| = \sqrt{14} \text{ units} \quad \text{and} \quad |\overrightarrow{BC}| = \sqrt{14} \text{ units} \quad \therefore \text{ABCD is a rhombus.}$$

$$\mathbf{c} \quad \overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{bmatrix} -5 \\ -1 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = (-5) \times 1 + (-1) \times 3 + 4(2) = 0$$

$\therefore \overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$ which illustrates that the diagonals of a rhombus are perpendicular.

$$\mathbf{9} \quad \mathbf{a} \quad \mathbf{a} = -\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

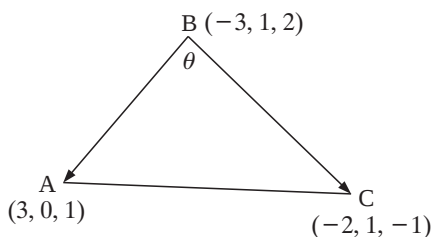
$$\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= (-1) + (-1) + 1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cos \theta &= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{-1}{\sqrt{3} \times \sqrt{3}} \\ &= -\frac{1}{3} \\ \therefore \theta &\doteq 109.5^\circ \end{aligned}$$

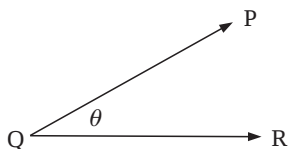
10 Given A(3, 0, 1), B(-3, 1, 2) and C(-2, 1, -1),

$$\overrightarrow{BC} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BA} = \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix}$$



$$\begin{aligned} \therefore \cos \theta &= \frac{\overrightarrow{BC} \bullet \overrightarrow{BA}}{|\overrightarrow{BC}| |\overrightarrow{BA}|} \\ &= \frac{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix}}{\sqrt{1+9} \times \sqrt{36+1+1}} \\ &= \frac{6+0+3}{\sqrt{10} \times \sqrt{38}} \\ &= \frac{9}{\sqrt{380}} \quad \text{and so } \theta \doteq 62.5^\circ \end{aligned}$$

If $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used we would find the exterior angle of the triangle at B, i.e., 117.5° .

11 a

$$\text{Now } \vec{QP} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{QR} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \cos \theta &= \frac{\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}}{\sqrt{1+16} \times \sqrt{4+9+9}} \\ &= \frac{2+0+12}{\sqrt{17} \times \sqrt{22}} \\ &= \frac{14}{\sqrt{374}} \end{aligned}$$

$$\therefore \theta = 43.6^\circ$$

$$\text{c } \vec{QP} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \quad \vec{QR} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \cos \theta &= \frac{\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}}{\sqrt{16+4} \times \sqrt{9+9+9}} \\ &= \frac{-12+0+6}{\sqrt{20} \times \sqrt{27}} \\ &= \frac{-6}{\sqrt{540}} \end{aligned}$$

$$\therefore \theta = 105.0^\circ$$

$$\text{b } \vec{QP} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \quad \vec{QR} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$$

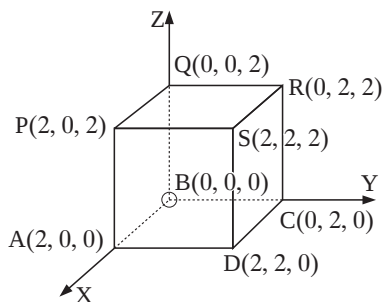
$$\begin{aligned} \therefore \cos \theta &= \frac{\sqrt{1+9} \times \sqrt{4+1+25}}{2+15} \\ &= \frac{\sqrt{10} \times \sqrt{30}}{17} \\ &= \frac{17}{\sqrt{300}} \end{aligned}$$

$$\therefore \theta = 11.0^\circ$$

$$\text{d } \vec{QP} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \vec{QR} = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$$

$$\begin{aligned} \cos \theta &= \frac{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}}{\sqrt{4+1} \times \sqrt{1+4+16}} \\ &= \frac{-2+2}{\sqrt{5} \times \sqrt{21}} \\ &= 0 \end{aligned}$$

$$\therefore \theta = 90^\circ$$

12**a** Suppose the origin is at B.

$$\text{Now } \vec{BA} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{BS} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \therefore \vec{BA} \cdot \vec{BS} &= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \\ &= 4+0+0 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \therefore \cos \angle ABS &= \frac{4}{\sqrt{4+0+0} \times \sqrt{4+4+4}} \\ &= \frac{4}{2 \times 2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\therefore \angle ABS \doteq 54.7^\circ$$

b Consider vectors away from B.

$$\overrightarrow{BR} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BP} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{BR} \bullet \overrightarrow{BP} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$= 0 + 0 + 4 = 4$$

$$\therefore \cos \angle RBP = \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}}$$

$$= \frac{4}{\sqrt{8} \times \sqrt{8}}$$

$$= \frac{1}{2} \quad \text{and so} \quad \angle RBP = 60^\circ$$

$$\mathbf{c} \quad \overrightarrow{BP} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BS} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \overrightarrow{BP} \bullet \overrightarrow{BS} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$= 4 + 0 + 4$$

$$= 8$$

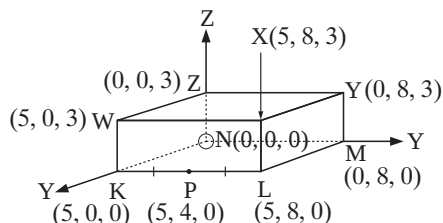
$$\therefore \cos \angle PBS = \frac{8}{\sqrt{4+4}\sqrt{4+4+4}}$$

$$= \frac{8}{\sqrt{96}}$$

$$\therefore \angle PBS \doteq 35.3^\circ$$

13 Suppose the origin is at N.

a



$$\overrightarrow{NY} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \quad \text{and} \quad \overrightarrow{NX} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$$\overrightarrow{NY} \bullet \overrightarrow{NX} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$$= 0 + 64 + 9$$

$$= 73$$

$$\therefore \cos \angle YNX = \frac{73}{\sqrt{64+9}\sqrt{25+64+9}} = \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\frac{73}{98}}$$

$$\therefore \angle YNX \doteq 30.3^\circ$$

$$\mathbf{b} \quad \overrightarrow{NY} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \quad \text{and} \quad \overrightarrow{NP} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

$$\overrightarrow{NY} \bullet \overrightarrow{NP} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

$$= 0 + 32 + 0$$

$$= 32$$

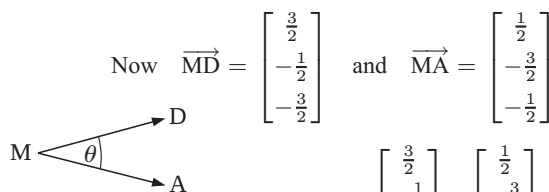
$$\therefore \cos \angle YNP = \frac{32}{\sqrt{64+9}\sqrt{25+16}}$$

$$= \frac{32}{\sqrt{73}\sqrt{41}}$$

$$\therefore \angle YNP \doteq 54.2^\circ$$

14 a M is the midpoint of BC $\therefore$ M is at $\left(\frac{2+1}{2}, \frac{2+3}{2}, \frac{2+1}{2}\right)$ i.e., $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$

b



$$\therefore \cos \theta = \frac{\overrightarrow{MD} \bullet \overrightarrow{MA}}{|\overrightarrow{MD}| |\overrightarrow{MA}|} = \frac{\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \bullet \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4} + \frac{9}{4}} \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}}$$

$$\therefore \cos \theta = \frac{\frac{3}{4} + \frac{3}{4} + \frac{3}{4}}{\sqrt{\frac{19}{4}} \sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\frac{\sqrt{201}}{4}} = \frac{9}{\sqrt{201}} \quad \text{and so} \quad \theta \doteq 51.5^\circ$$

$$\begin{aligned}
 15 \quad \mathbf{a} \quad \begin{bmatrix} 2 \\ t \\ t-2 \end{bmatrix} \bullet \begin{bmatrix} t \\ 3 \\ t \end{bmatrix} &= 0 \quad \therefore 2t + 3t + t(t-2) = 0 \\
 &\therefore 5t + t^2 - 2t = 0 \\
 &\therefore t^2 + 3t = 0 \\
 &\therefore t(t+3) = 0 \quad \text{and so } t = 0 \quad \text{or } t = -3
 \end{aligned}$$

$$\mathbf{b} \quad \text{Given that } \mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ r \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} \quad \text{are mutually perpendicular}$$

$$\therefore \mathbf{a} \bullet \mathbf{b} = 0, \quad \mathbf{b} \bullet \mathbf{c} = 0 \quad \text{and} \quad \mathbf{a} \bullet \mathbf{c} = 0$$

$$\begin{aligned}
 \therefore \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 2 \\ r \end{bmatrix} &= 0 & \therefore 2 + 4 + 3r &= 0 \\
 & & \therefore 3r &= -6 \\
 & & \therefore r &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{and} \quad \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} &= 0 & \therefore 2s + 2t - 2 &= 0 \\
 & & \therefore s + t &= 1 \quad \text{.....(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} &= 0 & \therefore s + 2t + 3 &= 0 \\
 & & \therefore s + 2t &= -3 \quad \text{.....(2)}
 \end{aligned}$$

$$(2) - (1) \text{ gives } t = -4 \quad \text{and so} \quad s = 5 \quad \text{i.e., } r = -2, \quad s = 5 \quad \text{and} \quad t = -4$$

16 a Choose any vector in the direction of

$$\text{the } X\text{-axis, e.g., } \mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned}
 \text{Then } \cos \theta &= \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|}} \\
 &= \frac{1}{\sqrt{1}\sqrt{1+4+9}} \\
 &= \frac{1}{\sqrt{14}} \quad \text{and so } \theta = 74.5^\circ
 \end{aligned}$$

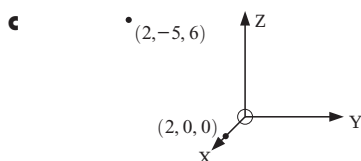
b A line parallel to the Y-axis is $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$\begin{aligned}
 \text{Then } \cos \theta &= \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}}{\sqrt{1}\sqrt{1+1+9}} \\
 &= \frac{1}{\sqrt{11}} \\
 \therefore \theta &\doteq 72.45^\circ
 \end{aligned}$$

17 $\mathbf{a} \bullet \mathbf{b}$ is a scalar, i.e., a number, so $\mathbf{a} \bullet \mathbf{b}$ dotted with $\mathbf{c}$ is not a vector dotted with a vector.
 $\therefore \mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ does not exist.

REVIEW SET 16A

$$1 \quad \mathbf{a} \quad \overrightarrow{PQ} = \begin{bmatrix} -1-2 \\ 7-(-5) \\ 9-6 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ 3 \end{bmatrix}$$



$$\begin{aligned}
 \mathbf{b} \quad PQ &= \sqrt{(-3)^2 + 12^2 + 3^2} \\
 &= \sqrt{162} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{distance} &= \sqrt{(2-2)^2 + (0-(-5))^2 + (0-6)^2} \\
 &= \sqrt{0+25+36} \\
 &= \sqrt{61} \text{ units}
 \end{aligned}$$

$$2 \quad \mathbf{a} \quad \mathbf{m} - \mathbf{n} + \mathbf{p} = \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 11 \end{bmatrix}$$

$$\mathbf{b} \quad 2\mathbf{n} - 3\mathbf{p} = 2 \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix} - \begin{bmatrix} -3 \\ 9 \\ 18 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -26 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{m} + \mathbf{p} = \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix} \quad \therefore |\mathbf{m} + \mathbf{p}| = \sqrt{25 + 0 + 49} = \sqrt{74} \text{ units}$$

$$3 \quad \overrightarrow{\text{CB}} = \overrightarrow{\text{CA}} + \overrightarrow{\text{AB}} = -\overrightarrow{\text{AC}} + \overrightarrow{\text{AB}} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 7 \end{bmatrix}$$

$$4 \quad \text{The vectors are parallel} \quad \therefore \begin{bmatrix} -12 \\ -20 \\ 2 \end{bmatrix} = k \begin{bmatrix} 3 \\ m \\ n \end{bmatrix} \quad \therefore \begin{aligned} 3k &= -12, & km &= -20, & kn &= 2 \\ \therefore k &= -4, & m &= 5, & n &= -\frac{1}{2} \end{aligned}$$

$$5 \quad \overrightarrow{\text{PQ}} = \begin{bmatrix} 4 - 6 \\ 6 - 8 \\ 8 - 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \quad \therefore \text{Both } \overrightarrow{\text{PQ}} \text{ and } \overrightarrow{\text{QR}} \text{ are parallel to } \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$\overrightarrow{\text{QR}} = \begin{bmatrix} 19 - 4 \\ 3 - 6 \\ 17 - 8 \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \quad \therefore \text{PQ} \parallel \text{QR with Q common to both.}$$

$\therefore$ P, Q, R are collinear.

$$\overrightarrow{\text{PQ}} : \overrightarrow{\text{QR}} = 2 \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} : 3 \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} = 2 : 3$$

$\therefore$ Q divides PR internally in the ratio 2 : 3.

6 As the vectors are perpendiculars,

$$\begin{aligned} \begin{bmatrix} -4 \\ t+2 \\ t \end{bmatrix} \bullet \begin{bmatrix} t \\ 1+t \\ -3 \end{bmatrix} &= 0 \\ \therefore 4t + (t+2)(1+t) - 3t &= 0 \\ \therefore -4t + t + t^2 + 2 + 2t - 3t &= 0 \\ \therefore t^2 - 4t + 2 &= 0 \end{aligned} \quad \therefore \begin{aligned} t &= \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2} \\ t &= \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \end{aligned}$$

$$\begin{aligned} 7 \quad \text{If } \theta \text{ is the angle then} \quad \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} &= \sqrt{4 + 16 + 9} \sqrt{1 + 1 + 9} \cos \theta \\ \therefore -2 - 4 + 9 &= \sqrt{29} \sqrt{11} \cos \theta \\ \therefore \frac{3}{\sqrt{29 \times 11}} &= \cos \theta \\ \therefore \theta &\doteq 80.3^\circ \end{aligned}$$

8 If D is the origin, DA the x -axis, DC the y -axis and DE the z -axis, then A is (4, 0, 0), C is (0, 8, 0) and G is (4, 8, 5)

$$\overrightarrow{\text{AG}} = \begin{bmatrix} 4 - 4 \\ 8 - 0 \\ 5 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix} \quad \overrightarrow{\text{AC}} = \begin{bmatrix} 0 - 4 \\ 8 - 0 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix}$$

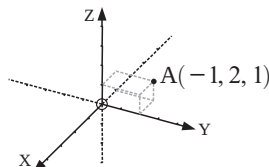
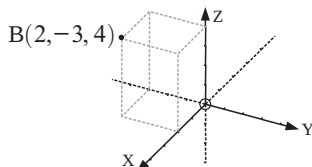
If the required angle is θ then

$$\begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix} = \sqrt{0+64+25}\sqrt{16+64+0} \cos \theta$$

$$\therefore 0+64+0 = \sqrt{89}\sqrt{80} \cos \theta$$

$$\therefore \cos \theta = \frac{64}{\sqrt{89 \times 80}}$$

$$\therefore \theta \doteq 40.7^\circ$$

REVIEW SET 16B**1 a****b****2 a**

$$\vec{PQ} = \begin{bmatrix} -4-2 \\ 4-3 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}$$

$$\mathbf{b} \quad PQ = |\vec{PQ}| = \sqrt{36+1+9} = \sqrt{46} \text{ units}$$

$$\mathbf{c} \quad M \text{ is } \left(\frac{2+(-4)}{2}, \frac{3+4}{2}, \frac{-1+2}{2} \right)$$

i.e., $\left(-1, \frac{7}{2}, \frac{1}{2} \right)$

$$\mathbf{3} \quad \mathbf{a} \quad \mathbf{p} \bullet \mathbf{q} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = -3 - 2 + 4 = -1$$

$$\mathbf{b} \quad \mathbf{p} + 2\mathbf{q} - \mathbf{r} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{p} \bullet \mathbf{r} = |\mathbf{p}| |\mathbf{r}| \cos \theta \quad \therefore \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \sqrt{1+4+1}\sqrt{1+1+4} \cos \theta$$

$$\therefore -1 + 2 + 2 = \sqrt{6}\sqrt{6} \cos \theta$$

$$\therefore 3 = 6 \cos \theta$$

$$\therefore \cos \theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

$$\mathbf{4} \quad \vec{KL} = \begin{bmatrix} -2-3 \\ 1-1 \\ 3-4 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{LK} = \begin{bmatrix} 3-(-2) \\ 1-1 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{KM} = \begin{bmatrix} 4-3 \\ 1-1 \\ 3-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{LM} = \begin{bmatrix} 4-(-2) \\ 1-1 \\ 3-3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{KL} \bullet \vec{KM} = |\vec{KL}| |\vec{KM}| \cos K$$

$$\therefore \vec{LK} \bullet \vec{LM} = |\vec{KL}| |\vec{KM}| \cos L$$

$$\therefore -5 + 0 + 1 = \sqrt{25+0+1}\sqrt{1+0+1} \cos K$$

$$\therefore 30 + 0 + 0 = \sqrt{25+0+1}\sqrt{36+0+0} \cos L$$

$$\therefore -4 = \sqrt{26}\sqrt{2} \cos K$$

$$\therefore 30 = \sqrt{26} \times 6 \cos L$$

$$\therefore \cos K = -\frac{4}{\sqrt{52}}$$

$$\therefore \frac{5}{\sqrt{26}} = \cos L$$

$$\therefore K \doteq 123.7^\circ$$

$$\therefore L \doteq 11.3^\circ$$

$$\text{and } M \doteq 180^\circ - 123.7^\circ - 11.3^\circ \doteq 45.0^\circ$$

$$\text{and } M \doteq 180^\circ - 123.7^\circ - 11.3^\circ \doteq 45.0^\circ$$

5 As the vectors are perpendicular $\begin{bmatrix} -4 \\ t \\ 1-t \end{bmatrix} \bullet \begin{bmatrix} t \\ t \\ -6-t \end{bmatrix} = 0$

$$\therefore -4t + t^2 - 6 - t + 6t + t^2 = 0$$

$$\therefore 2t^2 + t - 6 = 0$$

$$\therefore (2t-3)(t+2) = 0$$

$$\therefore t = \frac{3}{2} \text{ or } -2$$

6 a $\mathbf{x} \bullet \mathbf{y}$ $\mathbf{y} \bullet \mathbf{x}$

$$= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= 2 - 3 + 12 = 2 - 3 + 12$$

$$= 11 = 11$$

$\therefore \mathbf{x} \bullet \mathbf{y} = \mathbf{y} \bullet \mathbf{x}$ is verified

b $\mathbf{x} \bullet (\mathbf{y} - \mathbf{z})$ $\mathbf{x} \bullet \mathbf{y} - \mathbf{x} \bullet \mathbf{z}$

$$= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right) = 11 - \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = 11 - (-6 - 2 + 3)$$

$$= 11 - (-5)$$

$$= 16$$

$$= 8 - 1 + 9$$

$$= 16$$

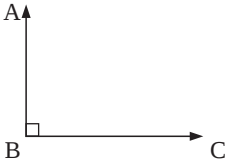
7 If the angle is θ then, $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \sqrt{9+1+4} \sqrt{4+25+1} \cos \theta$

$$\therefore 6 + 5 - 2 = \sqrt{14} \sqrt{30} \cos \theta$$

$$\therefore \frac{9}{\sqrt{14 \times 30}} = \cos \theta$$

$$\therefore \theta \doteq 63.95^\circ$$

8



$$\overrightarrow{BA} = \begin{bmatrix} 4 - -1 \\ 2 - 5 \\ -1 - 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -3 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 3 - -1 \\ -3 - 5 \\ c - 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ c - 2 \end{bmatrix}$$

But $\overrightarrow{BA} \bullet \overrightarrow{BC} = 0$

$$\therefore 20 + 24 - 3(c-2) = 0$$

$$\therefore 44 = 3(c-2)$$

$$\therefore 3c - 6 = 44$$

$$\therefore 3c = 50$$

$$\therefore c = \frac{50}{3}$$

Chapter 17

LINES IN THE PLANE AND IN SPACE

EXERCISE 17A

- 1 a i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ii $x = 3 + t, y = -4 + 4t, t \in \mathcal{R}$
 b i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + t \begin{bmatrix} -8 \\ 2 \end{bmatrix}$ ii $x = 5 - 8t, y = 2 + 2t, t \in \mathcal{R}$
 c i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ ii $x = -6 + 3t, y = 7t, t \in \mathcal{R}$
 d i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ii $x = -1 - 2t, y = 11 + t, t \in \mathcal{R}$

- 2 $x = -1 + 2\lambda, y = 4 - \lambda, \lambda \in \mathcal{R}$

When $\lambda = 0$, $x = -1 + 2(0) = -1$ and $y = 4 - 0 = 4 \therefore$ point is $(-1, 4)$

When $\lambda = 1$, $x = -1 + 2(1) = 1$ and $y = 4 - 1 = 3 \therefore$ point is $(1, 3)$

When $\lambda = 3$, $x = -1 + 2(3) = 5$ and $y = 4 - 3 = 1 \therefore$ point is $(5, 1)$

When $\lambda = -1$, $x = -1 + 2(-1) = -3$ and $y = 4 - (-1) = 5 \therefore$ point is $(-3, 5)$

When $\lambda = -4$, $x = -1 + 2(-4) = -9$ and $y = 4 - (-4) = 8 \therefore$ point is $(-9, 8)$

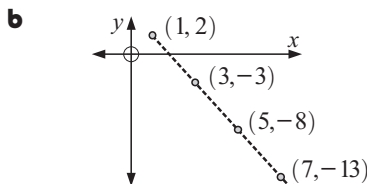
- 3 a If $t + 2 = 3$ and $1 - 3t = -2$, we get $t = 1$ and $-3t = -3$ i.e., $t = 1$
 Since $t = 1$ in each case, $(3, -2)$ lies on the line.

If $t + 2 = 0$ and $1 - 3t + 6$, we get $t = -2$ and $-3t = 5$ i.e., $t = -\frac{5}{3}$
 $\therefore (0, 6)$ does not lie on the line.

- b If $(k, 4)$ lies on $x = 1 - 2t, y = 1 + t$ then $k = 1 - 2t$ and $4 = 1 + t$
 $\therefore t = 3$ and $k = 1 - 6 = -5$, i.e., $k = -5$.

- 4 a $x(0) = 1$ and $y(0) = 2$,
 $\therefore$ the initial position is $(1, 2)$

- c In 1 second, the
 x -step is 2 and y -step is -5 , which is
 a distance of $\sqrt{2^2 + (-5)^2} = \sqrt{29}$
 $\therefore$ speed is $\sqrt{29}$ cm/sec.



EXERCISE 17B

- 1 a i when $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ ii The velocity vector is $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ iii The speed is $\sqrt{12^2 + 5^2} = 13$ m/s
 b i when $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$ ii The velocity vector is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ iii The speed is $\sqrt{3^2 + (-4)^2} = 5$ m/s
 c i when $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$ ii The velocity vector is $\begin{bmatrix} -6 \\ -4 \end{bmatrix}$ iii The speed is $\sqrt{(-6)^2 + (-4)^2} = \sqrt{36 + 16} = \sqrt{52}$ m/s

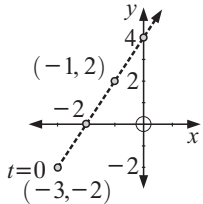
- 2 a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} + t \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ i velocity vector = $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$ ii speed = $\sqrt{8^2 + 4^2} = \sqrt{80}$ km/h

- b $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ i velocity vector = $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ ii speed = $\sqrt{6^2 + 2^2} = \sqrt{40}$ km/h

- c $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -12 \\ 15 \end{bmatrix} + t \begin{bmatrix} 7 \\ 24 \end{bmatrix}$ i velocity vector = $\begin{bmatrix} 7 \\ 24 \end{bmatrix}$ ii speed = $\sqrt{7^2 + 24^2} = 25$ km/h

- 3 a** $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ has length $\sqrt{4^2 + (-3)^2} = 5$
 $\therefore 30\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ has length 150
 $\therefore$ velocity vector is $\begin{bmatrix} 120 \\ -90 \end{bmatrix}$
- b** $\begin{bmatrix} 24 \\ 7 \end{bmatrix}$ has length $\sqrt{24^2 + 7^2} = 25$
 $\therefore \frac{1}{2}\begin{bmatrix} 24 \\ 7 \end{bmatrix}$ has length 12.5
 $\therefore$ velocity vector is $\begin{bmatrix} 12 \\ 3.5 \end{bmatrix}$
- c** $2\mathbf{i} + \mathbf{j} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has length $\sqrt{2^2 + 1^2} = \sqrt{5}$
 $\therefore 10\sqrt{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has length 50
 $\therefore$ velocity vector is $\begin{bmatrix} 20\sqrt{5} \\ 10\sqrt{5} \end{bmatrix}$
- d** $-3\mathbf{i} + 4\mathbf{j} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ has length $\sqrt{(-3)^2 + 4^2} = 5$
 $\therefore 20\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ has length 100
 $\therefore$ velocity vector is $\begin{bmatrix} -60 \\ 80 \end{bmatrix}$.

EXERCISE 17C

- 1 a** $x = -3 + 2t$, $y = -2 + 4t$, $t \geq 0$
 $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3+2t \\ -2+4t \end{bmatrix}$
- b** when $t = 2.5$,
 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3+5 \\ -2+10 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$
 $\therefore$ position is (2, 8).
- c i** When the car is due north,
 $x = 0$ and so $-3 + 2t = 0$
 $\therefore t = \frac{3}{2}$ sec.
- ii** When the car is due west,
 $y = 0$ and so $-2 + 4t = 0$
 $\therefore t = \frac{1}{2}$ sec
(O is the observation point)
- d** 
- 2 a** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $t \in \mathcal{R}$
- b** The direction vector is $\begin{bmatrix} 18-2 \\ 21-6 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$ $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} + \frac{t}{10}\begin{bmatrix} 20 \\ 15 \end{bmatrix}$, $t \in \mathcal{R}$
- c** Since $2\mathbf{i} + \mathbf{j} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has length $\sqrt{2^2 + 1^2} = \sqrt{5}$ then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $t \in \mathcal{R}$
- d** Since $3\mathbf{i} + 4\mathbf{j} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ has length $\sqrt{3^2 + 4^2} = 5$ then $3\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ has length 15
 $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + (t-1)\begin{bmatrix} 9 \\ 12 \end{bmatrix}$
or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} - \begin{bmatrix} 9 \\ 12 \end{bmatrix} + t\begin{bmatrix} 9 \\ 12 \end{bmatrix}$ i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -16 \end{bmatrix} + t\begin{bmatrix} 9 \\ 12 \end{bmatrix}$, $t \in \mathcal{R}$
- 3** $\mathbf{r} = \begin{bmatrix} -20 \\ 32 \end{bmatrix} + t\begin{bmatrix} 12 \\ -5 \end{bmatrix}$, where t is the time in hours since 6 am.
- a** At 6 am, $t = 0$ $\therefore \mathbf{r} = \begin{bmatrix} -20 \\ 32 \end{bmatrix}$ and $|\mathbf{r}| = \sqrt{(-20)^2 + (32)^2} \div 37.7$ $\therefore$ the ship is 37.7 km away
- b** Speed $= \left| \begin{bmatrix} 12 \\ -5 \end{bmatrix} \right| = \sqrt{12^2 + (-5)^2} = 13$ km/h
- c** When due north, $x = 0$. $\therefore -20 + 12t = 0$
 $\therefore t = \frac{5}{3}$ hours
 $\therefore t = 1$ hour 40 min, so the time is 7:40 am
- 4** Yacht A: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ Yacht B: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $t \geq 0$
- a** when $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$
 $\therefore$ A is at (4, 5) and B is at (1, -8)

b For A, the velocity vector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, and for B it is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{aligned} \text{c Speed of A} &= \sqrt{1^2 + (-2)^2} & \text{Speed of B} &= \sqrt{2^2 + 1^2} \\ &= \sqrt{5} \text{ km/h} & &= \sqrt{5} \text{ km/h} \end{aligned}$$

$$\begin{aligned} \text{d The distance between them is } D &= \sqrt{[(1+2t) - (4+t)]^2 + [(-8+t) - (5-2t)]^2} \\ &= \sqrt{(-3+t)^2 + (-13+3t)^2} \\ &= \sqrt{9-6t+t^2 + 169-78t+9t^2} \\ &= \sqrt{10t^2 - 84t + 178} \end{aligned}$$

This is a minimum when $10t^2 - 84t + 178$ is a minimum. This occurs when

$$t = \frac{-b}{2a} = \frac{84}{20} = 4.2 \text{ hours.} \quad \therefore \text{ the time is 4 h 12 min after 6 am i.e., 10:12 am.}$$

e A has direction vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and B has direction vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Since $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 - 2 = 0$, the paths of the yachts are at right angles to each other.

5 a P has position $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and at $t = 0$, the time is 1.34 pm

$$\therefore x_1(t) = -5 + 3t, \quad y_1(t) = 4 - t.$$

$$\text{b Speed} = \sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ km/min}$$

c Q fires its torpedo after a minutes.

$\therefore$ at time t , its torpedo has travelled for $(t - a)$ minutes.

$$\therefore \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \end{bmatrix} + (t - a) \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \quad t > a$$

$$\text{i.e., } x_2(t) = 15 - 4(t - a) \quad \text{and} \quad y_2(t) = 7 - 3(t - a)$$

d They meet when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$

$$\begin{aligned} \therefore -5 + 3t &= 15 - 4(t - a) & \text{and} & \quad 4 - t = 7 - 3(t - a) \\ \text{i.e., } 7t - 4a &= 20 & \text{and} & \quad 2t - 3a = 3 \end{aligned}$$

$$\begin{array}{rcl} \text{Solving simultaneously,} & 7t - 4a = 20 & \times 3 \\ & 2t - 3a = 3 & \times (-4) \end{array}$$

$$\begin{array}{rcl} \therefore & 21t - 12a = 60 \\ & -8t + 12a = -12 \\ \hline & 13t & = 48 \end{array}$$

$$\therefore t = \frac{48}{13} \quad \text{and} \quad 7\left(\frac{48}{13}\right) - 4a = 20$$

$$\text{i.e., } t \div 3.6923 \quad \therefore 5.8462 = 4a$$

$$\text{i.e., } t \div 3 \text{ min } 41.53 \quad \therefore a \div 1.4615 \div 1 \text{ min } 27.7 \text{ sec}$$

So, as $a \div 1.4615$, Q fired at 1:35:28 pm, and the explosion occurred at 1:37:42 pm.

EXERCISE 17D

1 a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $t \geq 0$ $\therefore$ its position is $(-3 + 2t, -2 + 4t)$.

b i It is due east when $y = 0$

$$\therefore -2 + 4t = 0$$

$$\therefore t = \frac{1}{2} \text{ sec}$$

ii It is due north when $x = 0$

$$\therefore -3 + 2t = 0$$

$$\therefore t = 1\frac{1}{2} \text{ sec}$$

c When $y = 0$, $t = \frac{1}{2}$ and $x = -3 + 2\left(\frac{1}{2}\right) = -2$

$$\text{When } x = 0, \quad t = 1\frac{1}{2} \quad \text{and} \quad y = -2 + 4\left(1\frac{1}{2}\right) = 4$$

$\therefore$ axis intercepts are $(-2, 0)$ and $(0, 4)$

- 2 a** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, $t \geq 0$ $\therefore$ its position is $(-2-t, 1-3t)$
- b** Crosses the x -axis when $y = 0$ $\therefore 1-3t = 0$ $\therefore t = \frac{1}{3}$ sec
- c** When $t = \frac{1}{3}$, the point is at $(-2\frac{1}{3}, 0)$.
- 3 a** $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ has length $\sqrt{(-3)^2 + 4^2} = 5$ $\therefore$ unit direction vector is $\frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$
 $\therefore$ the velocity vector is $\frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \times 10 = \begin{bmatrix} -6 \\ 8 \end{bmatrix} = -6\mathbf{i} + 8\mathbf{j}$

b $\therefore$ the liner's position is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} + t \begin{bmatrix} -6 \\ 8 \end{bmatrix}$, $t \geq 0$

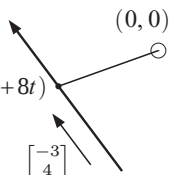
c The liner is due east when $y = 0$, $\therefore 0 = -6 + 8t$
 $\therefore t = \frac{3}{4}$ hour

d The liner is nearest to the fishing boat when $\vec{OP} \perp \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$\begin{aligned} \therefore \begin{bmatrix} 6-6t-0 \\ -6+8t-0 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 4 \end{bmatrix} &= 0 \quad P(6-6t, -6+8t) \\ \therefore -3(6-6t) + 4(-6+8t) &= 0 \\ \therefore -18 + 18t - 24 + 32t &= 0 \\ \therefore 50t &= 42 \\ \therefore t &= 0.84 \text{ hours} \end{aligned}$$

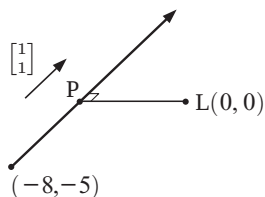
i.e., after 50 min 24 sec.

At this time, the liner is at $(6-6(0.84), -6+8(0.84))$ i.e., at $(0.96, 0.72)$.



- 4 a i** Initial position vector is $\begin{bmatrix} -8-0 \\ -5-0 \end{bmatrix} = -8\mathbf{i} - 5\mathbf{j}$
- ii** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $\sqrt{1^2 + 1^2} = \sqrt{2}$
 $\therefore 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $3\sqrt{2}$
 $\therefore$ the direction vector is $\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{i} + 3\mathbf{j}$
- iii** position vector at time t is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, or $(-8+3t)\mathbf{i} + (-5+3t)\mathbf{j}$

b



P is $(-8+3t, -5+3t)$

and $\vec{LP}$ is $\begin{bmatrix} -8+3t \\ -5+3t \end{bmatrix}$

Now when closest, $\vec{LP} \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$$\therefore \begin{bmatrix} -8+3t \\ -5+3t \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\therefore -8+3t-5+3t = 0$$

$$\therefore 6t = 13$$

$$\therefore t = \frac{13}{6} = 2\frac{1}{6}$$

i.e., at 2 hours 10 min

c When $t = \frac{13}{6}$, P is $(-8+3(\frac{13}{6}), -5+3(\frac{13}{6}))$ i.e., $(-\frac{3}{2}, \frac{3}{2})$

$$\begin{aligned} \therefore \text{the closest distance} &= \sqrt{\left(-\frac{3}{2}-0\right)^2 + \left(\frac{3}{2}-0\right)^2} \\ &= \sqrt{\frac{9}{4} + \frac{9}{4}} \\ &= \sqrt{\frac{9}{2}} \\ &\doteq 2.12 \text{ km} \end{aligned}$$

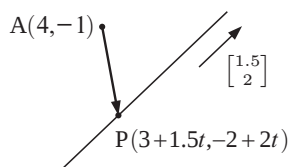
$\therefore$ the trawler would be breaking the law.

$$5 \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$

a When $t = 4$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ $\therefore$ the car is at (3, 6)

b Speed = $\sqrt{(1.5)^2 + 2^2} = 2.5$ m/s

c



$$\vec{AP} = \begin{bmatrix} -3 + 1.5t - 4 \\ -2 + 2t - (-1) \end{bmatrix} = \begin{bmatrix} -7 + 1.5t \\ -1 + 2t \end{bmatrix}$$

and for shortest AP, $\vec{AP} \bullet \begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = 0$

$$\therefore (-7 + 1.5t)1.5 + (-1 + 2t)2 = 0$$

$$\therefore -10.5 + 2.25t - 2 + 4t = 0$$

$$\therefore 6.25t = 12.5$$

$$\therefore t = 2 \text{ sec}$$

d $x = -3 + \frac{3}{2}t$ Equating t values, $\frac{2}{3}(x + 3) = \frac{1}{2}(y + 2)$

$$y = -2 + 2t$$

$$\therefore 4(x + 3) = 3(y + 2) \quad \{\times \text{ both sides by } 6\}$$

$$\therefore 4x + 12 = 3y + 6$$

$$\therefore 4x - 3y = -6$$

6 a $|b| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$

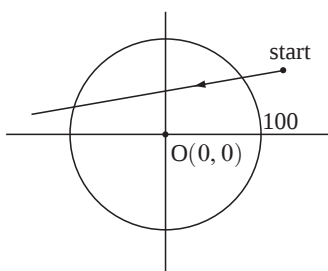
As the speed is $40\sqrt{10}$ km/h, the velocity vector is $40 \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -120 \\ -40 \end{bmatrix}$.

b $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix} + t \begin{bmatrix} -120 \\ -40 \end{bmatrix}$, $t \geq 0$ $\{t = 0 \text{ at } 12.00 \text{ noon}\}$

c At 1:00 pm, $t = 1$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 - 120 \\ 100 - 40 \end{bmatrix} = \begin{bmatrix} 80 \\ 60 \end{bmatrix}$

d The distance from $O(0, 0)$ to $P_1(80, 60)$ is $\left| \begin{bmatrix} 80 \\ 60 \end{bmatrix} \right| = \sqrt{80^2 + 60^2} = 100$ km,
which is when it becomes visible to radar. $\{\text{within } 100 \text{ km of } O(0, 0)\}$

e



A general point on the path is $P(200 - 120t, 100 - 40t)$.

Now $\vec{OP} = \begin{bmatrix} 200 - 120t \\ 100 - 40t \end{bmatrix}$,

and for the closest point $\vec{OP} \bullet \begin{bmatrix} -3 \\ -1 \end{bmatrix} = 0$

$$\therefore -3(200 - 120t) - 1(100 - 40t) = 0$$

$$\therefore -700 + 400t = 0$$

$$\therefore t = \frac{7}{4} = 1\frac{3}{4} \text{ hours}$$

The time when the aircraft is closest is 1:45 pm, and

at this time $\vec{OP} = \begin{bmatrix} 200 - 120(\frac{7}{4}) \\ 100 - 40(\frac{7}{4}) \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \end{bmatrix}$

$$\therefore d_{\min} = \sqrt{(-10)^2 + 30^2} \div 31.6 \text{ km}$$

f It disappears from radar when $|\vec{OP}| = 100$ and $t > 1\frac{3}{4}$

$$\therefore \sqrt{(200 - 120t)^2 + (100 - 40t)^2} = 100$$

$$\therefore 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1400t^2 = 10\,000$$

$$\therefore 16\,000t^2 - 56\,000t + 40\,000 = 0$$

$$\therefore 16t^2 - 56t + 40 = 0 \quad \{\div 1000\}$$

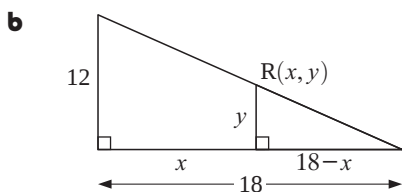
$$\therefore 2t^2 - 7t + 5 = 0 \quad \{\div 8\}$$

$$\therefore (2t - 5)(t - 1) = 0$$

$$\therefore t = \frac{5}{2} \quad \{\text{as } t > 1\frac{3}{4}\}$$

i.e., at $2\frac{1}{2}$ hours i.e., at 2:30 pm

- 7 a** when $x = 0$, $3y = 36$, $\therefore y = 12$ $\therefore$ B is $(0, 12)$
 when $y = 0$, $2x = 36$, $\therefore x = 18$ $\therefore$ A is $(18, 0)$



Using the similar triangles,

$$\frac{y}{18-x} = \frac{12}{18} = \frac{2}{3}$$

$$\therefore y = \frac{2}{3}(18-x)$$

$$\therefore R \text{ is } (x, 12 - \frac{2}{3}x)$$

c $\vec{PR} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x-0 \end{bmatrix} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x \end{bmatrix}$ and $\vec{AB} = \begin{bmatrix} 0-18 \\ 12-0 \end{bmatrix} = \begin{bmatrix} -18 \\ 12 \end{bmatrix}$

d R is located so the shortest distance PR is when $\vec{RP} \bullet \begin{bmatrix} -18 \\ 12 \end{bmatrix} = 0$

$$\therefore -18(x-4) + 12(12 - \frac{2}{3}x) = 0$$

$$\therefore -18x + 72 + 144 - 8x = 0$$

$$\therefore 26x = 216$$

$$\therefore x = \frac{108}{13}$$

$$\text{and } y = \frac{2}{3} \left(18 - \frac{108}{13} \right)$$

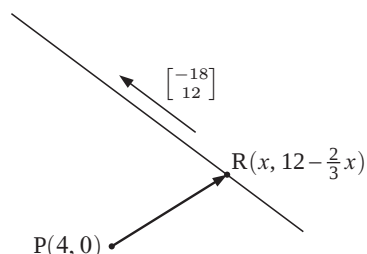
$$\therefore y = 12 - \frac{72}{13} = \frac{84}{13}$$

i.e., R is at $\left(\frac{108}{13}, \frac{84}{13} \right)$

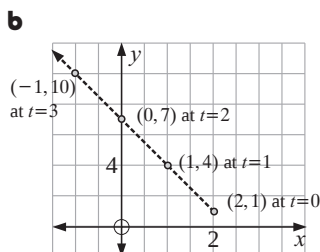
$$\text{and } \vec{PR} = \begin{bmatrix} \frac{108}{13} - 4 \\ 12 - \frac{2}{3} \left(\frac{108}{13} \right) \end{bmatrix} \div \begin{bmatrix} 4.308 \\ 6.4615 \end{bmatrix}$$

$$\therefore |\vec{PR}| \div \sqrt{(4.308)^2 + (6.4615)^2} \div 7.77 \text{ km,}$$

so the shortest distance is 7.77 km



- 8 a** $x(0) = 2$, $y(0) = 1$ $\therefore$ the particle's initial position is $(2, 1)$.



- c** If Q is a general point on the path, then Q is $(2-t, 1+3t)$

If R is $(0, 10)$ then $\vec{RQ} = \begin{bmatrix} 2-t-0 \\ 1+3t-10 \end{bmatrix} = \begin{bmatrix} 2-t \\ 3t-9 \end{bmatrix}$

For the shortest distance, $\vec{RQ} \bullet \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$

$$\therefore -1(2-t) + 3(3t-9) = 0$$

$$\therefore -2+t+9t-27 = 0$$

$$\therefore 10t = 29$$

$$\therefore t = 2.9 \text{ sec}$$

When $t = 2.9$, $\vec{RQ} = \begin{bmatrix} -0.9 \\ -0.3 \end{bmatrix}$ and $|\vec{RQ}| = \sqrt{(-0.9)^2 + (-0.3)^2} \div 0.949 \text{ cm}$

(or $\vec{RQ} = \begin{bmatrix} -\frac{9}{10} \\ -\frac{3}{10} \end{bmatrix}$ $\therefore |\vec{RQ}| = \sqrt{\frac{81}{100} + \frac{9}{100}} = \sqrt{\frac{90}{100}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}} \text{ cm}$)

- 9 a** $x(0) = 10$, $y(0) = 12$ $\therefore$ P is at $(10, 12)$

b The velocity vector is $\begin{bmatrix} a \\ -3 \end{bmatrix}$ and speed $= \sqrt{a^2 + 9} = 13$

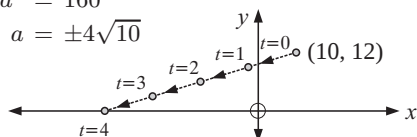
$$\therefore a^2 + 9 = 169$$

$$\therefore a^2 = 160$$

$$\therefore a = \pm 4\sqrt{10}$$

- c** For $a < 0$, i.e., $a = -4\sqrt{10}$,

$$x(t) = 10 - 4\sqrt{10}t, \quad y(t) = 12 - 3t.$$



- 10** For A, $x(t) = 3 - t$, $y(t) = 2t - 4$ For B, $x(t) = 4 - 3t$, $y(t) = 3 - 2t$
- a** When $t = 0$, $x(0) = 3$, $y(0) = -4$ $\therefore x(0) = 4$, $y(0) = 3$
 $\therefore$ A is at $(3, -4)$ $\therefore$ B is at $(4, 3)$
- b** The velocity vector of A is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and the velocity vector of B is $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$.
- c** If the angle is θ , $\begin{bmatrix} -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \sqrt{1+4}\sqrt{9+4}\cos\theta$
 $\therefore 3 - 4 = \sqrt{5}\sqrt{13}\cos\theta$
 $\therefore \frac{-1}{\sqrt{65}} = \cos\theta$ and so $\theta = 97.1^\circ$
- d** If D is the distance between them, then

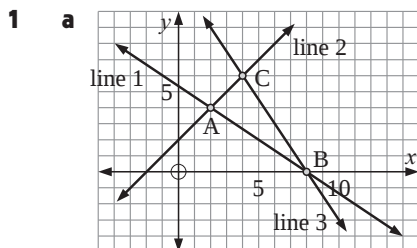
$$\begin{aligned} D &= \sqrt{[(4-3t) - (3-t)]^2 + [(2t-4) - (3-2t)]^2} \\ &= \sqrt{[1-2t]^2 + [-7+4t]^2} \\ &= \sqrt{1-4t+4t^2+49-56t+16t^2} \\ &= \sqrt{20t^2-60t+50} \end{aligned}$$

and D is a minimum when

$$t = -\frac{b}{2a} = \frac{60}{40} = 1\frac{1}{2}$$

i.e., $t = 1.5$ hours

EXERCISE 17E



b A is $(2, 4)$, B is $(8, 0)$, C is $(4, 6)$

c $BC = \sqrt{(8-4)^2 + (0-6)^2} = \sqrt{16+36} = \sqrt{52}$ units

$AC = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16} = \sqrt{52}$ units

$\therefore BC = AC$ and so $\triangle ABC$ is isosceles.

d Line 1 and Line 2 meet at A.

$$\therefore \begin{bmatrix} -1 \\ 6 \end{bmatrix} + \begin{bmatrix} 3r \\ -2r \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3r-s \\ -2r-s \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

i.e., $3r - s = 1$

and $2r + s = 4$

Adding, $5r = 5 \quad \therefore r = 1$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Line 1 and Line 3 meet at B.

$$\therefore \begin{bmatrix} -1 \\ 6 \end{bmatrix} + r \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3r+2t \\ -2r-3t \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \end{bmatrix}$$

$$\therefore 3r + 2t = 11 \quad \times 3$$

$$-2r - 3t = -9 \quad \times 2$$

$$\therefore 9r + 6t = 33$$

$$-4r - 6t = -18$$

Adding, $5r = 15$

$$\therefore r = 3$$

So, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \quad \checkmark$

Line 2 and Line 3 meet at C.

$$\therefore \begin{bmatrix} 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

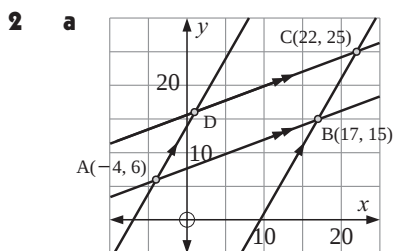
$$\therefore \begin{bmatrix} s+2t \\ s-3t \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

i.e., $s + 2t = 10$

$$-s + 3t = 5$$

Adding, $5t = 15 \quad \therefore t = 3$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$



b B(17, 15)
C(22, 25)
D(1, 16)

c Lines 1 and 4 meet at B

$$\begin{aligned}\therefore \begin{bmatrix} -4 \\ 6 \end{bmatrix} + r \begin{bmatrix} 7 \\ 3 \end{bmatrix} &= \begin{bmatrix} 22 \\ 25 \end{bmatrix} + u \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ \therefore \begin{bmatrix} 7r+u \\ 3r+2u \end{bmatrix} &= \begin{bmatrix} 26 \\ 19 \end{bmatrix} \\ \therefore 7r+u &= 26 \\ 3r+2u &= 19 \\ \therefore -14r-2u &= -52 \\ 3r+2u &= 19\end{aligned}$$

$$\text{Adding, } \begin{array}{r} -11r \quad \quad = -33 \\ \hline \therefore r = 3 \end{array}$$

$$\text{and } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 15 \end{bmatrix} \quad \checkmark$$

Lines 2 and 3 meet at D

$$\begin{aligned}\text{and } \begin{bmatrix} -4 \\ 6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 22 \\ 25 \end{bmatrix} + t \begin{bmatrix} -7 \\ -3 \end{bmatrix} \\ \therefore \begin{bmatrix} s+7t \\ 2s+3t \end{bmatrix} &= \begin{bmatrix} 26 \\ 19 \end{bmatrix} \\ \therefore -2s-14t &= -52 \\ \therefore 2s+3t &= 19\end{aligned}$$

$$\text{Adding, } \begin{array}{r} -11t \quad \quad = -33 \\ \hline \therefore t = 3 \end{array}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 25 \end{bmatrix} + 3 \begin{bmatrix} -7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \end{bmatrix} \quad \checkmark$$

3 a Lines 1 and 3 meet at A.

$$\begin{aligned}\therefore \begin{bmatrix} 0 \\ 2 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 7 \end{bmatrix} \\ \therefore \begin{bmatrix} 2r-t \\ r+t \end{bmatrix} &= \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ \therefore 2r-t &= 0 \\ r+t &= 3 \\ \text{Adding, } \begin{array}{r} 3r \quad \quad = 3 \\ \hline \therefore r &= 1 \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\end{array} \\ \therefore A \text{ is } (2, 3)\end{aligned}$$

Lines 2 and 3 meet at C.

$$\begin{aligned}\therefore \begin{bmatrix} 8 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \therefore \begin{bmatrix} -s-t \\ -2s+t \end{bmatrix} &= \begin{bmatrix} -8 \\ -1 \end{bmatrix} \\ \therefore -s-t &= -8 \\ -2s+t &= -1 \\ \text{Adding, } \begin{array}{r} -3s \quad \quad = -9 \\ \hline \therefore s &= 3 \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 8 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}\end{array} \\ \therefore C \text{ is } (5, 0)\end{aligned}$$

Lines 1 and 2 meet at B.

$$\begin{aligned}\therefore \begin{bmatrix} 0 \\ 2 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 8 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ \therefore \begin{bmatrix} 2r+s \\ r+2s \end{bmatrix} &= \begin{bmatrix} 8 \\ 4 \end{bmatrix} \\ \therefore -4r-2s &= -16 \\ r+2s &= 4 \\ \text{Adding, } \begin{array}{r} -3r \quad \quad = -12 \\ \hline \therefore r &= 4 \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}\end{array} \\ \therefore B \text{ is } (8, 6)\end{aligned}$$

b A (2, 3), B(8, 6), C(5, 0)

$$\begin{aligned}AB &= \sqrt{(8-2)^2 + (6-3)^2} \\ &= \sqrt{36+9} \\ &= \sqrt{45} \\ BC &= \sqrt{(5-8)^2 + (0-6)^2} \\ &= \sqrt{9+36} \\ &= \sqrt{45}\end{aligned}$$

The two equal sides are AB and BC and they have length $\sqrt{45}$ units.

4 a Lines QP and PR meet at P

$$\therefore \begin{bmatrix} 3 \\ -1 \end{bmatrix} + r \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix} + t \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 14r-5t \\ 10r+7t \end{bmatrix} = \begin{bmatrix} -3 \\ 19 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} 14r - 5t & = & -3 \quad \times 7 \\ 10r + 7t & = & 19 \quad \times 5 \end{array}$$

$$\therefore \begin{array}{rcl} 98r - 35t & = & -21 \\ 50r + 35t & = & 95 \end{array}$$

$$\text{Adding, } \begin{array}{rcl} 148r & = & 74 \end{array}$$

$$\therefore r = \frac{1}{2}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

$$\therefore P \text{ is } (10, 4)$$

Lines QP and PR meet at Q

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} + r \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 17 \\ -9 \end{bmatrix}$$

$$\therefore r \begin{bmatrix} 14 \\ 10 \end{bmatrix} = s \begin{bmatrix} 17 \\ -9 \end{bmatrix}$$

$$\therefore r = s = 0$$

$$\text{So, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\therefore Q \text{ is } (3, -1)$$

5 a Lines 1 and 2 meet at B

$$\therefore \begin{bmatrix} 2 \\ 5 \end{bmatrix} + r \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix} + s \begin{bmatrix} -8 \\ 32 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4r+8s \\ r-32s \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} 4r + 8s & = & 16 \\ r - 32s & = & 4 \end{array}$$

$$\therefore \begin{array}{rcl} 4r + 8s & = & 16 \\ r - 32s & = & 4 \end{array}$$

$$\therefore \begin{array}{rcl} r + 2s & = & 4 \\ -r + 32s & = & -4 \end{array}$$

$$\text{Adding, } \begin{array}{rcl} 34s & = & 0 \end{array}$$

$$\therefore s = 0$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix} \quad \therefore B \text{ is } (18, 9)$$

Lines 3 and 4 meet at D

$$\therefore \begin{bmatrix} 14 \\ 25 \end{bmatrix} + t \begin{bmatrix} -8 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + u \begin{bmatrix} -3 \\ 12 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -8t+3u \\ -2t-12u \end{bmatrix} = \begin{bmatrix} -11 \\ -24 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} -8t + 3u & = & -11 \\ t + 6u & = & 12 \end{array}$$

$$\therefore \begin{array}{rcl} -8t + 3u & = & -11 \\ t + 6u & = & 12 \end{array}$$

$$\therefore \begin{array}{rcl} 16t - 6u & = & 22 \\ t + 6u & = & 12 \end{array}$$

$$\text{Adding, } \begin{array}{rcl} 17t & = & 34 \end{array}$$

$$\therefore t = 2$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 25 \end{bmatrix} + 2 \begin{bmatrix} -8 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 21 \end{bmatrix}$$

$$\therefore D \text{ is } (-2, 21)$$

Lines QR and PR meet at R

$$\therefore \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 17 \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix} + t \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 17s-5t \\ -9s+7t \end{bmatrix} = \begin{bmatrix} -3 \\ 19 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} 17s - 5t & = & -3 \quad \times 7 \\ -9s + 7t & = & 19 \quad \times 5 \end{array}$$

$$\therefore \begin{array}{rcl} 17s - 5t & = & -3 \quad \times 7 \\ -9s + 7t & = & 19 \quad \times 5 \end{array}$$

$$\therefore \begin{array}{rcl} 119s - 35t & = & -21 \\ -45s + 35t & = & 95 \end{array}$$

$$\therefore \begin{array}{rcl} 119s - 35t & = & -21 \\ -45s + 35t & = & 95 \end{array}$$

$$\text{Adding, } \begin{array}{rcl} 74s & = & 74 \end{array}$$

$$\therefore s = 1$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 17 \\ -9 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \end{bmatrix}$$

$$\therefore R \text{ is } (20, -10)$$

$$\mathbf{b} \quad \overrightarrow{PQ} = \begin{bmatrix} 3-10 \\ -1-4 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$\overrightarrow{PR} = \begin{bmatrix} 20-10 \\ -10-4 \end{bmatrix} = \begin{bmatrix} 10 \\ -14 \end{bmatrix}$$

$$\text{and } \overrightarrow{PQ} \bullet \overrightarrow{PR} = -70 + 70 = 0$$

$$\mathbf{c} \quad PQ \perp PR \quad \text{i.e., } \angle QPR = 90^\circ$$

$$\begin{aligned} \mathbf{d} \quad \text{Area} &= \frac{1}{2} | \overrightarrow{PQ} \times \overrightarrow{PR} | \\ &= \frac{1}{2} \sqrt{49 + 25} \sqrt{100 + 196} \\ &= 74 \text{ units}^2 \end{aligned}$$

Lines 2 and 3 meet at C

$$\therefore \begin{bmatrix} 18 \\ 9 \end{bmatrix} + s \begin{bmatrix} -8 \\ 32 \end{bmatrix} = \begin{bmatrix} 14 \\ 25 \end{bmatrix} + t \begin{bmatrix} -8 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -8s+8t \\ 32s+2t \end{bmatrix} = \begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} -8s + 8t & = & -4 \\ 32s + 2t & = & 16 \end{array}$$

$$\therefore \begin{array}{rcl} -8s + 8t & = & -4 \\ 32s + 2t & = & 16 \end{array}$$

$$\text{Adding, } \begin{array}{rcl} 34s & = & 17 \end{array}$$

$$\therefore s = \frac{1}{2}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -8 \\ 32 \end{bmatrix} = \begin{bmatrix} 14 \\ 25 \end{bmatrix}$$

$$\therefore C \text{ is } (14, 25)$$

$$\mathbf{b} \quad \overrightarrow{AC} = \begin{bmatrix} 14-2 \\ 25-5 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

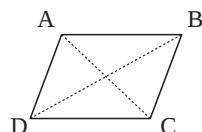
$$\overrightarrow{DB} = \begin{bmatrix} 18-2 \\ 9-21 \end{bmatrix} = \begin{bmatrix} 16 \\ -12 \end{bmatrix}$$

$$\mathbf{i} \quad | \overrightarrow{AC} | = \sqrt{12^2 + 20^2} = \sqrt{544} \text{ units}$$

$$\mathbf{ii} \quad | \overrightarrow{DB} | = \sqrt{16^2 + (-12)^2} = \sqrt{544} \text{ units}$$

$$\mathbf{iii} \quad \overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$$

- $\mathbf{c}$ The diagonals are perpendicular and equal in length, so, ABCD is a square.



EXERCISE 17F

- 1 a** The vector equation is $\begin{bmatrix} x-1 \\ y-3 \\ z+7 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad t \in \mathcal{R}$
- b** The vector equation is $\begin{bmatrix} x-0 \\ y-1 \\ z-2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad t \in \mathcal{R}$
- c** Since the line is parallel to the X -axis, it has direction vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \text{ the vector equation is } \begin{bmatrix} x+2 \\ y-2 \\ z-1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathcal{R}$$

- 2 a** The parametric equations are:
 $x = 5 + (-1)t, \quad y = 2 + 2t, \quad z = -1 + 6t$
 i.e., $x = 5 - t, \quad y = 2 + 2t, \quad z = -1 + 6t, \quad t \in \mathcal{R}$
- b** The parametric equations are:
 $x = 0 + 2t, \quad y = 2 + (-1)t, \quad z = -1 + 3t$
 i.e., $x = 2t, \quad y = 2 - t, \quad z = -1 + 3t, \quad t \in \mathcal{R}$
- c** Since the line is perpendicular to the XOY plane, it has direction vector $[0, 0, 1]$.
 $\therefore$ the parametric equations are:
 $x = 3 + 0t, \quad y = 2 + 0t, \quad z = -1 + 1t$
 i.e., $x = 3, \quad y = 2, \quad z = -1 + t, \quad t \in \mathcal{R}$

3 a $\overrightarrow{AB} = \begin{bmatrix} -1-1 \\ 3-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \therefore x = 1 - 2t, \quad y = 2 + t, \quad z = 1 + t, \quad t \in \mathcal{R}$

b $\overrightarrow{CD} = \begin{bmatrix} 3-0 \\ 1-1 \\ -1-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \quad \therefore x = 3t, \quad y = 1, \quad z = 3 - 4t, \quad t \in \mathcal{R}$

c $\overrightarrow{EF} = \begin{bmatrix} 1-1 \\ -1-2 \\ 5-5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \quad \therefore x = 1, \quad y = 2 - 3t, \quad z = 5, \quad t \in \mathcal{R}$

d $\overrightarrow{GH} = \begin{bmatrix} 5-0 \\ -1-1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \quad \therefore x = 5t, \quad y = 1 - 2t, \quad z = -1 + 4t, \quad t \in \mathcal{R}$

- 4** Given $x = 1 - t, \quad y = 3 + t, \quad z = 3 - 2t$:

a The line meets the XOY plane when $z = 0$ i.e., $3 - 2t = 0 \quad \therefore t = \frac{3}{2}$

Then $x = 1 - \frac{3}{2} = -\frac{1}{2}$ and $y = 3 + \frac{3}{2} = \frac{9}{2}$, i.e., the point is $(-\frac{1}{2}, \frac{9}{2}, 0)$

b The line meets the YOZ plane when $x = 0$ i.e., $1 - t = 0$

$\therefore t = 1$

Then $y = 3 + 1 = 4$ and $z = 3 - 2 = 1$, i.e., the point is $(0, 4, 1)$

c The line meets the XOZ plane when $y = 0$ i.e., $3 + t = 0 \quad \therefore t = -3$

Then $x = 1 - (-3) = 4$ and $z = 3 - 2(-3) = 9$, i.e., the point is $(4, 0, 9)$

- 5** Given a line with equations $x = 2 - t$, $y = 3 + 2t$ and $z = 1 + t$

the distance to the point $(1, 0, -2)$ is $\sqrt{(2 - t - 1)^2 + (3 + 2t - 0)^2 + (1 + t + 2)^2}$.

But this distance $= 5\sqrt{3}$ units

$$\therefore \sqrt{(1 - t)^2 + (3 + 2t)^2 + (t + 3)^2} = 5\sqrt{3}$$

$$\therefore (1 - t)^2 + (3 + 2t)^2 + (t + 3)^2 = 75$$

$$\therefore 1 - 2t + t^2 + 9 + 12t + 4t^2 + t^2 + 6t + 9 = 75$$

$$\therefore 6t^2 + 16t - 56 = 0$$

$$\therefore 3t^2 + 8t - 28 = 0$$

$$\therefore (3t + 14)(t - 2) = 0$$

$$\therefore t = -\frac{14}{3} \text{ or } t = 2$$

When $t = 2$, the point is $(0, 7, 3)$

and when $t = -\frac{14}{3}$, the point is $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$.

- 6 a** Let $A(1 + t, 2 - t, 3 + t)$ be a point on the given line.

$$\text{Then } \vec{PA} = \begin{bmatrix} 1 + t - 1 \\ 2 - t - 1 \\ 3 + t - 2 \end{bmatrix} = \begin{bmatrix} t \\ 1 - t \\ 1 + t \end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the direction vector of the line.

If $\vec{PA}$ and $\mathbf{v}$ are perpendicular, then $\vec{PA} \bullet \mathbf{v} = 0$

$$\therefore \begin{bmatrix} t \\ 1 - t \\ 1 + t \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$\therefore t - 1(1 - t) + 1(1 + t) = 0$$

$$\therefore t - 1 + t + 1 + t = 0$$

$$\therefore 3t = 0$$

$$\therefore t = 0$$

$\therefore A$ is at $(1, 2, 3)$ i.e., the foot of the perpendicular is $(1, 2, 3)$

- b** Let A be a point on the line such that $\vec{PA}$ is perpendicular to the line.

Then A is at $(1 + s, 2 - s, 2s)$ for some s .

$$\text{Now } \vec{PA} = \begin{bmatrix} 1 + s - 2 \\ 2 - s - 1 \\ 2s - 3 \end{bmatrix} = \begin{bmatrix} s - 1 \\ 1 - s \\ 2s - 3 \end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is the direction vector of the line.

$$\text{Since } \vec{PA} \bullet \mathbf{v} = 0, \quad \begin{bmatrix} s - 1 \\ 1 - s \\ 2s - 3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0$$

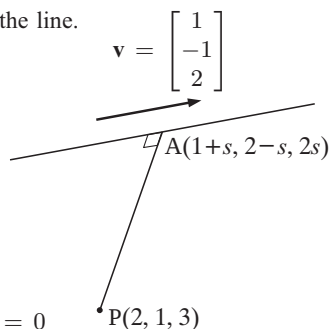
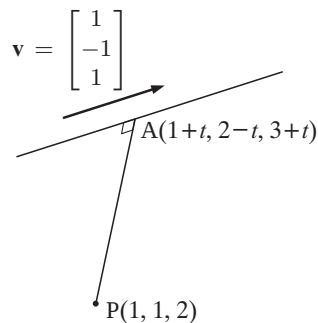
$$\therefore 1(s - 1) - 1(1 - s) + 2(2s - 3) = 0$$

$$\therefore s - 1 - 1 + s + 4s - 6 = 0$$

$$\therefore 6s = 8$$

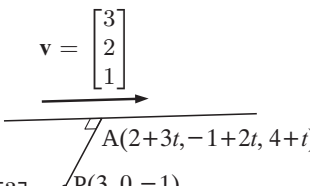
$$\therefore s = \frac{4}{3}$$

$\therefore A$ is at $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$ i.e., the foot of the perpendicular is $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$



- 7 a** Let $A(2 + 3t, -1 + 2t, 4 + t)$ be a point on the line such that $\vec{PA}$ is perpendicular to the line.

Now $\vec{PA} = \begin{bmatrix} 2 + 3t - 3 \\ -1 + 2t - 0 \\ 4 + t - (-1) \end{bmatrix} = \begin{bmatrix} 3t - 1 \\ 2t - 1 \\ t + 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$



is the direction vector of the line.

$\therefore$ since $\vec{PA} \bullet \mathbf{v} = 0$,

$$\begin{bmatrix} 3t - 1 \\ 2t - 1 \\ t + 5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\therefore 3(3t - 1) + 2(2t - 1) + 1(t + 5) = 0$$

$$\therefore 9t - 3 + 4t - 2 + t + 5 = 0$$

$$\therefore 14t = 0$$

$$\therefore t = 0$$

$\therefore$ A is at $(2, -1, 4)$

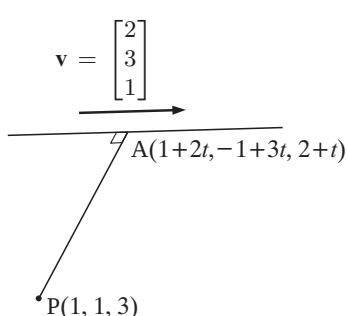
$$\begin{aligned} \therefore \text{the distance } d &= \sqrt{(2 - 3)^2 + (-1 - 0)^2 + (4 - (-1))^2} = \sqrt{1 + 1 + 25} \\ &= \sqrt{27} \\ &= 3\sqrt{3} \text{ units} \end{aligned}$$

- b** Let A be a point on the line such that $\vec{PA}$ is perpendicular to the line.

Then A is at $(1 + 2t, -1 + 3t, 2 + t)$ for some t .

Now $\vec{PA} = \begin{bmatrix} 1 + 2t - 1 \\ -1 + 3t - 1 \\ 2 + t - 3 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t - 2 \\ t - 1 \end{bmatrix}$

and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is the direction vector of the line.



$\therefore$ since $\vec{PA} \bullet \mathbf{v} = 0$,

$$\begin{bmatrix} 2t \\ 3t - 2 \\ t - 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0$$

$$\therefore 4t + 3(3t - 2) + 1(t - 1) = 0$$

$$\therefore 4t + 9t - 6 + t - 1 = 0$$

$$\therefore 14t = 7$$

$$\therefore t = \frac{1}{2}$$

$\therefore$ A is at $(2, \frac{1}{2}, \frac{5}{2})$

$$\begin{aligned} \therefore \text{the distance } d &= \sqrt{(2 - 1)^2 + (\frac{1}{2} - 1)^2 + (\frac{5}{2} - 3)^2} = \sqrt{1 + \frac{1}{4} + \frac{1}{4}} \\ &= \sqrt{\frac{3}{2}} \text{ units} \end{aligned}$$

EXERCISE 17G

- 1 a** Line 1 has direction vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $1 + 2t = -2 + 3s$ $2 - t = 3 - s$ $3 + t = 1 + 2s$
 $\therefore 2t - 3s = -3$ (1) $\therefore -t + s = 1$ (2) $\therefore t - 2s = -2$ (3)

Solving (2) and (3) simultaneously:

$$\begin{array}{r} -t + s = 1 \\ t - 2s = -2 \\ \hline -s = -1 \end{array} \therefore s = 1 \text{ and } t = 0$$

and in (1), LHS = $2t - 3s = 2(0) - 3(1) = -3$ ✓

$\therefore s = 1, t = 0$ satisfies all three equations

∴ the two lines meet at (1, 2, 3) {using $t = 0$ or $s = 1$ }

$$\begin{aligned} \text{The acute angle between the lines has } \cos \theta &= \frac{\left| \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right|}{\sqrt{6}\sqrt{14}} \\ &= \frac{|6 + 1 + 2|}{\sqrt{84}} \\ &= \frac{9}{\sqrt{84}} \quad \text{and so } \theta \doteq 10.9^\circ \end{aligned}$$

b Line 1 has direction vector $\begin{bmatrix} 2 \\ -12 \\ 12 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{aligned} \text{Now } -1 + 2t &= 4s - 3 & 2 - 12t &= 3s + 2 & 4 + 12t &= -s - 1 \\ \therefore 2t - 4s &= -2 & -12t - 3s &= 0 & 12t + s &= -5 \dots\dots (3) \\ \therefore t - 2s &= -1 \dots\dots (1) & s &= -4t \dots\dots (2) \end{aligned}$$

$$\begin{aligned} \text{Solving (1) and (2) simultaneously: } t - 2(-4t) &= -1 \\ \therefore 9t &= -1 \\ \therefore t &= -\frac{1}{9} \quad \text{and so } s = \frac{4}{9} \end{aligned}$$

$$\text{In (3), } 12t + s = 12\left(-\frac{1}{9}\right) + \frac{4}{9} = -\frac{12}{9} + \frac{4}{9} = -\frac{8}{9}, \text{ which is not } -5.$$

Since the system is inconsistent, the lines don't intersect.

∴ they are skew.

$$\begin{aligned} \text{The acute angle between the lines has } \cos \theta &= \frac{\left| \begin{bmatrix} 2 \\ -12 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} \right|}{\sqrt{2^2 + (-12)^2 + 12^2} \sqrt{4^2 + 3^2 + (-1)^2}} \\ &= \frac{|8 - 36 - 12|}{\sqrt{292}\sqrt{26}} \\ &= \frac{40}{\sqrt{7592}} \quad \text{and so } \theta \doteq 62.7^\circ \end{aligned}$$

c Line 1 has direction vector $\begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$$\text{As } \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \text{ the two lines are parallel. Hence, } \theta = 0^\circ.$$

d Line 1 has direction vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{aligned} \text{Now } t &= 1 + 3s \dots\dots (1) & 2 - t &= -2 - 2s & -2 + t &= 2s + \frac{1}{2} \\ \therefore -t + 2s &= -4 \dots\dots (2) & t - 2s &= 2\frac{1}{2} \dots\dots (3) \end{aligned}$$

$$\begin{aligned} \text{Solving (1) and (2) simultaneously, } -(1 + 3s) + 2s &= -4 \\ \therefore -1 - 3s + 2s &= -4 \\ \therefore -s &= -3 \\ \therefore s &= 3 \quad \text{and so } t = 1 + 3(3) = 10 \end{aligned}$$

$$\text{Substituting in (3), } t - 2s = 10 - 2(3) = 4 \neq 2\frac{1}{2}$$

Since the system is inconsistent, the lines do not meet. $\therefore$ they are skew.

$$\text{The acute angle between the lines has } \cos \theta = \frac{\left| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \right|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \frac{7}{\sqrt{3}\sqrt{17}}$$

$$\therefore \theta \doteq 11.4^\circ$$

e Line 1 has direction vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll} 1+t = 2+3s & 2-t = 3-2s & 3+2t = s-5 \\ t-3s = 1 \dots\dots (1) & -t+2s = 1 \dots\dots (2) & 3t-s = -8 \dots\dots (3) \end{array}$$

Solving (1) and (2) simultaneously, $t-3s = 1$

$$-t+2s = 1$$

$$\text{Adding, } -s = 2$$

$$\therefore s = -2 \text{ and } t-3(-2) = 1 \therefore t = -5$$

Checking in (3), $2t-s = 2(-5) - (-2) = -10+2 = -8 \checkmark$

Since $s = -2$, $t = -5$ satisfies all three equations, the lines meet.

They meet at $x = 1 + (-5)$, $y = 2 - (-5)$, $z = 3 + 2(-5)$ i.e., at $(-4, 7, -7)$

$$\text{The acute angle between the lines has } \cos \theta = \frac{\left| \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right|}{\sqrt{1+1+4}\sqrt{9+4+1}}$$

$$= \frac{|3+2+2|}{\sqrt{6}\sqrt{14}}$$

$$= \frac{7}{\sqrt{84}} \quad \text{and so } \theta \doteq 40.2^\circ$$

f Line 1 has direction vector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$

$$\text{Now } \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \text{ so the lines are parallel and hence } \theta = 0^\circ.$$

REVIEW SET 17A

1 a The vector equation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

b The parametric equations are

$$x = -6 + 4t, \quad y = 3 - 3t, \quad t \in \mathcal{R}$$

2 The vector equation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} + t \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad t \in \mathcal{R}$

3 $(-3, m)$ lies on the line, so $\begin{bmatrix} -3 \\ m \end{bmatrix} = \begin{bmatrix} 18 \\ -2 \end{bmatrix} + \begin{bmatrix} -7t \\ 4t \end{bmatrix}$

$$\therefore -3 = 18 - 7t \text{ and } m = -2 + 4t$$

$$\therefore 7t = 21$$

$$\text{i.e., } t = 3 \text{ and so } m = -2 + 4(3) = 10$$

$$\text{i.e., } m = 10$$

- 4 a** $x(0) = -4$ and $y(0) = 3$, so the initial position is $(-4, 3)$.
- b** $x(4) = -4 + 32$, $y(4) = 3 + 6(4)$, so at $t = 4$, the position is $(28, 27)$.
 $= 28$ $= 27$
- c** The velocity vector is $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$, so the speed is $\sqrt{8^2 + 6^2} = 10$ m/s **d** $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$
- 5** The direction vector is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ which has length $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ units
 $\therefore 2\sqrt{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ has length 20. So, the velocity vector is $\begin{bmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{bmatrix}$ or $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$
- 6 a i** The yacht is initially at $(-6, 10)$, so its initial position vector is $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$ i.e., $-6\mathbf{i} + 10\mathbf{j}$
- ii** $-\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$
 $\therefore 5(-\mathbf{i} - 3\mathbf{j})$ has length $5\sqrt{10}$
 $\therefore$ the direction vector is $-5\mathbf{i} - 15\mathbf{j}$
- iii** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \end{bmatrix} + t\begin{bmatrix} -5 \\ -15 \end{bmatrix}$ i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = -6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j})$
 $= (-6 - 5t)\mathbf{i} + (10 - 15t)\mathbf{j}$

b

beacon
 $O(0,0)$
 $P(-6-5t, 10-15t)$
 $\vec{OP} = \begin{bmatrix} -6-5t \\ 10-15t \end{bmatrix}$ and $\vec{OP} \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0$
 $\therefore -1(-6-5t) - 3(10-15t) = 0$
 $\therefore 6 + 5t - 30 + 45t = 0$
 $\therefore 50t = 24$
 $\therefore t = 0.48$ h
 (i.e., 28.8 min)

c When $t = 0.48$, $\vec{OP} = \begin{bmatrix} -6-5(0.48) \\ 10-15(0.48) \end{bmatrix} = \begin{bmatrix} -8.4 \\ 2.8 \end{bmatrix}$
 and $OP = \sqrt{(-8.4)^2 + (2.8)^2} \div 8.85$ km

As the closest distance is 8.85 km and the radius is 8 km, the yacht will miss the reef.

- 7 a** $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ where $t \geq 0$. When $t = 0$, the time is 2:17 pm
 i.e., $x_1(t) = 2 + t$, $y_1(t) = 4 - 3t$
- b** Likewise, $x_2(t) = 11 - (t + 2)$, $y_2(t) = 3 + a(t + 2)$
 i.e., $x_2(t) = 9 - t$ $y_2(t) = [3 + 2a] + at$
- c** They meet where $2 + t = 9 - t$ and $4 - 3t = [3 + 2a] + at$
 $\therefore 2t = 7$
 $\therefore t = \frac{7}{2}$ $\therefore$ the time would be 2:17 pm plus $3\frac{1}{2}$ min i.e., 2:20:30 pm
- d** When $t = \frac{7}{2}$,
 $4 - 3\left(\frac{7}{2}\right) = \left[3 + 2\left(\frac{7}{2}\right)\right] + a\left(\frac{7}{2}\right)$
 $\therefore -\frac{13}{2} = 10 + \frac{7a}{2}$
 $\therefore -13 = 20 + 7a$
 $\therefore 7a = -33$
 $\therefore a = -\frac{33}{7}$
- Y18 has velocity vector $\begin{bmatrix} -1 \\ -\frac{33}{7} \end{bmatrix}$
- with speed $= \sqrt{(-1)^2 + \left(-\frac{33}{7}\right)^2} \div 4.82$ km/min
 $\tan \alpha = \frac{1}{\frac{33}{7}} = \frac{7}{33}$
 $\therefore \alpha = \tan^{-1}\left(\frac{7}{33}\right) \div 12.0^\circ$
 $\therefore$ the direction is $180^\circ + \alpha^\circ = 192^\circ$
 i.e., the torpedo has speed 4.82 km/min and direction 192° .
-

- 8 a** Line 1 has direction vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and Line 4 has direction vector $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

Now $\begin{bmatrix} 5 \\ -2 \end{bmatrix} = -\begin{bmatrix} -5 \\ 2 \end{bmatrix}$, so Lines 1 and 4 are parallel, i.e., $KL \parallel MN$.

b $\overrightarrow{KL} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\overrightarrow{NK} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$, $\overrightarrow{MN} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ $\therefore \overrightarrow{KL} \bullet \overrightarrow{NK} = 20 - 20 = 0$,

and $\overrightarrow{NK} \bullet \overrightarrow{MN} = -20 + 20 = 0$

$\therefore NK$ is perpendicular to both KL and MN .

- c** Lines 1 and 3 meet at K

$$\therefore \begin{bmatrix} 2 \\ 19 \end{bmatrix} + p \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + r \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5p-4r \\ -2p-10r \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} 5p-4r & = & 1 \quad \times 5 \\ 2p+10r & = & 12 \quad \times 2 \end{array}$$

$$\therefore \begin{array}{rcl} 25p-20r & = & 5 \\ 4p+20r & = & 24 \end{array}$$

$$\hline 29p = 29$$

Adding, $29p = 29$

$$\therefore p = 1$$

So, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 19 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$

$\therefore K$ is (7, 17)

Lines 3 and 4 meet at N

$$\therefore \begin{bmatrix} 3 \\ 7 \end{bmatrix} + r \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + s \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4r+5s \\ 10r-2s \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} 4r+5s & = & 40 \quad \times 2 \\ 10r-2s & = & -16 \quad \times 5 \end{array}$$

$$\therefore \begin{array}{rcl} 8r+10s & = & 80 \\ 50r-10s & = & -80 \end{array}$$

$$\therefore 58r = 0$$

$$\therefore r = 0$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

i.e., N is (3, 7)

Lines 2 and 4 meet at M.

$$\therefore \begin{bmatrix} 33 \\ -5 \end{bmatrix} + r \begin{bmatrix} -11 \\ 16 \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + s \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -11r+5s \\ 16r-2s \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} -11r+5s & = & 10 \quad \times 2 \\ 16r-2s & = & -4 \quad \times 5 \end{array}$$

$$\therefore \begin{array}{rcl} -22r+10s & = & 20 \\ 80r-10s & = & -20 \end{array}$$

$$\hline 58r = 0$$

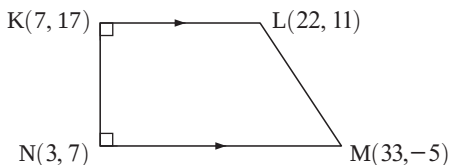
$$\therefore 58r = 0$$

$$\therefore r = 0$$

So, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 33 \\ -5 \end{bmatrix}$

$\therefore M$ is (33, -5)

d



$$\begin{aligned} KL &= \sqrt{(22-7)^2 + (11-17)^2} \\ &= \sqrt{225 + 36} \\ &= \sqrt{261} \text{ units} \end{aligned}$$

$$\begin{aligned} NK &= \sqrt{(33-3)^2 + (-5-7)^2} \\ &= \sqrt{900 + 144} \\ &= \sqrt{1044} \text{ units} \end{aligned}$$

$$\begin{aligned} KN &= \sqrt{(7-3)^2 + (17-7)^2} \\ &= \sqrt{16 + 100} \\ &= \sqrt{116} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= \left(\frac{\sqrt{261} + \sqrt{1044}}{2} \right) \times \sqrt{116} \\ &= 261 \text{ units}^2 \end{aligned}$$

REVIEW SET 17B

1 a $\overrightarrow{AB} = \begin{bmatrix} -1-3 \\ 2-2 \\ 4-1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$ $\therefore$ the equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}, \quad t \in \mathcal{R}$

- b** Any point on AB can be represented by a point $C(3 - 4t, 2, -1 + 5t)$.

If $AC = 2\sqrt{41}$, then

$$\begin{aligned}\sqrt{(3 - 4t - 3)^2 + (2 - 2)^2 + (-1 + 5t - (-1))^2} &= 2\sqrt{41} \\ \therefore \sqrt{16t^2 + 0 + 25t^2} &= 2\sqrt{41} \\ \therefore 41t^2 &= 4 \times 41 \\ \therefore t^2 &= 4 \\ \therefore t &= \pm 2\end{aligned}$$

So, C is $(-5, 2, 9)$ or $(11, 2, -11)$

2 a $\vec{AB} = \begin{bmatrix} 0 - 3 \\ 2 - -1 \\ -1 - 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$

- b** If P divides BA in the ratio $2 : 5$, then $\vec{BP} : \vec{PA} = 2 : 5$

If P is (a, b, c) then $\begin{bmatrix} a - 0 \\ b - 2 \\ c - -1 \end{bmatrix} : \begin{bmatrix} 3 - a \\ -1 - b \\ 1 - c \end{bmatrix} = 2 : 5$

$$5 \begin{bmatrix} a \\ b - 2 \\ c + 1 \end{bmatrix} = 2 \begin{bmatrix} 3 - a \\ -1 - b \\ 1 - c \end{bmatrix}$$

$$\begin{aligned}\therefore 5a &= 6 - 2a, & 5b - 10 &= -2 - 2b, & 5c + 5 &= 2 - 2c \\ \therefore 7a &= 6, & 7b &= 8, & 7c &= -3 \\ \therefore a &= \frac{6}{7}, & b &= \frac{8}{7}, & c &= -\frac{3}{7}\end{aligned}$$

So, P is $(\frac{6}{7}, \frac{8}{7}, -\frac{3}{7})$

3 a $\vec{PQ} = \begin{bmatrix} 4 - -1 \\ 0 - 2 \\ -1 - 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix}$ **b** The x -axis has direction vector $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and if the angle is θ ,

$$\begin{aligned}\begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \sqrt{25 + 4 + 16} \sqrt{1 + 0 + 0} \cos \theta \\ \therefore 5 &= \sqrt{45} \cos \theta \\ \therefore \cos \theta &= \frac{5}{\sqrt{45}} \\ \therefore \theta &\doteq 41.8^\circ\end{aligned}$$

4 a Let $\frac{x - 8}{3} = \frac{y + 9}{-16} = \frac{z - 10}{7} = t$

$$\therefore x = 8 + 3s, \quad y = -9 - 16s, \quad z = 10 + 7s$$

Line 1 has direction vector $\begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix}$, and Line 2 has direction vector $\begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix}$

Since one vector is not a scalar multiple of the other, the vector lines are not parallel.

Now $8 + 3s = 15 + 3t, \quad -9 - 16s = 29 + 8t, \quad 10 + 7s = 5 - 5t$

$$\therefore 3s - 3t = 7 \dots\dots (1) \quad 16s + 8t = -38 \dots\dots (2) \quad 7s + 5t = -5 \dots\dots (3)$$

$$\begin{array}{llll}
 \text{Solving (1) and (3),} & 3s - 3t = 7 & \times 5 & \\
 & 7s + 5t = -5 & \times 3 & \text{and } 3\left(\frac{5}{9}\right) - 3t = 7 \\
 \therefore & 15s - 15t = 35 & & \therefore \frac{5}{3} - 3t = 7 \\
 & 21s + 15t = -15 & & \therefore 5 - 9t = 21 \\
 \text{Adding,} & 36s & = 20 & \therefore 9t = -16 \\
 & \therefore s = \frac{5}{9} & & \therefore t = -\frac{16}{9}
 \end{array}$$

$$\text{and in (2), } 16s + 8t = 16\left(\frac{5}{9}\right) + 8\left(-\frac{16}{9}\right) = -5\frac{1}{3} \neq -38$$

$\therefore$ the equations do not have a common solution and so the two lines are skew.

b If θ is the angle,

$$\begin{aligned}
 \begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix} &= \sqrt{9 + 256 + 49} \sqrt{9 + 64 + 25} \cos \theta \\
 \therefore 9 - 128 - 35 &= \sqrt{314} \sqrt{98} \cos \theta \\
 \therefore \frac{-154}{\sqrt{314 \times 98}} &= \cos \theta \quad \therefore \theta \doteq 28.6^\circ
 \end{aligned}$$

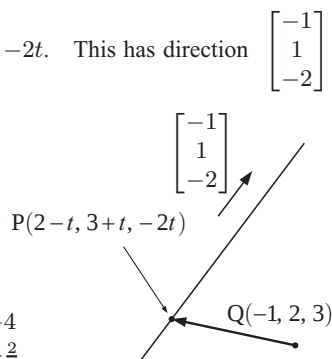
5 If $2 - x = y - 3 = -\frac{1}{2}z = t$, then $x = 2 - t$, $y = 3 + t$, $z = -2t$. This has direction $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

$$\vec{QP} = \begin{bmatrix} 2 - t - 1 \\ 3 + t - 2 \\ -2t - 3 \end{bmatrix} = \begin{bmatrix} 1 - t \\ 1 + t \\ -2t - 3 \end{bmatrix}$$

In order that $\vec{QP} \perp \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$,

$$\begin{bmatrix} 1 - t \\ 1 + t \\ -2t - 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = 0 \quad \therefore -3 + t + 1 + t + 4t + 6 = 0 \quad \therefore 6t = -4 \quad \therefore t = -\frac{2}{3}$$

So, P is $\left(2 - \frac{2}{3}, 3 - \frac{2}{3}, -2\left(-\frac{2}{3}\right)\right)$ i.e., $\left(\frac{8}{3}, \frac{7}{3}, \frac{4}{3}\right)$



6 a $\vec{PQ} = \begin{bmatrix} 3 - 2 \\ 4 - 0 \\ -2 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$ **b** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

and $|\vec{PQ}| = \sqrt{1 + 16 + 9} = \sqrt{26}$ units $\therefore x = 2 + t$, $y = 4t$, $z = 1 - 3t$, $t \in \mathcal{R}$

7 a Let $\frac{x-3}{2} = y-4 = \frac{z+1}{-2} = s \quad \therefore x = 3 + 2s$, $y = 4 + s$, $z = -1 - 2s$, $s \in \mathcal{R}$

Line 1 has direction vector $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Since one of these vectors is not a scalar multiple of the other, the lines are not parallel.

Now $3 + 2s = -1 + 3t$, $4 + s = 2 + 2t$, $-1 - 2s = 3 - t$
 $\therefore 2s - 3t = -4$ (1) $s - 2t = -2$ (2) $-2s + t = 4$ (3)

Solving (1) and (3), $2s - 3t = -4$ So, in (2), $s - 2t$
 $-2s + t = 4$ $= -2 - 2(0)$
 $\therefore -2t = 0$ $= -2 \quad \checkmark$
 $\therefore t = 0$ and $s = -2$

$\therefore s = -2$, $t = 0$ satisfies all three equations. $\therefore$ the lines intersect at $(-1, 2, 3)$.

b If θ is the angle between the lines, then

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \sqrt{4+1+4}\sqrt{9+4+1} \cos \theta$$

$$\therefore 6+2+2 = 3\sqrt{14} \cos \theta$$

$$\therefore \cos \theta = \frac{10}{3\sqrt{14}}$$

- 8** If A is the origin, AB the x -axis, AD the y -axis, and AP the z -axis, then Q is (4, 0, 7), D is (0, 10, 0) and M is (0, 5, 7).

Now $\overrightarrow{DQ} = \begin{bmatrix} 4-0 \\ 0-10 \\ 7-0 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 7 \end{bmatrix}$ and $\overrightarrow{DM} = \begin{bmatrix} 0-0 \\ 5-10 \\ 7-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$

So, if the required angle is θ ,

$$\begin{bmatrix} 4 \\ -10 \\ 7 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} = \sqrt{16+100+49}\sqrt{0+25+49} \cos \theta$$

$$\therefore 0+50+49 = \sqrt{165}\sqrt{74} \cos \theta$$

$$\therefore \frac{99}{\sqrt{165 \times 74}} = \cos \theta \quad \text{and so } \theta \doteq 26.4^\circ$$

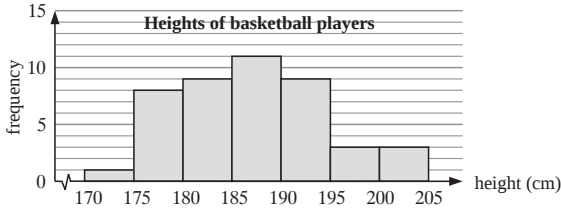
Chapter 18

DESCRIPTIVE STATISTICS

EXERCISE 18A

- 1 a Heights can take any value from 170 cm to 205 cm, including decimal values such as 181.372 cm, i.e., any real number between 170 and 205.

b



- c The modal class is the class occurring most often. This is '185 -'.

- d The distribution is slightly positively skewed (more of a 'tail' to the right (positive)).

- 2 a The data is continuous numerical. Actual time is continuous and could be measured to the nearest millisecond. After it has been rounded to the nearest minute it becomes discrete numerical data.

b

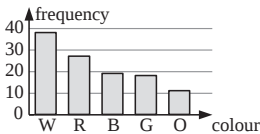
Stem	Leaf
0	3 6 8 8 8 8
1	0 0 0 0 2 2 2 4 4 4 4 5 5 5 5 6 6 6 6 7 8 8 8 8 9
2	0 0 0 1 2 4 5 5 5 6 7 7 8
3	1 2 2 2 3 4 5 7 8
4	0 2 5 5 5 6

1 | 2 means 12 minutes

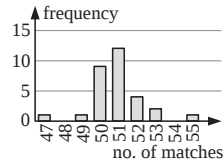
- c The distribution is positively skewed (i.e., skewed to the high end).

- d "The modal travelling time was between 10 and 20 minutes" if considering classes.
The mode is actually 10.

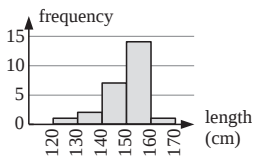
- 3 a The data is categorical, so a column graph should be used.



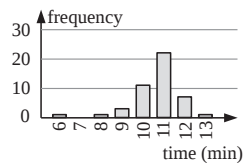
- b The data is discrete numerical, so a column graph should be used.



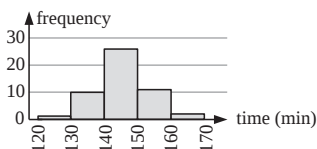
- c The data is continuous, so a histogram should be used.

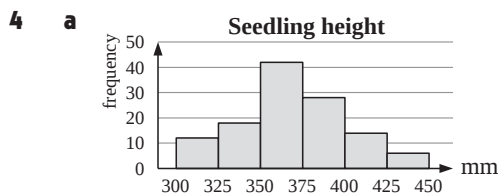


- d The data is discrete numerical, so a column graph should be used.



- e The data is continuous, so a histogram should be used.





- b** Number which are ≥ 400 mm is
 $14 + 6 = 20$ seedlings.
- c** $\% \text{ between } 349 \text{ and } 400 = \frac{42 + 28}{120} \times 100\%$
 $= \frac{70}{120} \times 100\%$
 $\div 58.3\%$

- d i** Number
 $= \frac{12 + 18 + 42 + 28}{120} \times 1462$
 $= \frac{100}{120} \times 1462$
 $\div 1218 \text{ seedlings}$
- ii** Number
 $= \frac{28 + 14}{120} \times 1462$
 $= \frac{42}{120} \times 1462$
 $\div 512 \text{ seedlings}$

- 5 a** Number in survey $= 15 + 20 + 30 + 40 + 25 + 10 = 140$
- b** Number less than 180 cm $= 15 + 20 + 30 = 65$
- c** $\% \text{ which were } \geq 175 \text{ cm} = \frac{40 + 25 + 10}{140} \times 100\%$
 $= \frac{75}{140} \times 100\%$
 $= 53.6\%$

EXERCISE 18B.1

- 1 a i** mean $= \frac{2+3+3+3+4+\dots+9+9}{23}$
 $= \frac{129}{23}$
 $\div 5.61$
- ii** median $= 12\text{th score (when in order)}$
 $= 6$
- iii** mode $= 6$ (6 occurs most often)
- b i** mean $= \frac{10+12+12+15+\dots+20+21}{15}$
 $= \frac{245}{15}$
 $\div 16.3$
- ii** median $= 8\text{th score (when in order)}$
 $= 17$
- iii** mode $= 18$
- c i** mean $= \frac{22.4+24.6+21.8+\dots+23.5}{11}$
 $= \frac{273}{11}$
 $\div 24.8$
- ii** median $= 6\text{th score (when in order)}$
 $= 24.9$
- iii** mode $= 23.5$
- d i** mean $= \frac{127+123+115+105+\dots+141}{21}$
 $= \frac{2700}{21}$
 $\div 128.6$
- ii** median $= 11\text{th score (when in order)}$
 $= 128$
- iii** mode $= 115, 127$ (bimodal)

- 2 a** mean of set A $= \frac{3+4+4+5+\dots+10}{13}$
 $= 6.46$
- mean of set B $= \frac{3+4+4+5+\dots+15}{13}$
 $= 6.85$
- b** median of set A $= 7\text{th score} = 7$ median of set B $= 7\text{th score} = 7$
- c** The data sets are the same except for the last value, and the last value of set A is less than that of set B. So, the mean of set A is less than that of set B.
- d** The middle value of both data sets is the same, so the median is the same.

$$3 \quad a \quad \text{mean} = \frac{23\,000 + 46\,000 + 23\,000 + \dots + 32\,000}{10} = \$29\,300$$

$$\text{median} = \text{middle score when in order of size} = \frac{\$23\,000 + \$24\,000}{2} = \$23\,500$$

$$\text{mode} = \$23\,000$$

b The mode is unsatisfactory because it is the lowest salary. It does not take the higher values into account.

c The median is too close to the lower end of the distribution and it does not take the higher values into account. So the median is not a satisfactory measure of the middle.

$$4 \quad a \quad \text{mean} = \frac{3 + 1 + 0 + 0 + \dots + 1 + 0 + 0}{31} = \frac{99}{31} \div 3.19$$

$$\text{median} = 16\text{th score (when in order)} = 0$$

$$\text{mode} = 0 \quad (\text{most frequently occurring score})$$

b The median is not in the centre. It ignores the high upper values of the distribution.

c The mode is the lowest value. It does not take the higher values into account.

d Yes, 42 and 21.

e No, as this would ignore actual valid data.

$$5 \quad a \quad \text{mean} = \frac{43 + 55 + 41 + 37}{4} = \frac{176}{4} = 44 \quad b \quad \text{another } 44$$

$$c \quad \text{new mean} = \frac{43 + 55 + 41 + 37 + 25}{5} = 40.2$$

d It will increase the new mean to 40.3 as 41 is greater than the old mean of 40.2.

$$\left\{ \frac{5 \times 40.2 + 41}{6} \div 40.3 \right\}$$

$$6 \quad \text{mean} = \frac{\text{total}}{10}, \therefore 11.6 = \frac{\text{total}}{10}, \therefore \text{total} = 11.6 \times 10 = 116$$

$$7 \quad \text{mean} = \frac{\text{total}}{12}, \therefore 262 = \frac{\text{total}}{12}, \therefore \text{total} = 262 \times 12 = 3144 \text{ km}$$

$$8 \quad \text{mean} = \frac{\text{total}}{12}, \therefore 15\,467 = \frac{\text{total}}{12}, \therefore \text{total} = 15\,467 \times 12 = \$185\,604$$

$$9 \quad \frac{5 + 9 + 11 + 12 + 13 + 14 + 17 + x}{8} = 12$$

$$\therefore \frac{81 + x}{8} = 12$$

$$\therefore 81 + x = 96$$

$$\therefore x = 15$$

$$10 \quad \frac{3 + 0 + a + a + 4 + a + 6 + a + 3}{9} = 4$$

$$\therefore \frac{4a + 16}{9} = 4$$

$$\therefore 4a + 16 = 36$$

$$\therefore 4a = 20$$

$$\therefore a = 5$$

$$11 \quad \frac{29 + 36 + 32 + 38 + 35 + 34 + 39 + x}{8} = 35 \quad \therefore \frac{243 + x}{8} = 35$$

$$\therefore 243 + x = 280$$

$$\therefore x = 37, \text{ so her 8th result was } 37$$

$$12 \quad a \quad \text{Total distance driven (first 5 days)} = 5 \times 256 = 1280 \text{ km}$$

$$b \quad \text{Total distance travelled (next 3 days)} = 172 \times 3 = 516 \text{ km}$$

$$c \quad \text{mean distance} = \frac{1280 + 516}{8} \text{ km} = 224.5 \text{ km}$$

13 Total for first 10 measurements = $10 \times 15.7 = 157$

Total for next 20 measurements = $20 \times 14.3 = 286$ $\therefore$ mean = $\frac{157 + 286}{30} \div 14.8$

14 Total for first 10 innings = $25.4 \times 10 = 254$ runs

$\therefore$ new average = $\frac{254 + 58 + 16}{12} \div 27.3$ runs

15 Scores were 5 7 9 9 10 a b where $a \leq b$ say.

mean = $\frac{5 + 7 + 9 + 9 + 10 + a + b}{7} = 8$

$\therefore \frac{40 + a + b}{7} = 8$

$\therefore 40 + a + b = 56$

$\therefore a + b = 16 \quad \{a \leq 12, b \leq 12\}$

Possibilities are:

a	5	6	7	8
b	11	10	9	8

$\times$ $\times$ $\checkmark$ $\times$
 $\uparrow$ $\uparrow$ $\uparrow$
 reject as modes are 8 and 9
 reject as modes are 9 and 10
 reject as modes are 5 and 9

So, the missing results are 7 and 9.

EXERCISE 18B.2

1 a The mode is 1 (occurs most often).

b The median is the average of the 15th and 16th scores = $\frac{1 + 1}{2} = 1$

c

x	f	fx
0	4	0
1	12	12
2	11	22
3	3	9
Σ	30	43

mean = $\frac{\Sigma fx}{\Sigma f}$
 $= \frac{43}{30}$
 $\div 1.43$

2 a i

x	f	fx
0	5	0
1	8	8
2	13	26
3	8	24
4	6	24
5	3	15
6	3	18
7	2	14
8	1	8
9	0	0
10	0	0
11	1	11
Σ	50	148

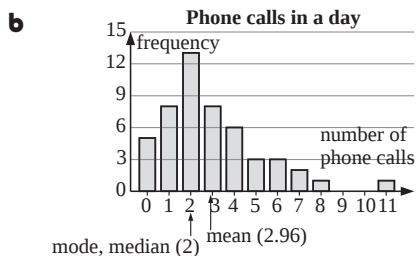
mean = $\frac{\Sigma fx}{\Sigma f}$
 $= \frac{148}{50}$
 $= 2.96$

ii median

= average of 25th and 26th
(when in order)

$= \frac{2 + 2}{2} \quad \left\{ \begin{array}{l} 13 \text{ scores are 1 or 0} \\ 26 \text{ scores are 2, 1 or 0} \end{array} \right\}$
 $= 2$

iii mode = 2 {occurs most often}



- c** The distribution is positively skewed. 11 is an outlier.
- d** The mean takes into account the larger numbers of phone calls.
- e** The mean as it best represents all the data.

3 a

Donation (\$)	Frequency
1	7
2	9
3	2
4	4
5	8

b Total number of donations = $\sum f$

$$= 7 + 9 + 2 + 4 + 8$$

$$= 30$$

c i

x	f	fx
1	7	7
2	9	18
3	2	6
4	4	16
5	8	40
Σ	30	87

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{87}{30} = 2.9$$

$\therefore$ mean donation is \$2.90

ii median donation = average of 15th and 16th (when in order)

$$= \frac{\$2 + \$2}{2} \quad \left\{ \begin{array}{l} 7 \text{ are } \$1 \\ 16 \text{ are } \$1 \text{ or } \$2 \end{array} \right\}$$

$$= \$2$$

iii mode donation = \$2 {occurs most often}

b The mode. (highest column)

4 a i mode = 49 {occurs most often}

ii median = average of 15th and 16th values (when in order)

$$= \frac{49 + 49}{2} \quad \{9 \text{ are } 47 \text{ or } 48 \text{ and the next } 11 \text{ are } 49\}$$

$$= 49$$

iii

x	f	fx
47	5	235
48	4	192
49	11	539
50	6	300
51	3	153
52	1	52
Σ	30	1471

$$\text{mean} = \frac{\sum fx}{\sum f}$$

$$= \frac{1471}{30}$$

$$\div 49.0$$

b No, as they claim the average (mean) is 50 matches/box.

c The sample of only 30 is not large enough. The company could have won its case by arguing that a larger sample would have found an average of 50 matches per box.

- 5 a i**
- | x | f | fx |
|----------|-----|------|
| 1 | 5 | 5 |
| 2 | 28 | 56 |
| 3 | 15 | 45 |
| 4 | 8 | 32 |
| 5 | 2 | 10 |
| 6 | 1 | 6 |
| Σ | 59 | 154 |
- $$\text{mean} = \frac{\sum fx}{\sum f} = \frac{154}{59} \div 2.61$$
- ii** mode = 2 {occurs most often}
- iii** median = 30th score (when in order) = 2
- b** This school has more children per family (2.61) than the average Australian family (2.2).
- c** Positive as the higher values are more spread out.

d The mean is higher than the mode and median.

- 6 a i** mean = $\frac{53 + 55 + 56 + 60 + \dots + 91}{17} = \frac{1175}{17} \div 69.1$
- ii** median = 9th score (when in order) = 67
- iii** mode = 73 (73 is the only one occurring more than once)
- b i** mean = $\frac{37 + 40 + 44 + 48 + \dots + 81}{23} = \frac{1347}{23} \div 5.86$
- ii** median = 12th score (when in order) = 5.8
- iii** mode = 6.7 {occurs most often}

7 a Without fertiliser

2	
3	
4	
5	
6	
7	
8	
9	

x	f	fx	cf
2	2	4	2
3	11	33	13
4	19	76	32
5	29	145	61
6	51	306	112
7	25	175	137
8	12	96	149
9	1	9	150

- i** mean = $\frac{\sum fx}{\sum f} = \frac{845}{150} \div 5.63$
- ii** mode = 6 {occurs most often}
- iii** median = average of 75th and 76th scores = $\frac{6 + 6}{2} = 6$

b With fertiliser

3	
4	
5	
6	
7	
8	
9	
10	
13	

x	f	fx	cf
3	4	12	4
4	13	52	17
5	11	55	28
6	28	168	56
7	48	336	104
8	27	216	131
9	14	126	145
10	4	40	149
13	1	13	150

- i** mean = $\frac{\sum fx}{\sum f} = \frac{1018}{150} \div 6.79$
- ii** mode = 7 {occurs most often}
- iii** median = average of 75th and 76th scores = $\frac{7 + 7}{2} = 7$
- c** The mean best represents the centre for this data.
- d** Yes, as 6.79 is significantly greater than 5.63.
- Note:** The total yield of the crop may not have improved as, for example, the number of pods per plant may have decreased when using the fertiliser.

$$8 \quad \text{Mean for Team A} = \frac{91 + 76 + 104 + \dots + 82}{12} = 91.25$$

$$\text{Mean for Team B} = \frac{87 + 104 + 112 + \dots + 108}{12} = 91.75$$

So, B has the higher average (but only just).

9 The mode(s) occurs most often.

a 49 **b** 144 and 147 (bimodal) **c** 25

$$10 \quad \begin{array}{ll} \mathbf{a} & \text{median} = 5\text{th score (when in order)} \\ & = 29 \\ \mathbf{b} & \text{median} = \frac{5\text{th score} + 6\text{th score}}{2} \\ & = \frac{107 + 107}{2} \\ & = 107 \\ \mathbf{c} & \text{median} = \frac{6\text{th score} + 7\text{th score}}{2} \\ & = \frac{146 + 153}{2} \\ & = 149.5 \end{array}$$

11

Siblings	f	fx
0	13	0
1	12	12
2	11	22
3	8	24
4	3	12
5	3	15
Σ	50	85

a modal number = 0

$$\mathbf{b} \quad \text{mean number} = \frac{\sum fx}{\sum f} = \frac{85}{50} = 1.7$$

$$\mathbf{c} \quad \begin{aligned} \text{median number} &= \frac{25\text{th} + 26\text{th}}{2} \\ &= \frac{1 + 2}{2} = 1.5 \end{aligned}$$

$$12 \quad \begin{aligned} \text{mean average rainfall} &= \frac{16 + 34 + 38 + \dots + 52 + 21}{12} \\ &= \frac{974}{12} \div 81.2 \text{ mm} \end{aligned}$$

$$13 \quad \begin{aligned} \mathbf{a} \quad \text{mean selling price} &= \frac{146\,400 + 127\,600 + 211\,000 + \dots + 162\,500}{10} \\ &= \$163\,770 \end{aligned}$$

$$\text{median selling price} = \frac{5\text{th} + 6\text{th}}{2} = \frac{146\,400 + 148\,000}{2} = \$147\,200$$

These figures differ by \$16 570. There are more selling prices at the lower end of the market (i.e., smaller prices).

b i Use the mean as it tends to inflate the average house value of that district.

ii Use the median as you want to buy at the lowest price possible.

14

x	f	fx
32	6	192
33	8	264
34	9	306
35	13	455
36	10	360
37	3	111
38	2	76
Σ	51	1764

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{1764}{51} \div 34.6$$

mode = 35 {occurs most often}

median = 26th score (when in order)
 = 35 {23 are 34 or less
 36 are 35 or less}

15 mean of all 27 is

$$\frac{12 \times 16.5 + 15 \times 18.6}{27} \div 17.7$$

$$16 \quad \text{a} \quad \text{mean birth weight} = \frac{75 + 70 + 80 + \dots + 83}{8} = \frac{567}{8} \div 70.9 \text{ grams}$$

$$\text{b} \quad \text{mean after 2 weeks} = \frac{210 + 200 + 200 + \dots + 230}{8} = \frac{1681}{8} \div 210 \text{ grams}$$

$$\text{c} \quad \text{mean increase} \div (210.13 - 70.88) \text{ grams} \div 139 \text{ grams}$$

17 15 of these 10.1, 10.4, 10.7, 10.9, 12 of these

Median = 16th score (when in order)
= 10.1 cm

18 Total for first 14 matches = 14×16.5 goals = 231 goals

$$\therefore \text{new average} = \frac{231 + 21 + 24}{16} = \frac{276}{16} = 17.25 \text{ goals/game}$$

19 The measurements are 7, 9, 11, 13, 14, 17, 19, a , b where $a \leq b$

$$\text{mean} = \frac{7 + 9 + 11 + 13 + \dots + a + b}{9} = \frac{90 + a + b}{9}$$

$$\therefore \frac{90 + a + b}{9} = 12$$

$$\therefore 90 + a + b = 108$$

$$\therefore a + b = 18$$

7 9 11 13 14 17 19

If $b \geq 13$, then $a \leq 5$ and the median = 13 (×)

If $b = 12$, then $a = 6$ and the median = 12 (✓)

The remaining cases are

a	7	8	9
b	11	10	9
median	11	11	11

So the other two data values are 6 and 12.

20 a Brand A

x	f	fx
46	1	46
47	1	47
48	2	96
49	7	343
50	10	500
51	20	1020
52	15	780
53	3	159
55	1	55
Σ	60	3046

$$\begin{aligned} \text{mean} &= \frac{\sum fx}{\sum f} \\ &= \frac{3046}{60} \\ &\div 50.8 \end{aligned}$$

Brand B

x	f	fx
48	3	144
49	17	833
50	30	1500
51	7	357
52	2	104
53	1	53
54	1	54
Σ	61	3045

$$\begin{aligned} \text{mean} &= \frac{\sum fx}{\sum f} \\ &= \frac{3045}{61} \\ &\div 49.9 \end{aligned}$$

b Based on average contents the CPS should not prosecute either manufacturer. To the nearest match, the average contents of A is 51 and B, 50.

21 Brand X

x	f	fx
370	2	740
380	5	1900
390	32	12 480
400	35	14 000
410	20	8200
420	5	2100
430	1	430
Σ	100	39 850

$$\begin{aligned}
 &\text{mean weight} \\
 &= \frac{\Sigma fx}{\Sigma f} \\
 &= \frac{39\,850}{100} \\
 &\div 399 \text{ g}
 \end{aligned}$$

Brand Y

x	f	fx
380	3	1140
390	21	8190
400	43	17 200
410	24	9840
420	7	2940
430	2	860
Σ	100	40 170

$$\begin{aligned}
 &\text{mean weight} \\
 &= \frac{\Sigma fx}{\Sigma f} \\
 &= \frac{40\,170}{100} \\
 &\div 402 \text{ g}
 \end{aligned}$$

So, the magazine would recommend Brand Y.

22 a i median salary

$$\begin{aligned}
 &= \frac{10\text{th} + 11\text{th}}{2} \quad \{\text{when in order}\} \\
 &= \frac{35\,000 + 28\,000}{2} \\
 &= \$31\,500
 \end{aligned}$$

ii modal salary

$$= \$28\,000 \quad \{\text{occurs most often}\}$$

iii

x	f	fx
50 000	1	50 000
42 000	3	126 000
35 000	6	210 000
28 000	10	280 000
Σ	20	666 000

$$\begin{aligned}
 &\text{mean} \\
 &= \frac{\Sigma fx}{\Sigma f} \\
 &= \frac{666\,000}{20} \\
 &= \$33\,300
 \end{aligned}$$

b The mean as it is the highest value.

EXERCISE 18B.3**1 a**

x (midpoint)	f	fx
3	7	21
8	12	96
13	15	195
18	10	180
23	11	253
Σ	55	745

$$\bar{x} = \frac{745}{55} \div 13.5$$

b

x (midpoint)	f	fx
41	2	82
44	1	44
47	5	235
50	6	300
53	12	636
56	3	168
Σ	29	1465

$$\bar{x} = \frac{1465}{29} \div 50.5$$

2

x (midpoint)	f	fx
4.5	2	9
14.5	5	72.5
24.5	7	171.5
34.5	27	931.5
44.5	9	400.5
Σ	50	1585

$$\begin{aligned}
 \therefore \text{mean result} &= \frac{1585}{50} \\
 &= 31.7
 \end{aligned}$$

$$\mathbf{3 \quad a} \quad \bar{x} = \frac{15 + 8 + 6 + 10 + \dots + 14 + 26}{50} = \frac{681}{50} \div 13.6 \text{ goals}$$

b i

Group	Tally	midpoint (x)	frequency (f)	$f x$
0 - 4		2	5	10
5 - 9		7	9	63
10 - 14		12	14	168
15 - 19		17	13	221
20 - 24		22	6	132
25 - 29		27	3	81
		Σ	50	675

$$\begin{aligned} \text{Approx. mean} \\ &= \frac{675}{50} \\ &= 13.5 \text{ goals} \end{aligned}$$

ii

Group	Tally	midpt. (x)	freq. (f)	$f x$
0 - 8		4	11	44
9 - 16		12.5	23	287.5
17 - 24		20.5	13	266.5
25 - 30		27.5	3	82.5
		Σ	50	680.5

$$\begin{aligned} \text{Approx. mean} \\ &= \frac{680.5}{50} \\ &\div 13.6 \text{ goals} \end{aligned}$$

c The approximations are very good as estimations of the actual mean, 13.6 goals.

4

midpoint (x)	f	$f x$
412	2	824
437	7	3059
462	15	6930
487	31	15 097
512	27	13 824
537	12	6444
562	4	2248
587	1	587
Σ	99	49 013

$$\begin{aligned} \text{Approximate mean} \\ &= \frac{\Sigma f x}{\Sigma f} \\ &= \frac{49\,013}{99} \\ &\div 495 \text{ mm} \end{aligned}$$

5
a 70

b $\div 411\,000$ litres

c

midpoint (x)	f	$f x$
2499.5	4	9998
3499.5	4	13 998
4499.5	9	40 495.5
5499.5	14	76 993
6499.5	23	149 488.5
7499.5	16	119 992
Σ	70	410 965

$$\begin{aligned} \text{Approximate mean} \\ &= \frac{\Sigma f x}{\Sigma f} \\ &= \frac{410\,965}{70} \\ &\div 5870 \text{ litres} \end{aligned}$$

6
a $5 + 10 + 25 + 40 + 10 + 15 + 10 + 10 = 125$ people

b

midpoint (x)	frequency (f)	$f x$
85	5	425
95	10	950
105	25	2625
115	40	4600
125	10	1250
135	15	2025
145	10	1450
155	10	1550
Σ	125	14 875

$$\begin{aligned} \text{Approximate mean} \\ &= \frac{\Sigma f x}{\Sigma f} \\ &= \frac{14\,875}{125} \\ &\div 119 \text{ people} \end{aligned}$$

c $\frac{15}{125}$ scored < 100
i.e., $\frac{3}{25}$ scored < 100

- d** 20% of 125 people = 25 people and 90 people scored < 130 for the test

$\therefore$ estimate is $130 + \frac{10}{15} \times 10 \div 137$ marks

- 7 a** $10 + 15 + 20 + 30 + 15 + 5 = 95$ students

midpoint (x)	f	fx
47.5	10	475
52.5	15	787.5
57.5	20	1150
62.5	30	1875
67.5	15	1012.5
72.5	5	362.5
Σ	95	5662.5

$$\begin{aligned} \text{Approx. mean} &= \frac{\Sigma fx}{\Sigma f} \\ &= \frac{5662.5}{95} \\ &\div 59.6 \text{ kg} \end{aligned}$$

- c** $10 + 15 = 25$ students

d $\frac{15 + 20}{95} \times 100\% \div 36.8\%$

e $\frac{10 + 15 + 20}{95} = \frac{45}{95} = \frac{9}{19} (\div 47.4\%)$

EXERCISE 18C

- 1 a** The total frequency = $25 + 11 + 8 + 5 + 4 + 1 = 54$
 $\therefore$ the median is the average of the 27th and 28th scores

$\therefore$ median = $\frac{2 + 2}{2} = 2$

- b** The total frequency = $1 + 3 + 11 + 12 + 8 + 2 = 37$

$\therefore$ the median is the 19th score

$\therefore$ median = 8 {15 scores are 5, 6 or 7}

$\left(\frac{37 + 1}{2} = 19 \right)$

- 2** Total frequency = $67 + 35 + 17 + 8 + 11 + 2 + 1 = 141$

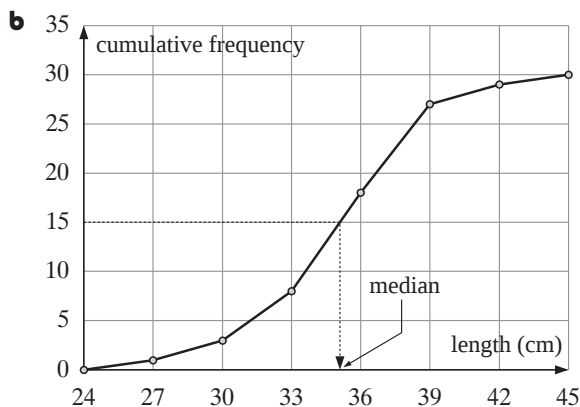
$\therefore$ the median is the 71st score

$\therefore$ median = 1 error {67 scores are 0s}

$\left(\frac{141 + 1}{2} = 71 \right)$

3 a

Length (x cm)	Freq.	C. freq.
$24 \leq x < 27$	1	1
$27 \leq x < 30$	2	3
$30 \leq x < 33$	5	8
$33 \leq x < 36$	10	18
$36 \leq x < 39$	9	27
$39 \leq x < 42$	2	29
$42 \leq x < 45$	1	30



- c** median $\div 35$

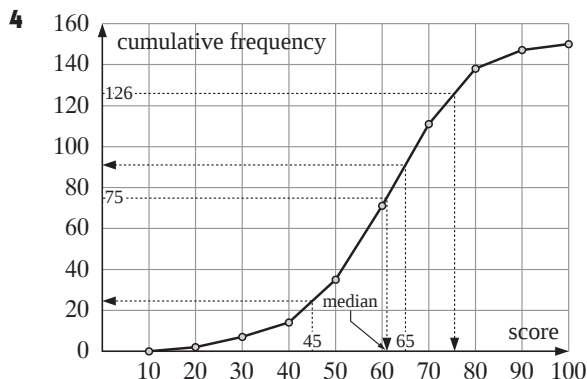
- d** There are 30 data values. So, the median is the average of the 15th and 16th scores (when in order).

In order they are:

24 27 28 30 31 31 32 32 33 33 33 33 34 34 34 35 35 35 36 etc.

median = $\frac{34 + 35}{2} = 34.5$

So, the median from the graph is a good approximation.



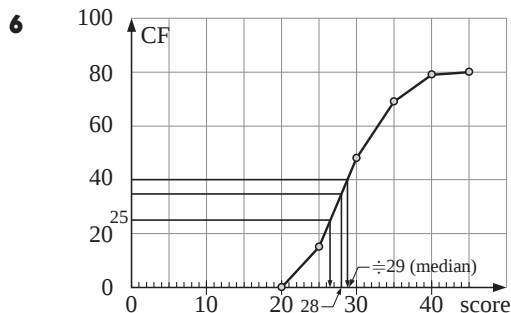
- a** Total frequency = 150
 $\therefore$ median $\div 61$ {from the graph}
- b** When the score = 65,
 CF $\div 91$ {from the graph}
 i.e., about 91 students scored 65 or less.
- c** $\div 36 + 40 = 76$ students scored between 50 and 70

- d** For a pass mark of 45, CF $\div 24.5$
 $\therefore$ 24 or 25 students failed the exam.

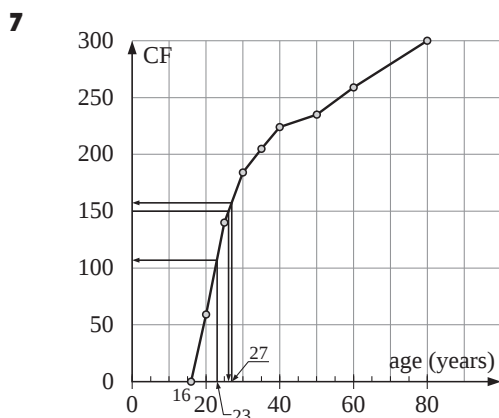
- e** 84% of 150 = 126
 So, for a CF of 126, the score value is 76.
 So, the minimum credit mark would be 76.

- 5 a** Total frequency = $1 + 1 + 0 + 3 + \dots + 0 + 1 = 50$
 $\therefore$ the median is the average of the 25th and 26th scores
- $\left. \begin{array}{l} 23 \text{ are } 7\frac{1}{2} \text{ or less} \\ 40 \text{ are } 8 \text{ or less} \end{array} \right\} \therefore \text{median} = \frac{8+8}{2} = 8$

- b i** $13 + 17 + 7 + 2 + 0 + 1 = 40$ people
ii $1 + 1 + 0 + 3 + 5 + 13 + 17 = 40$ are 8 or less

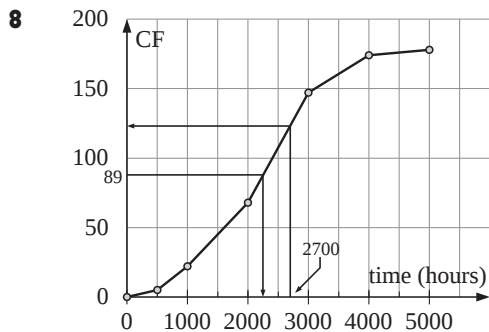


- a** median $\div 29$ min {from the graph}
- b** when the score is 28, CF $\div 35$
 i.e., 35 runners were faster than 28 min
- c** when CF = 25 {25 faster}
 the score is $\div 26.5$ min



Total frequency = 300

- a** median $\div 26$ years {from the graph}
- b** when the age = 23, CF $\div 108$
 and $\frac{108}{300} \times 100\% \div 36\%$
- c i** when age is 27, CF $\div 158$
 $\therefore P(\leq 27) = \frac{158}{300} = 0.527$
- ii** when age is 27, CF $\div 158$
 when age is 28, CF $\div 167$
 i.e., 9 died
 $\therefore P(\text{aged } 27) \div \frac{9}{300} \div 0.030$



Total frequency = 178

a median $\div$ 2300 hours

b For a life of 2700 hours

CF $\div$ 123

$$\text{and } \frac{123}{178} \times 100\% \div 69\%$$

So, about 69% have a life $\leq$ 2700 h

c For a life of 1500 h, CF $\div$ 45

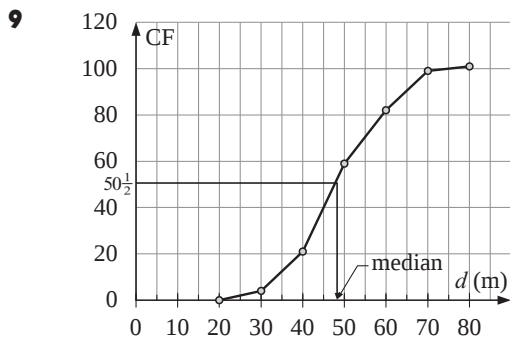
For a life of 2500 h, CF $\div$ 107.5

i.e., $107.5 - 45 = 62.5$

$\therefore$ 62 or 63 failed

d When life = 800, CF $\div$ 15

$$\therefore P(\text{failed before 800}) \div \frac{15}{178} \div 0.084$$



a median = $50\frac{1}{2}$ th score
 $\div$ 48 m

b When $d = 35$, CF $\div$ 12.5

i.e., 12 or 13 students

c When $d = 45$, CF $\div$ 40

When $d = 65$, CF $\div$ 90.5

$$\therefore \text{number of students} \div 90.5 - 40 \div 50.5$$

i.e., about 50 or 51 of them

d When $d = 53$, CF $\div$ 66. So, $101 - 66 = 35$ were considered.

e When $d = 32$, CF $\div$ 7. So, $P(\leq 32) = \frac{7}{101}$

$$\div 0.07 \text{ or } 7\%$$

EXERCISE 18D.1

1 a 2 3 3 3 4 4 4 5 5 5 6 6 6 6 6 7 7 8 8 8 9 9 ($n = 23$)

$\uparrow$ $\uparrow$ $\uparrow$
 Q_1 median Q_3

i median = 6

ii $Q_1 = 4$,
 $Q_3 = 7$

iii range
 $= 9 - 2$
 $= 7$

iv IQR
 $= Q_3 - Q_1$
 $= 7 - 4$
 $= 3$

b 10 12 12 14 15 15 16 16 17 18 18 18 18 19 20 21 22 24 ($n = 18$)

$\uparrow$ $\uparrow$ $\uparrow$
 Q_1 median Q_3

i median = 17.5

ii $Q_1 = 15$,
 $Q_3 = 19$

iii range
 $= 24 - 10$
 $= 14$

iv IQR
 $= Q_3 - Q_1$
 $= 19 - 15$
 $= 4$

c 21.8 22.4 23.5 23.5 24.6 24.9 25.0 25.3 26.1 26.4 29.5 ($n = 11$)

↑
 Q_1

↑
median

↑
 Q_3

i median = 24.9

ii $Q_1 = 23.5$,
 $Q_3 = 26.1$

iii range
= $29.5 - 21.8$
= 7.7

iv IQR
= $Q_3 - Q_1$
= $26.1 - 23.5$
= 2.6

d 103 105 115 115 119 123 124 125 127 127 128 129 130 133 135 140 141 ($n = 21$)
142 145 146 148

Q_1
↑
median

Q_3

i median = 128

ii $Q_1 = 121$,
 $Q_3 = 140.5$

iii range
= $148 - 103$
= 45

iv IQR
= $Q_3 - Q_1$
= $140.5 - 121$
= 19.5

2 0 0 0 0.8 1.4 1.5 1.6 1.9 2.1 2.2 2.7 3.0 3.4 3.6 3.8 3.8 4.5 4.8 5.2 5.2

↑ min ↑ Q_1 ↑ median ↑ Q_3 ↑ max

a median = $\frac{2.2 + 2.7}{2}$
= 2.45

$Q_1 = 1.45$
 $Q_3 = 3.8$

b range = $5.2 - 0$
= 5.2

IQR = $Q_3 - Q_1$
= $3.8 - 1.45$
= 2.35

c **i** the median, i.e., 2.45 min **ii** Q_3 , i.e., 3.8 min

iii The minimum waiting time was 0 minutes and the maximum waiting time was 5.2 minutes. The waiting time was spread over 5.2 minutes.

3 3 4 7 9 10 13 14 16 17 18 20 20 23 25 26 29 29 29 31 33 37 38 42 ($n = 23$)

↑ min ↑ Q_1 ↑ median ↑ Q_3 ↑ max

a min = 3

b max = 42

c median = 20

d $Q_1 = 13$

e $Q_3 = 29$

f range = $42 - 3$
= 39

g IQR = $Q_3 - Q_1$
= $29 - 13$
= 16

4 109 111 113 114 114 118 119 122 122 124 124 126 128 129 129 131 132 ($n = 20$)
135 138 138

min
↓
↑ max ↑ Q_1 ↑ median ↑ Q_3

a **i** median = 124 cm

ii $Q_3 = 130$ cm,
 $Q_1 = 116$ cm

b **i** 124 cm tall

ii 130 cm tall

c **i** range = $138 - 109$
= 29 cm

ii IQR = $Q_3 - Q_1$
= $130 - 116$
= 14 cm

d the IQR, i.e., over 14 cm

5 a Without fertiliser - See **Exercise 18B.2** solution to question **7**.

i range = $9 - 2 = 7$

ii median = 6

iii lower quartile = 38th score = 5

iv upper quartile = 113th score = 7

v interquartile range = $7 - 5 = 2$

b With fertiliser

i range = $13 - 3 = 10$

ii median = 7

iii lower quartile = 38th score = 6

iv upper quartile = 113th score = 8

v interquartile range = $8 - 6 = 2$

EXERCISE 18D.2

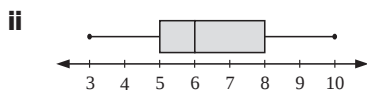
1 a i median = 35 **ii** max. value = 78 **iii** min. value = 13
iv $Q_3 = 53$ **v** $Q_1 = 26$

b i range = $78 - 13 = 65$ **ii** $IQR = Q_3 - Q_1 = 53 - 26 = 27$

2 a highest mark = 98 **b** lowest mark = 25 **c** the median which is 70
d Q_3 which is 85 **e** $Q_1 = 55$ and $Q_3 = 85$ **f** range = $98 - 25 = 73$
g $IQR = Q_3 - Q_1 = 85 - 55 = 30$

3 a i 3 4 5 5 5 6 6 6 7 7 8 8 9 10
 $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
min Q_1 median Q_3 max

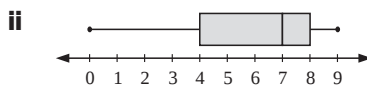
So, min = 3, $Q_1 = 5$, median = 6, $Q_3 = 8$, max = 10



iii range = $10 - 3 = 7$ **iv** $IQR = Q_3 - Q_1 = 8 - 5 = 3$

b i 0 1 2 3 4 5 6 6 7 7 7 8 8 8 8 8 8 9 9
 $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
min Q_1 median Q_3 max

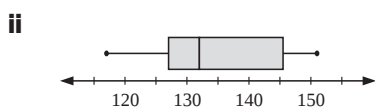
So, min = 0, $Q_1 = 4$, median = 7, $Q_3 = 8$, max = 9



iii range = $9 - 0 = 9$ **iv** $IQR = Q_3 - Q_1 = 8 - 4 = 4$

c i 117 120 123 126 126 128 130 131 131 131 133 135 135 137 144 147 147
149 149 151
 $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
max Q_1 median Q_3

So, min = 117, $Q_1 = 127$, median = 132, $Q_3 = 145.5$, max = 151



iii range = $151 - 117 = 34$ **iv** $IQR = Q_3 - Q_1 = 145.5 - 127 = 18.5$

4 a

Stem	Leaf	
88	3 5 9	min
89	2 2 5 7 8 8 9 9 9	Q ₁
90	0 0 1 1 2 3 3 3 5 5 6 6 6	median
91	0 1 1 1 2 3 3 4 6 7 7 8 9	
92	0 1 4 7	Q ₂
		max

88 | 3 represents 883 g

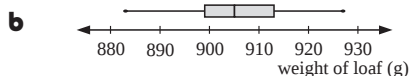
$n = 42$, so, median = ave. of 21st and 22nd

$$= \frac{905 + 905}{2}$$
$$= 905$$

$$\begin{aligned} Q_1 &= \text{average of 10th and 11th} \\ &= \frac{899 + 899}{2} \\ &= 899 \end{aligned}$$

$$\begin{aligned} Q_3 &= \text{average of 31st and 32nd} \\ &= \frac{913 + 913}{2} \\ &= 913 \end{aligned}$$

So, $\min = 883$ g, $Q_1 = 899$ g, $\text{median} = 905$ g, $Q_3 = 913$ g, $\max = 927$ g



c i $\text{IQR} = Q_3 - Q_1 = 913 - 899 = 14 \text{ g}$

ii range = max – min = 927 – 883 = 44 g

d **i** the median, i.e., 905 g **ii** less than 900 g $\Rightarrow \leq 899$ g, so 25%

iii the IQR, i.e., 14 g **iv** Q_1 or less, i.e., 899 g

e slightly skewed to the left, i.e., a little negatively skewed

Statistic	Year 9	Year 12
min value	1	6
Q ₁	5	10
median	7.5	14
Q ₃	10	16
max value	12	17.5

b For year 9 group

i $\text{range} = 12 - 1$
 $= 11$

ii $\text{IQR} = 10 - 5$
 $= 5$

For year 12 group

i $\text{range} = 17.5 - 6$
 $\quad \quad = 11.5$

ii $\text{IQR} = 16 - 10$
 $= 6$

c **i** True, indicated by the median.

ii True, as Q_1 for year 9 = 5 and min for year 12 = 6.

[illegible]
$$\uparrow$$
$$\min$$
$$\begin{array}{c} \text{ } \\ \uparrow \\ Q_1 \end{array}$$

↑
median

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ Q_3 \end{array}$$
$$\uparrow$$
$$\max$$

a median = 6, $O_1 = 5$, $O_3 = 8$

b $\text{IQR} = 8 - 5$
 $= 3$



7 a

<i>Number of bolts</i>	33	34	35	36	37	38	39	40
<i>Frequency</i>	1	5	7	13	12	8	0	1

↑ ↑ ↑
min median max
is one
of these

$$\begin{array}{lcl} Q_1 = 12\text{th} & \left(\frac{47+1}{4} = 12 \right) & Q_3 = 36\text{th} \\ = 35 & & = 37 \end{array}$$

$$\begin{aligned} Q_3 &= 36\text{th} \\ &= 37 \end{aligned}$$

So, $\min = 33$, $Q_1 = 35$, $\text{median} = 36$, $Q_3 = 37$, $\max = 40$

There are 47 scores

$$\therefore \text{median} = 24\text{th}$$

$$\left(\frac{47+1}{2} = 24\right)$$

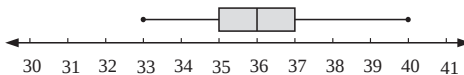
14 scores are 35 or less

27 scores are 36 or less

$\therefore$ median is 36

b i $\text{range} = 40 - 33 = 7$

ii $\text{IQR} = 37 - 35 = 2$



8 a For $h = 5$, $\text{CF} \div 10 \therefore 10$ seedlings have height 5 cm or less

b For $h = 8$, $\text{CF} \div 43 \therefore \% \text{ taller than } 8 \text{ cm} = \frac{60 - 43}{60} \times 100\% \div 28.3\%$

c Approx. median occurs at $\text{CF} = 30$, i.e., median $\div 7$ cm.

d $\text{IQR} = Q_3 - Q_1 = (h \text{ when } \text{CF} = 45) - (h \text{ when } \text{CF} = 15)$
 $\div 8.4 - 5.8$
 $\div 2.6 \text{ cm}$

e 90th percentile occurs when $\text{CF} = 90\% \text{ of } 60 = 54$

$\therefore 90\text{th percentile} = 10$

This means that 90% of the seedlings have a height of 10 cm or less.

9 a The lower quartile occurs when $\text{CF} = 25\% \text{ of } 80 = 20$

$\therefore Q_1 = 27 \text{ min}$

b The median occurs when $\text{CF} = 50\% \text{ of } 80 = 40$

$\therefore \text{median} = 29 \text{ min}$

c The upper quartile occurs when $\text{CF} = 75\% \text{ of } 80 = 60$

$\therefore Q_3 \div 31\frac{1}{2} \text{ min}$

d $\text{IQR} = Q_3 - Q_1 = 31\frac{1}{2} - 27 = 4\frac{1}{2} \text{ min}$

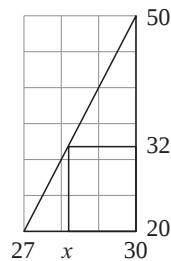
e For the 40th percentile, $\text{CF} = 40\% \text{ of } 80 = 32$

From the similar Δ s, $\frac{x - 27}{12} = \frac{3}{30}$

$\therefore x - 27 = 1.2$

$\therefore x = 28.2$

So about 28 min 10 sec.



10 a Highest CF value = 480 $\therefore 480$ sat for the exam.

b Highest mark = 120 {assuming 'best' student got full marks}

c The median occurs when $\text{CF} \div 50\% \text{ of } 480 = 240$

$\therefore \text{median} \div 84 \text{ marks}$

d $\text{IQR} = Q_3 - Q_1 = (\text{mark when } \text{CF} = 360) - (\text{mark when } \text{CF} = 120)$

$\div 99 - 71$

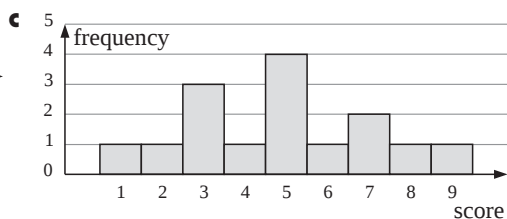
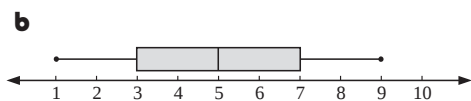
$\div 28 \text{ marks}$

e For the 85th percentile, $\text{CF} = 85\% \text{ of } 480 = 408$

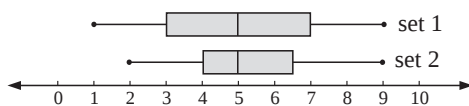
$\therefore 85\text{th percentile} \div 107 \text{ marks}$

EXERCISE 18E.1

1 a $\bar{x} \div 4.87$, $\text{Min}_x = 1$, $Q_1 = 3$, $Q_2 = 5$, $Q_3 = 7$, $\text{Max}_x = 9$

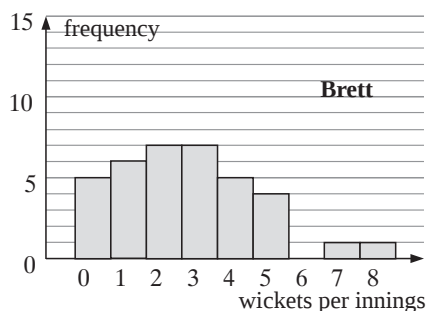
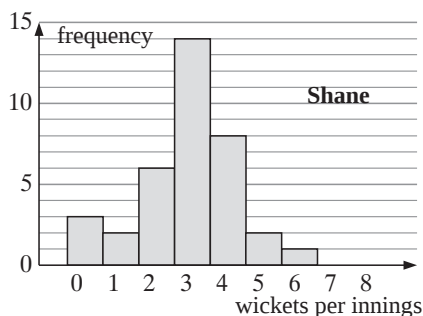


d $\bar{x} \div 5.24$, $\text{Min}_x = 2$, $Q_1 = 4$, $Q_2 = 5$, $Q_3 = 6.5$, $\text{Max}_x = 9$

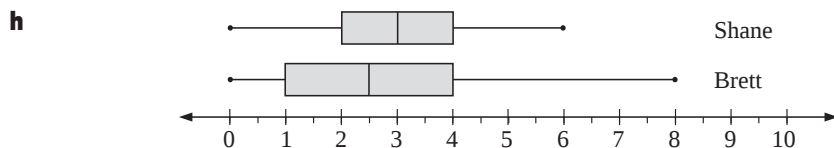


EXERCISE 18E.2

1 a discrete **c**



- d** There are no outliers for Shane. Brett has outliers of 7 and 8 which must not be removed.
- e** Shane's distribution is reasonably symmetrical. Brett's distribution is positively skewed.
- f** Shane has a higher mean ($\div 2.89$ wickets) compared with Brett ($\div 2.67$ wickets). Shane has a higher median (3 wickets) compared with Brett (2.5 wickets). Shane's modal number of wickets is 3 (14 times) compared with Brett, who has a bi-modal distribution of 2 and 3 (7 times each).
- g** Shane's range is 6 wickets, compared with Brett's range of 8 wickets. Shane's IQR is 2 wickets, compared with Brett's IQR of 3 wickets. Brett's wicket taking shows greater spread or variability.



- j** Generally, Shane takes more wickets than Brett and is a more consistent bowler.

2 a continuous

- c** For the 'New type' globes, 191 hours could be considered an outlier. However, it could be a genuine piece of data, so we will include it in the analysis.

d

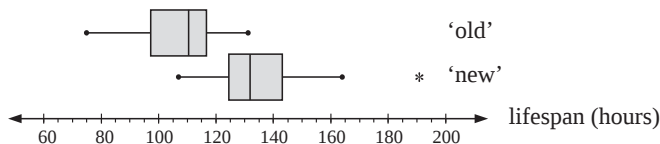
	Old type	New type
Mean	107	134
Median	110.5	132
Range	56	84
IQR	19	18.5

The mean and median are $\div 25\%$ and $\div 19\%$ higher for the 'new type' of globe compared with the 'old type'.

The range is higher for the 'new type' of globe (but has been affected by the 191 hours).

The IQR for each type of globe is almost the same.

e



f For the 'old type' of globe, the data is bunched to the right of the median, hence the distribution is negatively skewed. For the 'new type' of globe, the data is bunched to the left of the median, hence the distribution is positively skewed.

g The manufacturer's claim, that the 'new type' of globe has a 20% longer life than the 'old type' seems to be backed up by the 25% higher mean life and 19.5% higher median life.

3

	Set 1	Set 2		Set 1	Set 2
Mean	21.95	21.81	Q ₃	22.0	21.9
Min _x	21.6	21.5	Max _x	22.2	22.2
Q ₁	21.9	21.7	Range	0.6	0.7
Median	21.9	21.8	IQR	0.1	0.2

Set 2 has more data points than set 1.

The 5-number summary for each set of data is reasonably similar, however the spread of the middle 50% of data for set 2 is double that for set 1.

EXERCISE 18F

1 a Sally

x	$(x - \bar{x})^2$
23	4
17	64
31	36
25	0
25	0
19	36
28	9
32	49
200	198

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} & s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \frac{200}{8} & &= \sqrt{\frac{198}{8}} \\ &= 25 & &\div 4.97\end{aligned}$$

Joanne

x	$(x - \bar{x})^2$
9	$(21.5)^2$
29	$(1.5)^2$
41	$(10.5)^2$
26	$(4.5)^2$
14	$(16.5)^2$
44	$(13.5)^2$
38	$(7.5)^2$
43	$(12.5)^2$
244	1262

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} & s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \frac{244}{8} & &= \sqrt{\frac{1262}{8}} \\ &= 30.5 & &\div 12.56\end{aligned}$$

b The standard deviation. The lower the value of s , the more consistent.

2 a Glen has mean = $\frac{0 + 10 + 1 + 9 + 11 + 0 + 8 + 5 + 6 + 7}{10} = \frac{57}{10} = 5.7$

Shane has mean = $\frac{4 + 3 + 4 + 1 + 4 + 11 + 7 + 6 + 12 + 5}{10} = \frac{57}{10} = 5.7$

Glen's range = $11 - 0 = 11$ Shane's range = $12 - 1 = 11$

- b** We suspect Glen's as he has more high and low values.

c Glen

x	$(x - \bar{x})^2$
0	$(5.7)^2$
10	$(4.3)^2$
1	$(4.7)^2$
9	$(3.3)^2$
11	$(5.3)^2$
0	$(5.7)^2$
8	$(2.3)^2$
5	$(0.7)^2$
6	$(0.3)^2$
7	$(1.3)^2$
	154.9

$$\begin{aligned}
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{154.9}{10}} \\
 &\div 3.9 \\
 &\quad \uparrow \\
 &\quad \text{greater} \\
 &\quad \text{variability}
 \end{aligned}$$

Shane

x	$(x - \bar{x})^2$
4	$(1.7)^2$
3	$(2.7)^2$
4	$(1.7)^2$
1	$(4.7)^2$
4	$(1.7)^2$
11	$(5.3)^2$
7	$(1.3)^2$
6	$(0.3)^2$
12	$(6.3)^2$
5	$(0.7)^2$
	108.1

$$\begin{aligned}
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{108.1}{10}} \\
 &\div 3.29
 \end{aligned}$$

- d** The standard deviation as it takes all values into account, not just the lowest and highest.

- 3 a** We suspect variability in standard deviation since the factors may change every day.

- b i** sample mean **ii** sample standard deviation **c** less variability

4

a

x	$(x - \bar{x})^2$
79	10^2
64	5^2
59	10^2
71	2^2
68	1^2
68	1^2
74	5^2
483	256

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{483}{7} \\
 &= 69 \\
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{256}{7}} \\
 &\div 6.05
 \end{aligned}$$

b

x	$(x - \bar{x})^2$
89	10^2
74	5^2
69	10^2
81	2^2
78	1^2
78	1^2
84	5^2
553	256

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{553}{7} \\
 &= 79 \\
 s &\div 6.05
 \end{aligned}$$

- c** The distribution has simply shifted by 10 kg. The mean increases by 10 kg and the standard deviation remains the same.

5

a

x	$(x - \bar{x})^2$
0.8	$(0.21)^2$
1.1	$(0.09)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
1.2	$(0.19)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
0.7	$(0.31)^2$
1.0	$(0.01)^2$
1.1	$(0.09)^2$
10.4	0.289

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{10.1}{10} \\
 &= 1.01 \text{ kg} \\
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{0.289}{10}} \\
 &\div 0.17 \text{ kg}
 \end{aligned}$$

b

x	$(x - \bar{x})^2$
1.6	$(0.42)^2$
2.2	$(0.18)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
2.4	$(0.38)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
1.4	$(0.62)^2$
2.0	$(0.02)^2$
2.2	$(0.18)^2$
20.2	1.156

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{20.2}{10} \\
 &= 2.02 \text{ kg} \\
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{1.156}{10}} \\
 &= 0.34 \text{ kg}
 \end{aligned}$$

c Doubling the values doubles the mean and standard deviation.

$$\begin{aligned}
 \text{6 mean} &= \frac{3+7+a+4+b}{5} & s &= \sqrt{\frac{(3-5)^2 + (7-5)^2 + (a-5)^2 + (4-5)^2 + (b-5)^2}{5}} \\
 &= \frac{14+a+b}{5} & &= \sqrt{\frac{4+4+(a-5)^2+1+(b-5)^2}{5}} \\
 \therefore \frac{14+a+b}{5} &= 5 & &= \sqrt{\frac{9+(a-5)^2+(b-5)^2}{5}} \quad \{\text{using (1)}\} \\
 \therefore 14+a+b &= 25 & &= \sqrt{\frac{9+a^2-10a+25+36-12a+a^2}{5}} \\
 \therefore a+b &= 11 & &= \sqrt{\frac{2a^2-22a+70}{5}} \\
 \therefore b &= 11-a & & \\
 &\dots (1) & & \\
 \text{So, } \sqrt{\frac{2a^2-22a+70}{5}} &= \sqrt{2} & \therefore 2a^2-22a+70 &= 10 \\
 & & \therefore 2a^2-22a+60 &= 0 \\
 & & \therefore a^2-11a+30 &= 0 \\
 & & \therefore (a-5)(a-6) &= 0 \\
 \therefore a &= 5 \text{ or } 6 \text{ and corresponding values of } b \text{ are } 6 \text{ or } 5 \text{ and as } a > b, a = 6 \text{ and } b = 5
 \end{aligned}$$

7

x	f	fx	$f(x-\bar{x})^2$
0	14	0	41.61
1	18	18	9.44
2	13	26	0.99
3	5	15	8.14
4	3	12	15.54
5	2	10	21.46
6	2	12	36.57
7	1	7	27.84
Σ	58	100	161.59

$$\begin{aligned}
 \bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \\
 &= \frac{100}{58} & &= \sqrt{\frac{161.59}{58}} \\
 &\div 1.724 \text{ children} & &\div 1.67 \text{ children}
 \end{aligned}$$

8

a

x	f	fx	$f(x-\bar{x})^2$
0	1	0	26.01
1	0	0	0
2	1	2	9.61
3	1	3	4.41
4	2	8	2.42
5	6	30	0.06
6	5	30	4.05
7	3	21	10.83
8	1	8	8.41
Σ	20	102	65.80

$$\begin{aligned}
 \bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \\
 &= \frac{102}{20} & &= \sqrt{\frac{65.80}{20}} \\
 &= 5.1 & &\div 1.81
 \end{aligned}$$

b

x	f	fx	$f(x - \bar{x})^2$
11	2	22	24.22
12	1	12	6.15
13	4	52	8.76
14	5	70	1.15
15	6	90	1.62
16	4	64	9.24
17	2	34	12.70
18	1	18	12.39
Σ	25	362	76.23

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} \\ &= \frac{362}{25} \\ &= 14.48 \\ &\div 14.5\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \sqrt{\frac{76.25}{25}} \\ &\div 1.75\end{aligned}$$

9

x	f	fx	$f(x - \bar{x})^2$
33	1	33	18.24
35	5	175	25.78
36	7	252	11.31
37	13	481	0.95
38	12	456	6.38
39	8	312	23.92
40	2	80	14.90
Σ	48	1789	101.48

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} \\ &= \frac{1789}{48} \\ &\div 37.3\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \sqrt{\frac{101.48}{48}} \\ &\div 1.45\end{aligned}$$

10

Midpoint (x)	f	fx	$f(x - \bar{x})^2$
40.5	1	40.5	52.85
42.5	1	42.5	27.77
44.5	3	133.5	32.08
46.5	7	325.5	11.29
48.5	11	533.5	5.86
50.5	5	252.5	37.26
52.5	2	105	44.75
Σ	30	1433	211.86

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} \\ &= \frac{1433}{30} \\ &= 47.7666 \dots \\ &\div 47.8 \text{ cm}\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &\div \sqrt{\frac{211.86}{30}} \\ &\div 2.657 \dots \\ &\div 2.66 \text{ cm}\end{aligned}$$

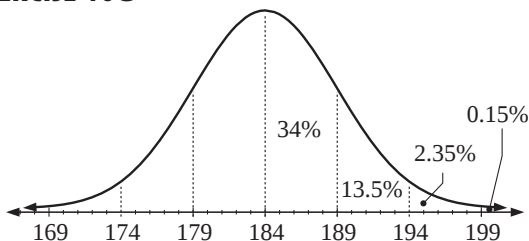
11

Midpoint (x)	f	fx	$f(x - \bar{x})^2$
364.995	17	6204.9	10 885.83
374.995	38	14 249.8	8901.23
384.995	47	18 094.8	1322.72
394.995	57	22 514.7	1256.45
404.995	18	7289.9	3886.97
414.995	10	4149.9	6098.43
424.995	10	4250.0	12 037.43
434.995	3	1305.0	5992.93
Σ	200	78 059	50 381.99

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} \\ &= \frac{78\,059}{200} \\ &\div \$390.30\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{n}} \\ &= \sqrt{\frac{50\,381.99}{200}} \\ &\div \$15.87\end{aligned}$$

12 a $\bar{x} = 26.5$, $s \div 2.29$ **b** $\bar{x} \div 13.2$, $s \div 4.65$ **c** $\bar{x} \div 170$, $s \div 9.84$ **d** $\bar{x} = 0$, $s = 2$ **13** $\bar{x} = 55 \text{ L}$, $s \div 10.9 \text{ L}$ **14 a** $\bar{x} \div 33.7$, $s \div 1.11$ **b** $\bar{x} \div 17.4$, $s \div 0.123$ **c** $\bar{x} \div 34.6$, $s \div 10.4$ **15** $\bar{x} \div \$18.60$, $s \div \$8.33$

EXERCISE 18G**1**

a $50\% + 34\% = 84\%$

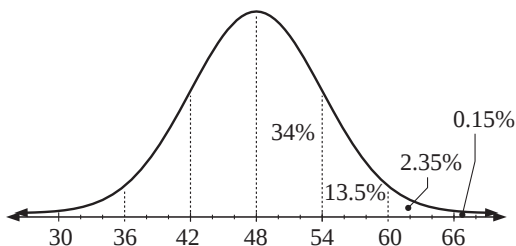
So, 16% are above 189.

b $50\% + 34\% = 84\%$

So, 84% are above 179.

c $2 \times 34\% + 2 \times 13.5\% + 2.35\%$
 $\div 97.4\%$

d 0.15%

2

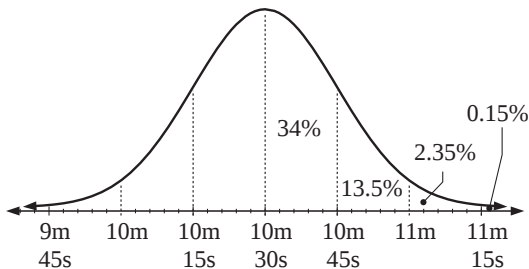
$100\% - 50\% - 34\% = 16\%$

and $20 \times 16\%$

$= 20 \times 0.16$

$= 3.2$

On 3 occasions.

3

c $200 \times 68\%$
 $= 136 \text{ lifesavers}$

a $200 \times (50 + 34 + 13.5)\%$

$= 200 \times 95.5\%$

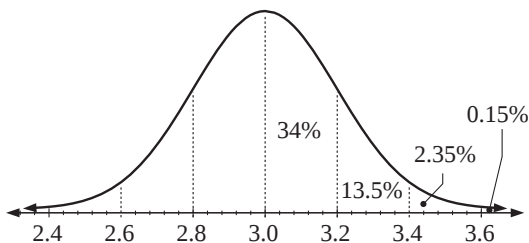
$\div 195$

$\therefore 200 - 195 = 5$ took > 11 minutes

b $200 \times (0.15 + 2.35 + 13.5)\%$

$= 200 \times 16\%$

$= 32 \text{ lifesavers}$

4

a $545 \times (50 + 34)\%$

$= 545 \times 0.84$

$\div 458 \text{ babies}$

b $545 \times (68 + 13.5)\%$

$= 545 \times 81.5\%$

$\div 444 \text{ babies}$

REVIEW SET 18A**1**

a i many ii n.a. iii categorical

b i many ii breaths per minute iii quantitative discrete

c i infinitely many ii centimetres, metres, etc. iii quantitative continuous

2

a Diameter of bacteria colonies

0 | 4 8 9

1 | 3 5 5 7

2 | 1 1 5 6 8 8

3 | 0 1 2 3 4 5 5 6 6 7 7 9

4 | 0 1 2 7 9

leaf unit: 0.1 cm

b Using technology

i median = 3.15 cm

ii range = $4.9 - 0.4 = 4.5$ cm

c The distribution has a large tail on the left.
 So, it is negatively skewed.

3 a Using technology

	<i>Girls</i>	<i>Boys</i>
shape	pos. skewed	approx. symm.
centre (median)	36.3 sec	34.9 sec
spread (range)	7.7 sec	4.9 sec

- b** The girls' distribution is positively skewed and the boys' distribution is approximately symmetrical. The median swim times for boys is 1.4 seconds lower than for girls but the range of the girls' swim times is 2.8 seconds higher than for boys. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

4 a highest = 97.5 m, lowest = 64.6 m

- b** The range = $97.5 - 64.5 = 33$

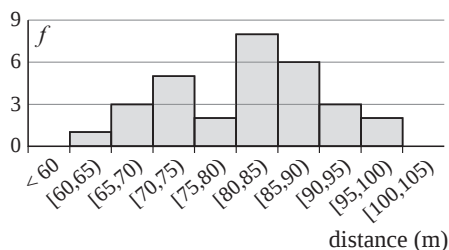
So, if intervals of length 5 are used we need about 7 of them.

We choose 60 -, 65 -, 70 -, 75 -, 80 -, etc.

c

<i>A frequency distribution table for distances thrown by Thabiso</i>		
distance (m)	tally	freq. (<i>f</i>)
60 -		1
65 -		3
70 -		5
75 -		2
80 -		8
85 -		6
90 -		3
95 < 100		2
	Total	30

d i/ii Frequency histogram displaying the distance Thabiso throws a cricket ball



e Using technology

- i** $\bar{x} \div 81.1$ m **ii** median $\div 83.0$ m

5 a Reading from the box-plots

	<i>A</i>	<i>B</i>
Min	11	11.2
Q_1	11.6	12
Median	12	12.6
Q_3	12.6	13.2
Max	13	13.8

b i

$$\begin{aligned} \text{range of } A &= 13 - 11 \\ &= 2 \end{aligned}$$

ii IQR of *A*

$$\begin{aligned} &= 12.6 - 11.6 \\ &= 1.0 \end{aligned}$$

range of *B*

$$\begin{aligned} &= 13.8 - 11.2 \\ &= 2.6 \end{aligned}$$

IQR of *B*

$$\begin{aligned} &= 13.2 - 12 \\ &= 1.2 \end{aligned}$$

- c i** The members in squad *A* generally ran faster times.
ii The times in squad *B* were more varied.

6 Using technology **a i** 101.5 **ii** 98 **iii** 105.5 **b** 7.5 **c** $\bar{x} = 100.2$, $s \div 7.59$

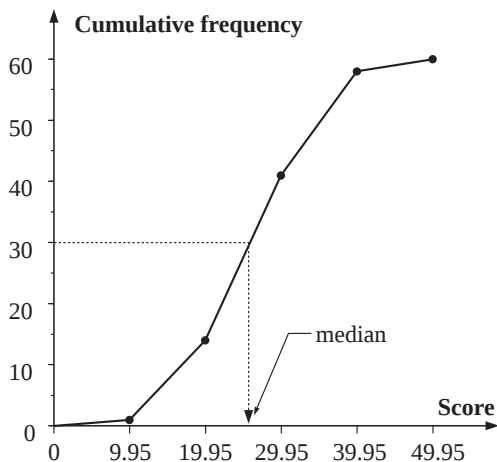
- 7 a** The mean length is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.
b Similar reasons as for **a**.

8 This question could be done using technology or

litres (x)	f	fx	$f(x - \bar{x})^2$
17	5	85	1299.27
22	13	286	1607.50
27	17	459	636.72
32	29	928	36.38
37	27	999	406.47
42	18	756	1419.38
47	7	329	1348.58
Σ	116	3842	6754.30

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \frac{3842}{116} & & \div \sqrt{\frac{6754.30}{116}} \\ &\div 33.12 & & \\ &\div 33.1 \text{ litres} & & \div 7.63 \text{ litres}\end{aligned}$$

9 a



b median $\div 25.9$ (see graph)

c $\text{IQR} = Q_3 - Q_1$
 $= (\text{score for CF of } 45)$
 $\quad - (\text{score for CF of } 15)$
 $\div 32 - 20 \div 12$

d

	f	midpt x	fx	$(x - \bar{x})^2$
0 – 9.9	1	4.95	4.95	441
10 – 19.9	13	14.95	194.35	121
20 – 29.9	27	24.95	673.65	1
30 – 39.9	17	34.95	594.15	81
40 – 49.9	2	44.95	89.9	361
	60		1557	

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum x} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \frac{1557}{60} \div 26.0 & & = \sqrt{\frac{4140}{60}} \div 8.31\end{aligned}$$

10 $\frac{6 + x + 9 + y + 3 + 11}{6} = 7$ {as the mean is 7}

$$\therefore 29 + x + y = 42$$

$$\therefore x + y = 13$$

$$\therefore y = 13 - x \quad \dots (1)$$

$$\text{Also } \sqrt{\frac{(6-7)^2 + (x-7)^2 + (9-7)^2 + (y-7)^2 + (3-7)^2 + (11-7)^2}{6}} = \frac{5}{\sqrt{3}}$$

$$\therefore \frac{1 + (x-7)^2 + 4 + (y-7)^2 + 16 + 16}{6} = \frac{25}{3}$$

$$\therefore 37 + (x-7)^2 + (6-x)^2 = 50$$

$$\therefore 37 + x^2 - 14x + 49 + 36 - 12x + x^2 - 50 = 0$$

$$\therefore 2x^2 - 26x + 72 = 0$$

$$\therefore x^2 - 13x + 36 = 0$$

$$\therefore (x-9)(x-4) = 0$$

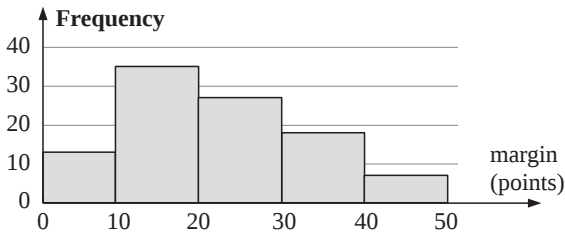
$$\therefore x = 9 \text{ or } 4$$

and corresponding y -values are 4 or 9

But $x < y$ $\therefore x = 4, y = 9$.

REVIEW SET 18B

1



2 Using technology:

a $\bar{x} \div 3.69$, mode = 4, median = 4 b $\bar{x} = 56$, bi-modal (58 and 63), median = 58

3 Using technology, $\bar{x} \div 49.6$, $s \div 1.60$ does not justify claim. A much greater sample is needed.

4 Use technology or

Midpoint (x)	f	fx
274.5	14	3843
324.5	34	11 033
374.5	68	25 466
424.5	72	30 564
474.5	54	25 623
524.5	23	12 063.5
574.5	7	4021.5
Σ	272	112 613.5

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{112\,613.5}{272}$$

$\div 414$ customers

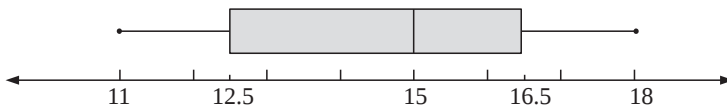
5 116 118 120 122 127 128 132 135 ($n = 8$)

$\uparrow$ min $\uparrow$ Q_1 $\uparrow$ median $\uparrow$ Q_3 $\uparrow$ max
 116 118 120 122 127 128 132 135

$$\begin{aligned} \text{range} &= 135 - 116 = 19 & Q_1 &= \frac{118 + 120}{2} = 119 & Q_3 &= \frac{128 + 132}{2} = 130 & s &\div 6.38 \quad \{\text{technology}\} \end{aligned}$$

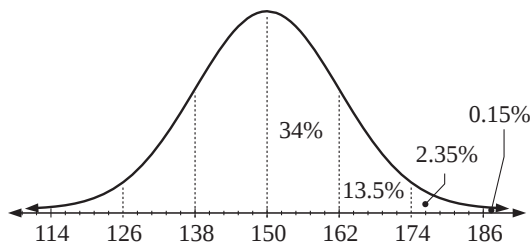
6 11 12 12 13 14 14 15 15 15 16 17 17 18

$\uparrow$ min $\uparrow$ Q_1 $\uparrow$ median $\uparrow$ Q_3 $\uparrow$ max
 11 12 12 13 14 14 15 15 15 16 17 17 18



7 Using technology with x values 74.995, 84.995, 94.995, etc.

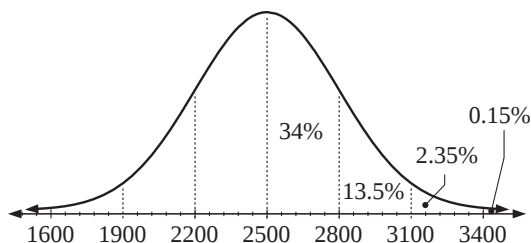
$\bar{x} = \$103.50$ and $s = \$19.40$

8

- a** 68% lie here
b $2 \times (34 + 13.5) = 95\%$ lie here
c $2 \times 34 + 13.5 = 81.5\%$ lie here
d 13.5% lie here

9 Using technology

a $\bar{x} \doteq 46.0, s \doteq 1.50$ **b** $\bar{x} \doteq 149.5, s \doteq 14.0$

10

- a** $(50 - 34 - 13.5)\% = 2.5\%$
 days are less than 1900
b $(50 + 34)\% = 84\%$
c $(2 \times 34 + 13.5)\% = 81.5\%$

11 $\frac{3 + a + 6 + b + 13}{5} = 6.8 \quad \therefore \frac{a + b + 22}{5} = 6.8$

$$\therefore a + b + 22 = 34$$

$$\therefore a + b = 12$$

$$\therefore b = 12 - a \quad \dots (1)$$

Now $\sqrt{\frac{(3 - 6.8)^2 + (a - 6.8)^2 + (6 - 6.8)^2 + (b - 6.8)^2 + (13 - 6.8)^2}{5}} = \sqrt{12.56}$

$$\therefore \frac{(-3.8)^2 + (a - 6.8)^2 + (-0.8)^2 + (12 - a - 6.8)^2 + (6.2)^2}{5} = 12.56$$

$$\therefore 53.52 + (a - 6.8)^2 + (5.2 - a)^2 = 62.8$$

$$\therefore 53.52 + a^2 - 13.6a + 46.24 + 27.04 - 10.4a + a^2 - 62.8 = 0$$

$$\therefore 2a^2 - 24a + 64 = 0$$

$$\therefore a^2 - 12a + 32 = 0$$

$$\therefore (a - 4)(a - 8) = 0$$

$$\therefore a = 4 \text{ or } 8 \text{ and } b = 8 \text{ or } 4$$

$$\text{But } a > b, \therefore a = 8, b = 4$$

Chapter 19

PROBABILITY

EXERCISE 19A

- 1 a** P (inside a square)

$$\div \frac{113}{145}$$

$$\div 0.7793 \dots$$

$$\div 0.78$$

- b** P (on a line)

$$\div \frac{32}{145}$$

$$\div 0.2206 \dots$$

$$\div 0.22$$

- 2** Total frequency = $17 + 38 + 19 + 4 = 78$

- a** P (20 to 39 seconds)

$$= \frac{38}{78}$$

$$\div 0.487$$

- b** P (> 60 seconds)

$$= \frac{4}{78}$$

$$\div 0.051$$

- c** P (between 20 and 59 sec (inc))

$$= \frac{38 + 19}{78}$$

$$\div 0.731$$

- 3**

Calls/day	No. of days
0	2
1	7
2	11
3	8
4	7
5	4
6	3
7	0
8	1

- a** Survey lasted $2 + 7 + 11 + 8 + 7 + 4 + 3 + 0 + 1$
= 43 days

- b i** P (0 calls)

$$= \frac{2}{43}$$

$$\div 0.047$$

- ii** P (≥ 5 calls)

$$= \frac{4 + 3 + 0 + 1}{43}$$

$$\div 0.186$$

- iii** P (< 3 calls)

$$= \frac{2 + 7 + 11}{43}$$

$$\div 0.465$$

- 4** Total frequency

$$= 37 + 81 + 48 + 17 + 6 + 1$$

$$= 190$$

- a** P (4 days gap)

$$= \frac{17}{190}$$

$$\div 0.089$$

- b** P (at least 4 days gap)

$$= \frac{17 + 6 + 1}{190}$$

$$\div 0.126$$

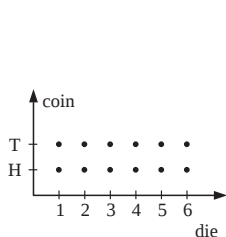
EXERCISE 19B

- 1 a** {A, B, C, D} **b** {BB, BG, GB, GG}

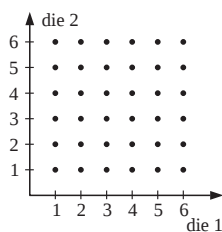
- c** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

- d** {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

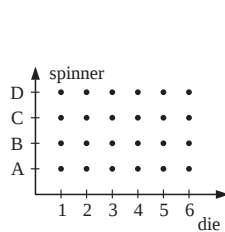
- 2 a**



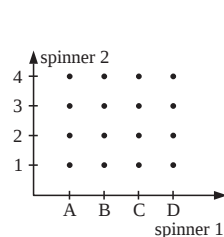
- b**

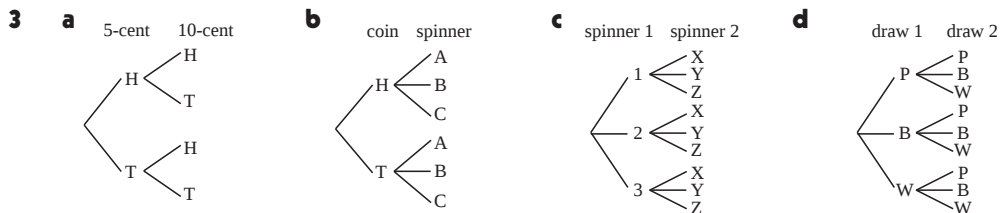


- c**



- d**



**EXERCISE 19C**

1 Total number of marbles = $5 + 3 + 7 = 15$

a P(red)

$$= \frac{3}{15}$$

$$= \frac{1}{5}$$

b P(green)

$$= \frac{5}{15}$$

$$= \frac{1}{3}$$

c P(blue)

$$= \frac{7}{15}$$

d P(not red)

$$= \frac{5 + 7}{15}$$

$$= \frac{12}{15} \text{ or } = \frac{4}{5}$$

e P(neither green nor blue)

$$= P(\text{red})$$

$$= \frac{1}{5}$$

f P(green or red)

$$= \frac{5 + 3}{15}$$

$$= \frac{8}{15}$$

2 a 8 are brown and so 4 are white.

b i P(brown)

$$= \frac{8}{12}$$

$$= \frac{2}{3}$$

ii P(white)

$$= \frac{4}{12}$$

$$= \frac{1}{3}$$

3 8 have *one*
19 have *two*
5 have *three*

a P(no first name)

$$= \frac{0}{32}$$

$$= 0$$

b P(one first name)

$$= \frac{8}{32}$$

$$= \frac{1}{4}$$

c P(two first names)

$$= \frac{19}{32}$$

d P(three first names)

$$= \frac{5}{32}$$

4 a P(multiple of 4)

$$= P(4, 8, 12, 16, 20, 24, 28, 32, 36)$$

$$= \frac{9}{36}$$

$$= \frac{1}{4}$$

b P(between 6 and 9 inclusive)

$$= P(6, 7, 8 \text{ or } 9)$$

$$= \frac{4}{36}$$

$$= \frac{1}{9}$$

c P(> 20)

$$= P(21, 22, 23, 24, \dots, 35, 36)$$

$$= \frac{36 - 20}{36}$$

$$= \frac{16}{36}$$

$$= \frac{4}{9}$$

d P(9)

$$= \frac{1}{36}$$

e P(multiple of 13)

$$= P(13 \text{ or } 26)$$

$$= \frac{2}{36}$$

$$= \frac{1}{18}$$

f P(odd multiple of 3)

$$= P(3, 9, 15, 21, 27, \text{ or } 33)$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & P(\text{on Tuesday}) = \frac{1}{7} \\
 \mathbf{b} \quad & P(\text{on a weekend}) = \frac{2}{7} \\
 \mathbf{c} \quad & P(\text{in July}) = \frac{4 \times 31}{365 \times 3 + 366} \quad \{\text{over a 4 year period}\} \\
 & = \frac{124}{1461} \\
 \mathbf{d} \quad & P(\text{in January or February}) \\
 & = \frac{4 \times 31 + 3 \times 28 + 1 \times 29}{3 \times 365 + 1 \times 366} \quad \{\text{over a 4 year period}\} \\
 & = \frac{237}{1461}
 \end{aligned}$$

6 Let A denote Antti, K denote Kai and N denote Neda.

Possible orders are: {AKN, ANK, KAN, KNA, NAK, NKA}

$$\begin{aligned}
 \mathbf{a} \quad & P(\text{A in middle}) = \frac{2}{6} \\
 & = \frac{1}{3} \\
 \mathbf{b} \quad & P(\text{A at left end}) = \frac{2}{6} \\
 & = \frac{1}{3} \\
 \mathbf{c} \quad & P(\text{A at right end}) = \frac{2}{6} \\
 & = \frac{1}{3} \\
 \mathbf{d} \quad & P(\text{K and N are together}) = \frac{4}{6} = \frac{2}{3}
 \end{aligned}$$

7 Let G denote ‘a girl’ and B denote ‘a boy’.

a Possible orders are: {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{i} \quad & P(\text{all boys}) = P(\text{BBB}) = \frac{1}{8} \\
 \mathbf{ii} \quad & P(\text{all girls}) = P(\text{GGG}) = \frac{1}{8} \\
 \mathbf{iii} \quad & P(\text{boy, then girl, then girl}) = P(\text{BGG}) = \frac{1}{8} \\
 \mathbf{iv} \quad & P(\text{2 girls and a boy}) = P(\text{GGB or GBG or BGG}) = \frac{3}{8} \\
 \mathbf{v} \quad & P(\text{girl is eldest}) = P(\text{GGG or GBG or GBB or GGB}) = \frac{4}{8} \\
 \mathbf{vi} \quad & P(\text{at least one boy}) = \frac{7}{8} \quad \{\text{all except GGG}\} \\
 & = \frac{1}{2}
 \end{aligned}$$

8 a {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{i} \quad & P(\text{A sits on one end}) = \frac{12}{24} = \frac{1}{2} \\
 \mathbf{ii} \quad & P(\text{B sits on one of the two middle seats}) = \frac{12}{24} = \frac{1}{2} \\
 \mathbf{iii} \quad & P(\text{A and B are together}) = \frac{12}{24} = \frac{1}{2} \\
 \mathbf{iv} \quad & P(\text{A, B and C are together}) = \frac{12}{24} = \frac{1}{2}
 \end{aligned}$$

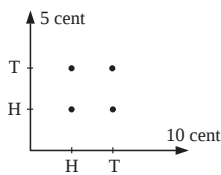
9 {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, THTH, HTTH, TTHH, TTTH, TTHT, THTT, TTTT}

$$\begin{aligned}
 \mathbf{a} \quad & P(\text{all heads}) = \frac{1}{16} \\
 \mathbf{b} \quad & P(\text{2H and 2T}) = \frac{6}{16} = \frac{3}{8} \\
 \mathbf{c} \quad & P(\text{more H than T}) = \frac{5}{16}
 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & P(\text{at least one T}) \\ &= \frac{15}{16} \quad \{\text{excludes HHHH}\} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & P(\text{exactly one H}) = \frac{4}{16} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad P(\text{hits the bulls-eye}) &= \frac{\text{area of bulls-eye}}{\text{area of whole target}} \\ &= \frac{\pi \times 20^2}{\pi \times 30^2} \\ &= \frac{400\pi}{900\pi} \\ &= \frac{4}{9} \end{aligned}$$

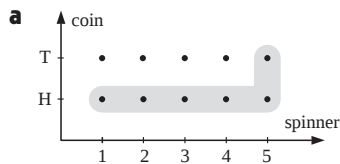
EXERCISE 19D**1**

$$\begin{aligned} \mathbf{a} \quad & P(2 \text{ heads}) \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(2 \text{ tails}) \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & P(\text{at least one H}) \\ &= P(\text{HT or TH or HH}) \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & P(\text{exactly 1 head}) \\ &= P(\text{HT or TH}) \\ &= \frac{2}{4} \quad \text{or} \quad \frac{1}{2} \end{aligned}$$

2**b** There are $2 \times 5 = 10$ possible outcomes.

$$\begin{aligned} \mathbf{c} \quad \mathbf{i} \quad & P(\text{T and 3}) \\ &= \frac{1}{10} \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & P(\text{H and even}) \\ &= P(\text{H2 or H4}) \\ &= \frac{2}{10} \\ &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & P(\text{an odd}) \\ &= P(\text{H1, T1, H3, T3, H5, T5}) \\ &= \frac{6}{10} \\ &= \frac{3}{5} \end{aligned}$$

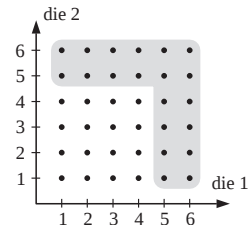
$$\begin{aligned} \mathbf{iv} \quad & P(\text{H or 5}) \\ &= \frac{6}{10} \\ &= \frac{3}{5} \\ &\{\text{those shaded}\} \end{aligned}$$

3

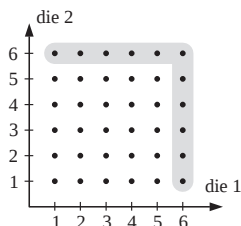
$$\begin{aligned} \mathbf{a} \quad & P(\text{two 3s}) \\ &= P((3, 3)) \\ &= \frac{1}{36} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(5 \text{ and a } 6) \\ &= P((5, 6), (6, 5)) \\ &= \frac{2}{36} \\ &= \frac{1}{18} \end{aligned}$$

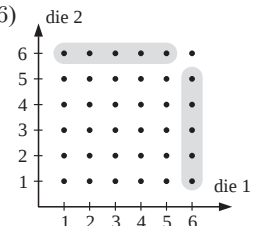
$$\begin{aligned} \mathbf{c} \quad & P(5 \text{ or a } 6) \\ &= \frac{20}{36} \\ &= \frac{5}{9} \end{aligned}$$



$$\begin{aligned} \mathbf{d} \quad & P(\text{at least one } 6) \\ &= \frac{11}{36} \end{aligned}$$

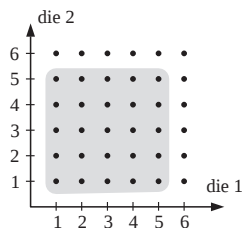


$$\begin{aligned} \mathbf{e} \quad & P(\text{exactly one } 6) \\ &= \frac{10}{36} \\ &= \frac{5}{18} \end{aligned}$$



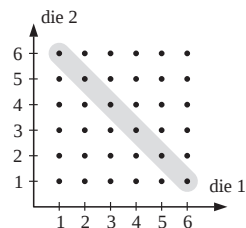
f $P(\text{no sixes})$

$$= \frac{25}{36}$$


g $P(\text{sum of 7})$

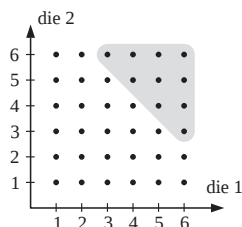
$$= \frac{6}{36}$$

$$= \frac{1}{6}$$


h $P(\text{sum} > 8)$

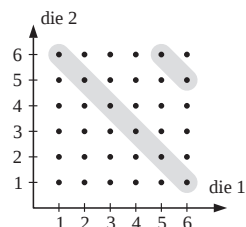
$$= \frac{10}{36}$$

$$= \frac{5}{18}$$


i $P(\text{sum of 7 or 11})$

$$= \frac{6 + 2}{36}$$

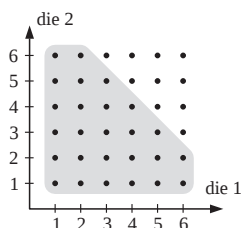
$$= \frac{2}{9}$$


j $P(\text{sum no more than 8})$

$$= P(\text{sum} \leq 8)$$

$$= \frac{26}{36}$$

$$= \frac{13}{18}$$



EXERCISE 19E.1

1 a $P(\text{rains on any one day})$

$$= \frac{6}{7}$$

c $P(\text{rains on 3 successive days})$

$$= P(R \text{ and } R \text{ and } R)$$

$$= \frac{6}{7} \times \frac{6}{7} \times \frac{6}{7} \text{ or } \frac{216}{343}$$

b $P(\text{rains on 2 successive days})$

$$= P(R \text{ and } R)$$

$$= \frac{6}{7} \times \frac{6}{7}$$

$$= \frac{36}{49}$$

2 a $P(H, \text{ then } H, \text{ then } H)$

$$= P(H \text{ and } H \text{ and } H)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

b $P(T, \text{ then } H, \text{ then } T)$

$$= P(T \text{ and } H \text{ and } T)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

3 Let A be the event of photocopier A malfunctioning and B be the event of photocopier B malfunctioning.

a $P(\text{both malfunction})$

$$= P(A \text{ and } B)$$

$$= 0.08 \times 0.12$$

$$= 0.0096$$

b $P(\text{both work})$

$$= P(A' \text{ and } B')$$

$$= 0.92 \times 0.88$$

$$= 0.8096$$

4 a $P(\text{they will be happy})$

$$= P(B, \text{ then } G, \text{ then } B, \text{ then } G)$$

$$= P(B \text{ and } G \text{ and } B \text{ and } G)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{16}$$

b $P(\text{they will be unhappy})$

$$= 1 - P(\text{they will be happy})$$

$$= 1 - \frac{1}{16}$$

$$= \frac{15}{16}$$

- 5** Let J be the event of Jiri hitting the target and
B be the event of Benita hitting the target.

a	$P(\text{both hit})$	b	$P(\text{both miss})$	c	$P(\text{J hits and B misses})$
	$= P(JB)$		$= P(J'B')$		$= P(JB')$
	$= 0.7 \times 0.8$		$= 0.3 \times 0.2$		$= 0.7 \times 0.2$
	$= 0.56$		$= 0.06$		$= 0.14$
d	$P(\text{B hits and J misses})$		$= P(BJ')$		
			$= 0.8 \times 0.3$		
			$= 0.24$		

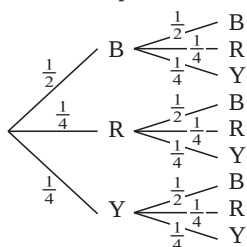
- 6** Let H be the event the archer hits the target.

$$\therefore P(H) = \frac{2}{5}, \quad P(H') = \frac{3}{5}$$

a	$P(3 \text{ hits})$	b	$P(2 \text{ hits then a miss})$	c	$P(\text{all misses})$
	$= P(HHH)$		$= P(HHH')$		$= P(H'H'H')$
	$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$		$= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5}$		$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$
	$= \frac{8}{125}$		$= \frac{12}{125}$		$= \frac{27}{125}$

EXERCISE 19E.2

1 a	$P(\text{both red})$	b	$P(\text{GR})$	c	$P(\text{a green and a red})$
	$= P(RR)$		$= \frac{3}{10} \times \frac{7}{9}$		$= P(\text{GR or RG})$
	$= \frac{7}{10} \times \frac{6}{9}$		$= \frac{7}{30}$		$= \frac{3}{10} \times \frac{7}{9} + \frac{7}{10} \times \frac{3}{9}$
	$= \frac{7}{15}$				$= \frac{7}{15}$
2 a	$P(\text{all strawberry creams})$	b	$P(\text{none is a strawberry cream})$		
	$= P(SSS)$		$= P(S'S'S')$		
	$= \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10}$		$= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$		
	$= \frac{14}{55}$		$= \frac{1}{55}$		
3 a	$P(\text{wins first prize})$	b	$P(\text{wins 1st and 2nd})$	c	$P(\text{wins all 3})$
	$= \frac{3}{100}$		$= P(WW)$		$= P(WWW)$
			$= \frac{3}{100} \times \frac{2}{99}$		$= \frac{3}{100} \times \frac{2}{99} \times \frac{1}{98}$
d	$P(\text{wins none of them})$		$\div 0.000\,606$		$\div 0.000\,006\,18$
	$= P(W'W'W')$				
	$= \frac{97}{100} \times \frac{96}{99} \times \frac{95}{98}$				
	$\div 0.912$				
4 a	$P(\text{contains the captain})$	b	$P(\text{does not contain captain or vice captain})$		
	$= P(CC'C' \text{ or } C'CC' \text{ or } C'C'C)$		$= P(OOO)$		
	$= \frac{1}{7} \times \frac{6}{6} \times \frac{5}{5} + \frac{6}{7} \times \frac{1}{6} \times \frac{5}{5} + \frac{6}{7} \times \frac{5}{6} \times \frac{1}{5}$		$= \frac{5}{7} \times \frac{4}{6} \times \frac{3}{5}$		
	$= 3 \left(\frac{30}{7 \times 6 \times 5} \right)$		$= \frac{60}{210}$		
	$= \frac{3}{7}$		$= \frac{2}{7}$		
			$\therefore P(\text{does contain captain or vice captain})$		
			$= 1 - \frac{2}{7}$		
			$= \frac{5}{7}$		

EXERCISE 19F
1 a 1st spin 2nd spin

b P(both black) **c** P(both yellow)

$$= P(BB)$$

$$= \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}$$

$$= P(YY)$$

$$= \frac{1}{4} \times \frac{1}{4}$$

$$= \frac{1}{16}$$

d P(both different)

$$= P(BR \text{ or } BY \text{ or } RB \text{ or } RY \text{ or } YB \text{ or } YR)$$

$$= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4}$$

$$= \frac{4}{8} + \frac{2}{16}$$

$$= \frac{5}{8}$$

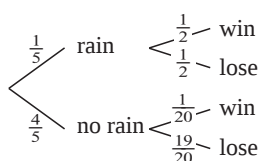
e P(B appears on either spin)

$$= P(BB \text{ or } BR \text{ or } BY \text{ or } RB \text{ or } YB)$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2}$$

$$= 4\left(\frac{1}{8}\right) + \frac{1}{4}$$

$$= \frac{3}{4}$$

2


P(M wins)

$$= P(\text{rain and win or no rain and win})$$

$$= \frac{1}{5} \times \frac{1}{2} + \frac{4}{5} \times \frac{1}{20}$$

$$= \frac{1}{10} \times \frac{10}{10} + \frac{4}{100}$$

$$= \frac{14}{100}$$

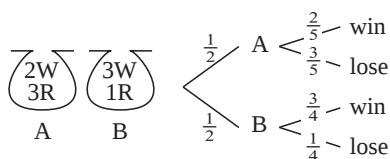
$$= \frac{7}{50}$$

3 P(next is spoiled) = P(from A and spoiled or from B and spoiled)

$$= 0.4 \times 0.05 + 0.6 \times 0.02$$

$$= 0.020 + 0.012$$

$$= 0.032 \quad (3.2\%)$$

4


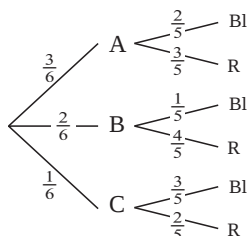
P(red)

$$= P(A \text{ and red or } B \text{ and red})$$

$$= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{3}{10} + \frac{1}{8}$$

$$= \frac{17}{40}$$

5

a P(blue) = P(A and Bl or B and Bl or C and Bl)

$$= \frac{3}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5} + \frac{1}{6} \times \frac{3}{5}$$

$$= \frac{11}{30}$$

b P(red) = 1 - P(blue)

$$= 1 - \frac{11}{30}$$

$$= \frac{19}{30}$$

EXERCISE 19G**1****a** P(different colours)

$$\begin{aligned}
 &= P(PG \text{ or } GP) \\
 &= \frac{2}{7} \times \frac{5}{7} + \frac{5}{7} \times \frac{2}{7} \\
 &= \frac{20}{49}
 \end{aligned}$$

b P(different colours)

$$\begin{aligned}
 &= P(PG \text{ or } GP) \\
 &= \frac{2}{7} \times \frac{5}{6} + \frac{5}{7} \times \frac{2}{6} \\
 &= \frac{20}{42} \\
 &= \frac{10}{21}
 \end{aligned}$$

2**a** P(both odd)

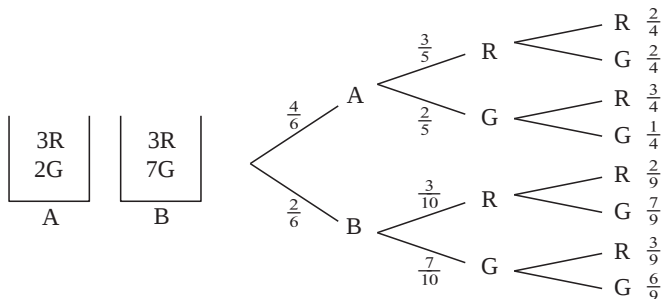
$$\begin{aligned}
 &= P(\text{odd and odd}) \\
 &= \frac{3}{5} \times \frac{2}{4} \\
 &= \frac{3}{10}
 \end{aligned}$$

b P(both even)

$$\begin{aligned}
 &= P(\text{even and even}) \\
 &= \frac{2}{5} \times \frac{1}{4} \\
 &= \frac{1}{10}
 \end{aligned}$$

c P(one odd and other even)

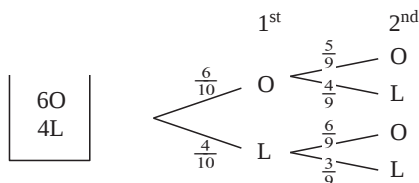
$$\begin{aligned}
 &= 1 - P(\text{both odd}) - P(\text{both even}) \\
 &= 1 - \frac{3}{10} - \frac{1}{10} \\
 &= \frac{6}{10} \\
 &= \frac{3}{5}
 \end{aligned}$$

3**a** P(both green)

$$\begin{aligned}
 &= P(AGG \text{ or } BGG) \\
 &= \frac{4}{6} \times \frac{2}{5} \times \frac{1}{4} + \frac{2}{6} \times \frac{7}{10} \times \frac{6}{9} \\
 &= \frac{1}{15} + \frac{7}{45} \\
 &= \frac{3}{45} + \frac{7}{45} \\
 &= \frac{10}{45} \\
 &= \frac{2}{9}
 \end{aligned}$$

b P(different in colour)

$$\begin{aligned}
 &= 1 - P(\text{both green}) - P(\text{both red}) \\
 &= 1 - \frac{2}{9} - P(\text{ARR or BRR}) \\
 &= \frac{7}{9} - \left(\frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{2}{6} \times \frac{3}{10} \times \frac{2}{9} \right) \\
 &= \frac{7}{9} - \left(\frac{1}{5} + \frac{1}{45} \right) \\
 &= \frac{7}{9} - \left(\frac{9}{45} + \frac{1}{45} \right) \\
 &= \frac{7}{9} - \frac{2}{9} \\
 &= \frac{5}{9}
 \end{aligned}$$

4**a** P(both O)

$$\begin{aligned}
 &= \frac{6}{10} \times \frac{5}{9} \\
 &= \frac{1}{3}
 \end{aligned}$$

b P(both L)

$$\begin{aligned}
 &= \frac{4}{10} \times \frac{3}{9} \\
 &= \frac{2}{15}
 \end{aligned}$$

c P(OL)

$$\begin{aligned}
 &= \frac{6}{10} \times \frac{4}{9} \\
 &= \frac{4}{15}
 \end{aligned}$$

d P(LO)

$$\begin{aligned}
 &= \frac{4}{10} \times \frac{6}{9} \\
 &= \frac{4}{15}
 \end{aligned}$$

$$\begin{aligned}
 &\frac{1}{3} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15} \\
 &= \frac{5}{15} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15} \\
 &= \frac{15}{15} \text{ which is } 1
 \end{aligned}$$

The answer must be 1 as the four categories **a**, **b**, **c**, **d** are all the possibilities that could occur.

5 a sample 1 sample 2

b $P(\text{both have two yolks})$
 $= P(DD)$
 $= \frac{3}{9} \times \frac{2}{8}$
 $= \frac{1}{12}$

c $P(\text{both have one yolk})$
 $= P(SS)$
 $= \frac{6}{9} \times \frac{5}{8}$
 $= \frac{10}{24}$
 $= \frac{5}{12}$

6 a chocolate 1 chocolate 2

b $P(\text{both hard})$
 $= P(HH)$
 $= \frac{10}{25} \times \frac{9}{24}$
 $= \frac{3}{20}$

c $P(\text{both soft})$
 $= P(SS)$
 $= \frac{15}{25} \times \frac{14}{24}$
 $= \frac{7}{20}$

7 1st 2nd 3rd

a $P(\text{all red})$
 $= P(RRR)$
 $= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4}$
 $= \frac{1}{5}$

b $P(\text{only two are red})$
 $= P(RRB \text{ or } RBR \text{ or } BRR)$
 $= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{4}{6} \times \frac{2}{5} \times \frac{3}{4} + \frac{2}{6} \times \frac{4}{5} \times \frac{3}{4}$
 $= 3 \times \left(\frac{24}{6 \times 5 \times 4} \right)$
 $= \frac{3}{5}$

c $P(\text{at least two are red})$
 $= P(\text{all red or only two are red})$
 $= \frac{1}{5} + \frac{3}{5} \quad \{\text{from a and b}\}$
 $= \frac{4}{5}$

8

$P(\text{marble from B is W}) = P(RW \text{ or } WW)$
 $= \frac{3}{5} \times \frac{3}{9} + \frac{2}{5} \times \frac{5}{9}$
 $= \frac{19}{45}$
 {paths ticked}

9 1st draw 2nd draw

a $P(\text{wins both})$
 $= P(WW)$
 $= \frac{2}{100} \times \frac{1}{99}$
 $\div 0.000\ 202$

b $P(\text{wins neither})$
 $= P(LL)$
 $= \frac{98}{100} \times \frac{97}{99}$
 $\div 0.960$

$$\begin{aligned}
 \text{c } P(\text{wins at least one prize}) &= 1 - P(\text{wins neither}) \\
 &= 1 - \frac{98}{100} \times \frac{97}{99} \\
 &\doteq 0.0398
 \end{aligned}$$

EXERCISE 19H

1 a The rule is: *add* the two terms directly above the new row with 1s at either end.

$$\text{b } 1 \ 7 \ 21 \ 35 \ 35 \ 21 \ 7 \ 1 \quad \text{c } 1 \ 7 \ 21 \ 35 \ 35 \ 21 \ 7 \ 1$$

2

$$\begin{array}{cccccccc}
 & & & 1 & & 1 & & \\
 & & 1 & 2 & 1 & & & \\
 & 1 & 3 & 3 & 1 & & & \\
 & & 1 & 4 & 6 & 4 & 1 & \\
 & 1 & 5 & 10 & 10 & 5 & 1 & \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1
 \end{array}$$

$$\text{a } C_2^3 = 3$$

$$\text{b } C_5^7 = 21$$

$$\text{c } C_2^5 = 10$$

$$\text{d } C_1^8 = 8$$

$$\text{e } C_0^4 = 1$$

$$\text{f } C_6^6 = 1$$

$$\text{3 a } C_5^9 = 126 \quad \text{b } C_3^{14} = 364 \quad \text{c } C_1^{40} = 40 \quad \text{d } C_2^3 = 3$$

$$\text{e } C_{10}^{40} = 847\,660\,528 \quad \text{f } C_{20}^{40} \doteq 1.3784 \times 10^{11}$$

4

a	HHHHH	HHHHT	HHHTT	HHTTT	HTTTT	TTTTT
		HHHTH	HHTHT	HTHTT	THTTT	
		HHTHH	HTHHT	THHTT	TTHTT	
		HTHHH	THHHT	HTTHT	TTTHT	
		THHHH	HHTTH	THTHT	TTTTH	
			HTHTH	TTHHT		
			THHTH	HTTTH		
			HTTTH	THTTH		
			THTTH	TTHTH		
			TTHHH	TTTHH		

$$\text{b } \text{ i } 1 \quad \text{ii } 5 \quad \text{iii } 10 \quad \text{iv } 10 \quad \text{v } 5 \quad \text{vi } 1$$

$$\text{5 a } C_{10}^{18} = 43\,758 \quad \text{b } C_{14}^{23} = 817\,190$$

6

$$\begin{aligned}
 \text{a } (p+q)^3 &= (p+q)(p+q)^2 \\
 &= (p+q)(p^2 + 2pq + q^2) \\
 &= p^3 + 2p^2q + pq^2 \\
 &\quad + p^2q + 2pq^2 + q^3 \\
 &= p^3 + 3p^2q + 3pq^2 + q^3
 \end{aligned}$$

Coefficients are: 1, 3, 3, 1

$$\begin{aligned}
 \text{b } (p+q)^4 &= (p+q)(p+q)^3 \\
 &= (p+q)(p^3 + 3p^2q + 3pq^2 + q^3) \quad \{\text{from a}\} \\
 &= p^4 + 3p^3q + 3p^2q^2 + pq^3 \\
 &\quad + p^3q + 3p^2q^2 + 3pq^3 + q^4 \\
 &= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4
 \end{aligned}$$

Coefficients are: 1, 4, 6, 4, 1

c From **a** and **b** we notice that for $(p+q)^n$ the coefficients are the n th row of Pascal's triangle. The powers of p decrease by 1 as the powers of q increase by 1.

$$\text{i } p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$

$$\text{ii } p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

$$\begin{aligned}
 \text{d } C_0^4 + C_1^4 p^3q + C_2^4 p^2q^2 + C_3^4 pq^3 + C_4^4 q^4 &= (p+q)^4 \quad \{\text{Binomial expansion in reverse}\} \\
 &= 1^4 \\
 &= 1
 \end{aligned}$$

7 a $(p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

b $\xrightarrow{\text{P(3 heads)}} 4p^3q$
 $= 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) \quad \{\text{as } p = q = \frac{1}{2}\}$
 $= \frac{1}{4}$

8 a $(p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

b $\xrightarrow{\text{P(4H and 1T)}} 5p^4q$
 $= 5p^4q$
 $= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)$
 $= \frac{5}{32}$

c $\xrightarrow{\text{P(2H and 3T)}} 10p^2q^3$
 $= 10p^2q^3$
 $= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$
 $= \frac{10}{32}$
 $= \frac{5}{16}$

9 a $\left(\frac{2}{3} + \frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$

b $P(S) = \frac{2}{3}, P(S') = \frac{1}{3}$ S' represents a non-strawberry cream (or an almond centre)

i $P(\text{all } S)$
 $= \left(\frac{2}{3}\right)^4$
 $= \frac{16}{81}$

ii $P(\text{two of each})$
 $= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$
 $= \frac{8}{27}$

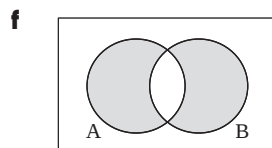
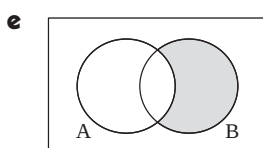
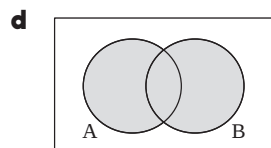
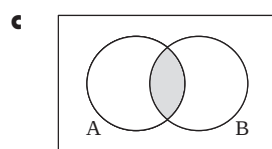
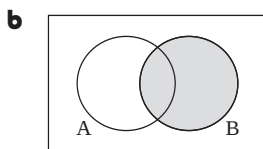
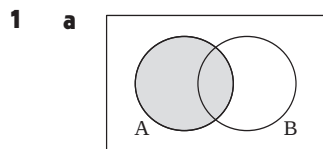
iii $P(\text{at least 2 strawberry creams})$
 $= P(\text{all } S \text{ or } 3S, 1T \text{ or } 2S, 2T)$
 $= \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)$
 $= \frac{16}{81} + \frac{32}{81} + \frac{24}{81}$
 $= \frac{72}{81}$
 $= \frac{8}{9}$

10 a $\left(\frac{3}{4} + \frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

b i $P(2 \text{ 'flat backs'})$
 $= P(2F \text{ and } 3F's)$
 $= 10 \times \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$
 $= \frac{135}{512}$

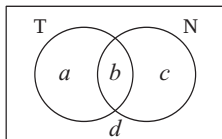
ii $P(\text{at least 3 are 'flat backs'})$
 $= P(3F, 2F' \text{ or } 4F, 1F' \text{ or } 5F)$
 $= 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$
 $= \frac{90 + 15 + 1}{4^5}$
 $= \frac{106}{1024}$
 $= \frac{53}{512}$

EXERCISE 19I



- 2** **a** Total number in the class = $3 + 4 + 5 + 17 = 29$
b Number who study both = 17 {the intersection}
c Number who study at least one = $5 + 17 + 4 = 26$ {the union}
d Number who study only Chem. = 5

- 3** **a** Total number in the survey = $37 + 9 + 15 + 4 = 65$
b Number who liked both = 9 {the intersection}
c Number who liked neither = 4
d Number who liked exactly one = $37 + 15 = 52$

4

T represents those playing tennis
 N represents those playing netball

$$\begin{aligned}\therefore a + b + c + d &= 40 \\ a + b &= 19 \\ b + c &= 20 \\ d &= 8\end{aligned}$$

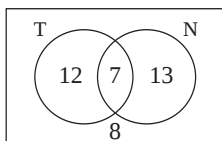
$$\text{So, } a + b + c = 32$$

$$\therefore 19 + c = 32 \quad \text{and} \quad a + 20 = 32$$

$$\therefore c = 13 \quad \text{and} \quad a = 12$$

$$\text{Hence, } 12 + b + 13 + 8 = 40$$

$$\therefore b = 7$$



a $P(\text{plays tennis})$

$$\begin{aligned}&= \frac{12 + 7}{40} \\ &= \frac{19}{40}\end{aligned}$$

b $P(\text{does not play netball})$

$$\begin{aligned}&= \frac{12 + 8}{40} \\ &= \frac{1}{2}\end{aligned}$$

c $P(\text{plays at least one})$

$$\begin{aligned}&= \frac{12 + 7 + 13}{40} \\ &= \frac{32}{40} \\ &= \frac{4}{5}\end{aligned}$$

d $P(\text{plays one and only one})$

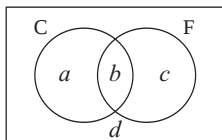
$$\begin{aligned}&= \frac{12 + 13}{40} \\ &= \frac{25}{40} \\ &= \frac{5}{8}\end{aligned}$$

e $P(\text{plays netball, but not tennis})$

$$= \frac{13}{40}$$

f $P(\text{plays tennis given plays netball})$

$$\begin{aligned}&= \frac{7}{7 + 13} \\ &= \frac{7}{20}\end{aligned}$$

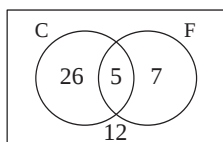
5

C represents men who gave chocolates
 F represents men who gave flowers

$$\begin{aligned}\therefore a + b + c + d &= 50 \\ a + b &= 31 \\ b + c &= 12 \\ b &= 5\end{aligned}$$

$$\text{Thus } c = 7, \quad a = 26 \quad \text{and} \quad 26 + 5 + 7 + d = 50$$

$$\therefore d = 12$$



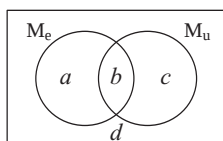
$$\begin{aligned} \mathbf{a} \quad P(C \text{ or } F) &= \frac{26 + 5 + 7}{50} \\ &= \frac{38}{50} \\ &= \frac{19}{25} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(C \text{ but not } F) &= \frac{26}{50} \\ &= \frac{13}{25} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad P(\text{neither } C \text{ nor } F) &= \frac{12}{50} \\ &= \frac{6}{25} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad P(F \text{ given that } C') &= \frac{7}{7 + 12} \\ &= \frac{7}{19} \end{aligned}$$

6

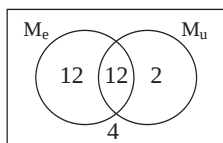


$$\begin{aligned} a + b + c + d &= 30 \\ a + b &= 24 \\ b &= 12 \\ a + b + c &= 26 \end{aligned}$$

$$\therefore 26 + d = 30 \quad \text{i.e., } d = 4$$

$$24 + c = 26 \quad \text{i.e., } c = 2$$

$$\text{and } a + 12 + 2 = 26 \quad \therefore a = 12$$



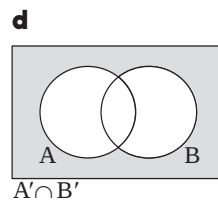
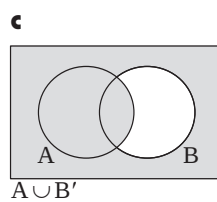
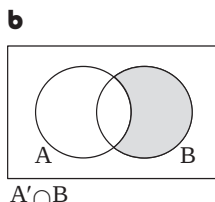
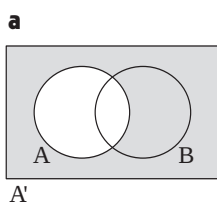
$$\begin{aligned} \mathbf{a} \quad P(M_u) &= \frac{14}{30} \\ &= \frac{7}{15} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(M_u, \text{ but not } M_e) &= \frac{2}{30} \\ &= \frac{1}{15} \end{aligned}$$

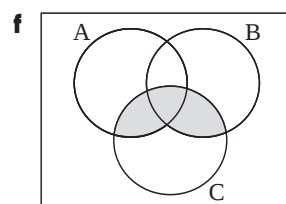
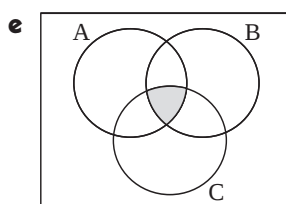
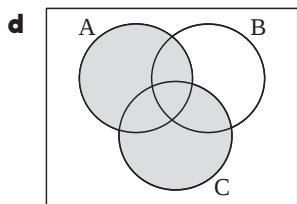
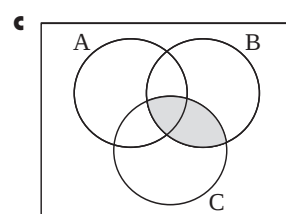
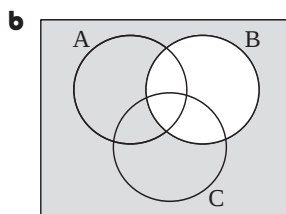
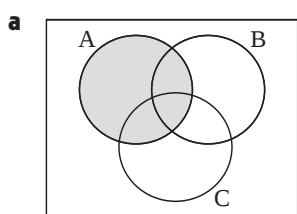
$$\begin{aligned} \mathbf{c} \quad P(\text{neither } M_u \text{ nor } M_e) &= \frac{4}{30} \\ &= \frac{2}{15} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad P(M_e \text{ given } M_u) &= \frac{12}{14} \\ &= \frac{6}{7} \end{aligned}$$

7



8



9 a

$A \cap B$ — $(A \cap B)'$ is shaded

So $A' \cup B'$ is the region containing either type of shading.

Thus, as the regions are the same, $(A \cap B)' = A' \cup B'$ is verified.

b

$A \cup (B \cap C)$ consists of the shaded region

As the two regions are identical

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ is verified.

$(A \cup B) \cap (A \cup C)$ consists of the 'double shaded' region.

c

$A \cap (B \cup C)$ consists of the double shaded region

$(A \cap B) \cup (A \cap C)$ consists of the region shaded. (all forms and)

As the regions are identical, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is verified.

10 a $A = \{7, 14, 21, 28, 35, \dots, 98\}$
 $B = \{5, 10, 15, 20, 25, \dots, 95\}$

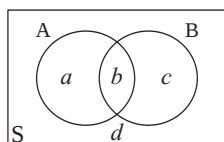
i as $98 = 7 \times 14$, $n(A) = 14$ **ii** as $95 = 5 \times 19$, $n(B) = 19$

iii $A \cap B = \{35, 70\}$ $\therefore n(A \cap B) = 2$

iv $A \cup B = \{5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, 42, 45, 49, 50, 55, 56, 60, 63, 65, 70, 75, 77, 80, 84, 85, 90, 91, 95, 98\}$
 $\therefore n(A \cup B) = 31$

b $n(A) + n(B) - n(A \cap B)$
 $= 14 + 19 - 2$
 $= 31$
 $= n(A \cup B) \quad \checkmark$

11



$$\begin{aligned}
 & n(A) + n(B) - n(A \cap B) \\
 &= a + b + b + c - b \\
 &= a + b + c \\
 &= n(A \cup B)
 \end{aligned}$$

12

a i

$$\begin{aligned}
 & P(B) \\
 &= \frac{n(B)}{n(S)} \\
 &= \frac{b + c}{a + b + c + d}
 \end{aligned}$$

ii

$$\begin{aligned}
 & P(A \text{ and } B) \\
 &= \frac{n(A \cap B)}{n(S)} \\
 &= \frac{b}{a + b + c + d}
 \end{aligned}$$

iii

$$\begin{aligned}
 & P(A \text{ or } B) \\
 &= \frac{n(A \cup B)}{n(S)} \\
 &= \frac{a + b + c}{a + b + c + d}
 \end{aligned}$$

iv

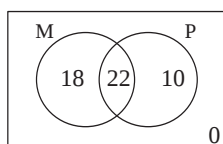
$$\begin{aligned}
 & P(A) + P(B) - P(A \text{ and } B) \\
 &= \frac{a + b + b + c - b}{a + b + c + d} \\
 &= \frac{a + b + c}{a + b + c + d}
 \end{aligned}$$

$$\mathbf{b} \quad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \quad \{\text{using iii and iv}\}$$

EXERCISE 19J

1

a



So 22 study both.

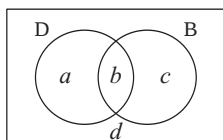
b i

$$\begin{aligned}
 & P(M \text{ but not } P) \\
 &= \frac{18}{50} \\
 &= \frac{9}{25}
 \end{aligned}$$

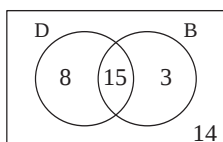
ii

$$\begin{aligned}
 & P(P \text{ given } M) \\
 &= \frac{22}{18 + 22} \\
 &= \frac{22}{40} \\
 &= \frac{11}{20}
 \end{aligned}$$

2



$$\begin{aligned}
 a + b + c + d &= 40 \quad \dots (1) & \therefore d = 14 \quad \{\text{using (1) and (4)}\} \\
 a + b &= 23 \quad \dots (2) & 23 + c = 26 \quad \text{and} \quad a + 18 = 26 \\
 b + c &= 18 \quad \dots (3) & \therefore c = 3 \quad \text{and} \quad a = 8 \\
 a + b + c &= 26 \quad \dots (4) & \text{Thus } b = 18 - c = 15
 \end{aligned}$$



a

$$\begin{aligned}
 & P(D \text{ and } B) \\
 &= \frac{15}{40} \\
 &= \frac{3}{8}
 \end{aligned}$$

b

$$\begin{aligned}
 & P(\text{neither } D \text{ nor } B) \\
 &= \frac{14}{40} \\
 &= \frac{7}{20}
 \end{aligned}$$

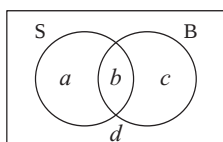
c

$$\begin{aligned}
 & P(D, \text{ but not } B) \\
 &= \frac{8}{40} \\
 &= \frac{1}{5}
 \end{aligned}$$

d

$$\begin{aligned}
 & P(B \text{ given } D) \\
 &= \frac{15}{23}
 \end{aligned}$$

3



$$\begin{aligned}
 a + b + c + d &= 50 & \therefore c = 17, \quad a = 18 \\
 a + b &= 23 & \text{and } 18 + 5 + 17 + d = 50 \\
 b + c &= 22 & \therefore d = 10 \\
 b &= 5
 \end{aligned}$$

a

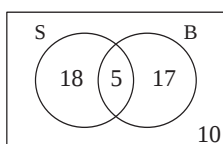
$$\begin{aligned}
 & P(\text{not } B) \\
 &= P(B') \\
 &= \frac{28}{50} \\
 &= \frac{14}{25}
 \end{aligned}$$

b

$$\begin{aligned}
 & P(B \text{ or } S) \\
 &= \frac{18 + 5 + 17}{50} \\
 &= \frac{40}{50} \\
 &= \frac{4}{5}
 \end{aligned}$$

c

$$\begin{aligned}
 & P(\text{neither } B \text{ nor } S) \\
 &= \frac{10}{50} \\
 &= \frac{1}{5}
 \end{aligned}$$



d $P(B, \text{ given } S)$

$$= \frac{5}{18+5}$$

$$= \frac{5}{23}$$

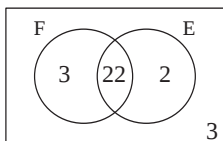
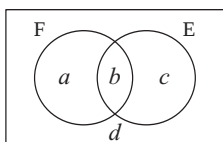
e $P(S, \text{ given } B')$

$$= \frac{18}{18+10}$$

$$= \frac{18}{28}$$

$$= \frac{9}{14}$$

4



$$a + b + c + d = 30$$

$$a + b = 25$$

$$b + c = 24$$

$$d = 3$$

$$\therefore a + b + c = 27$$

$$\text{and so, } 25 + c = 27 \quad \text{and} \quad a + 24 = 27$$

$$\therefore c = 2 \quad \text{and} \quad a = 3$$

$$\text{Also, } b = 25 - a = 22$$

a $P(E, \text{ given } F)$

$$= \frac{22}{3+22}$$

$$= \frac{22}{25}$$

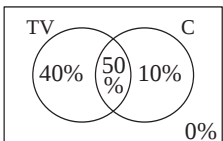
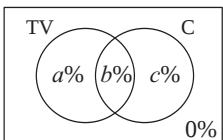
b $P(F, \text{ given } E')$

$$= \frac{3}{3+3}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

5



$$a + b + c = 100$$

$$a + b = 90$$

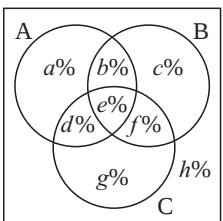
$$b + c = 60$$

$$\therefore c = 10 \quad \text{and} \quad a = 40$$

$$\therefore b = 50$$

$$P(\text{TV, given } C) = \frac{50}{50+10} = \frac{5}{6}$$

6



$$a + b + c + d + e + f + g + h = 100$$

$$a + b + d + e = 20$$

$$b + c + e + f = 16$$

$$d + e + f + g = 14$$

$$b + e = 8$$

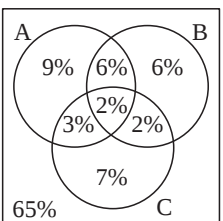
$$d + e = 5$$

$$e + f = 4$$

$$e = 2$$

$$\text{i.e., } e = 2, \quad f = 2, \quad d = 3, \quad b = 6,$$

$$\begin{cases} a + 6 + 3 + 2 = 20 \\ 6 + c + 2 + 2 = 16 \\ 3 + 2 + 2 + g = 14 \end{cases} \quad \therefore \begin{cases} a = 9 \\ c = 6 \\ g = 7 \end{cases}$$



a $P(\text{none})$

$$= \frac{65}{100}$$

$$= \frac{13}{20}$$

b $P(\text{at least one})$

$$= 1 - P(\text{none})$$

$$= 1 - \frac{13}{20}$$

$$= \frac{7}{20}$$

c $P(\text{exactly one})$

$$= \frac{9+6+7}{100}$$

$$= \frac{22}{100}$$

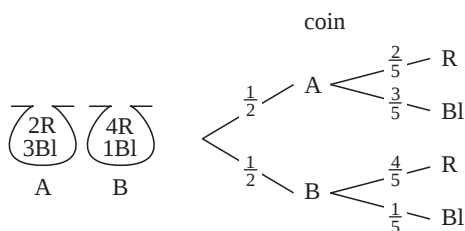
$$= \frac{11}{50}$$

$$\begin{aligned}
 \mathbf{d} \quad P(A \text{ or } B) &= \frac{9+6+6+3+2+2}{100} \\
 &= \frac{28}{100} \\
 &= \frac{7}{25}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad P(A, \text{ given at least one}) &= \frac{9+6+2+3}{35} \\
 &= \frac{20}{35} \\
 &= \frac{4}{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad P(C, \text{ given } A \text{ or } B \text{ or both}) &= \frac{3+2+2}{9+6+6+3+2+2} \\
 &= \frac{7}{28} \\
 &= \frac{1}{4}
 \end{aligned}$$

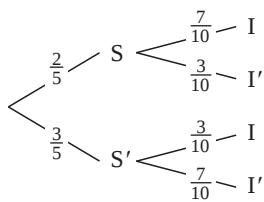
7



$$\begin{aligned}
 \mathbf{a} \quad P(R) &= \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{5} \\
 &= \frac{6}{10} \\
 &= \frac{3}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad P(B|R) &= \frac{P(B \cap R)}{P(R)} \\
 &= \frac{\frac{1}{2} \times \frac{4}{5}}{\frac{3}{5}} \\
 &= \frac{2}{3}
 \end{aligned}$$

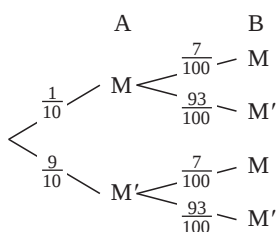
8



$$\begin{aligned}
 \mathbf{a} \quad P(I) &= \frac{2}{5} \times \frac{7}{10} + \frac{3}{5} \times \frac{3}{10} \\
 &= \frac{23}{50} \quad (\text{or } 0.46)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad P(S|I) &= \frac{P(S \cap I)}{P(I)} \\
 &= \frac{\frac{2}{5} \times \frac{7}{10}}{\frac{23}{50}} \\
 &= \frac{14}{23}
 \end{aligned}$$

9

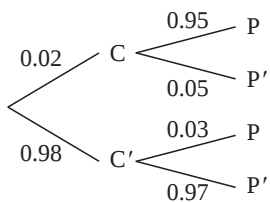


$$\begin{aligned}
 &P(B \mid \text{at least one malfunctions}) \\
 &= \frac{P(B \cap \text{at least one malfunctions})}{P(\text{at least one malfunctions})} \\
 &= \frac{\frac{1}{10} \times \frac{7}{100} + \frac{9}{10} \times \frac{7}{100}}{\frac{1}{10} \times \frac{7}{100} + \frac{1}{10} \times \frac{93}{100} + \frac{9}{10} \times \frac{7}{100}} \\
 &= \frac{7 + 63}{7 + 93 + 63} \\
 &= \frac{70}{163}
 \end{aligned}$$

$$\mathbf{10} \quad P(B) = 0.5, \quad P(G) = 0.6, \quad P(G|B) = 0.9$$

$$\begin{aligned}
 \mathbf{a} \quad &P(\text{both eat}) \\
 &= P(B \cap G) \\
 &= P(G|B) \times P(B) \quad \left\{ \text{as } P(G|B) = \frac{P(G \cap B)}{P(B)} \right\} \\
 &= 0.9 \times 0.5 \\
 &= 0.45
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad &P(B|G) \\
 &= \frac{P(B \cap G)}{P(G)} \\
 &= \frac{0.45}{0.6} \\
 &= 0.75 \\
 \mathbf{c} \quad &P(\text{at least one eats}) \\
 &= P(B \cup G) \\
 &= P(B) + P(G) - P(B \cap G) \\
 &= 0.5 + 0.6 - 0.45 \\
 &= 0.65
 \end{aligned}$$

11

a

$$\begin{aligned} P(P) &= 0.02 \times 0.95 + 0.98 \times 0.03 \\ &= 0.0484 \end{aligned}$$

b

$$\begin{aligned} P(C|P) &= \frac{P(C \cap P)}{P(P)} \\ &= \frac{0.02 \times 0.95}{0.0484} \\ &\doteq 0.3926 \end{aligned}$$

12 The coins are H, H T, T and H, T.

Any one of these 6 faces could be seen uppermost, $\therefore P(\text{falls H}) = \frac{3}{6} = \frac{1}{2}$

$$\begin{aligned} \text{Now } P(\text{HH coin} | \text{falls H}) &= \frac{P(\text{HH coin} \cap \text{falls H})}{P(\text{falls H})} \\ &= \frac{P(\text{HH})}{P(\text{falls H})} \\ &= \frac{\frac{1}{3}}{\frac{1}{2}} \\ &= \frac{2}{3} \end{aligned}$$

EXERCISE 19K

1

$$\begin{aligned} P(R \cap S) &= P(R) + P(S) - P(R \cup S) \\ &= 0.4 + 0.5 - 0.7 \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} \text{Also, } P(R) \times P(S) &= 0.4 \times 0.5 \\ &= 0.2 \end{aligned}$$

So, $P(R \cap S) = P(R) \times P(S) \Rightarrow R$ and S are independent events.

2

$$\begin{aligned} \text{a } P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= \frac{2}{5} + \frac{1}{3} - \frac{1}{2} \\ &= \frac{7}{30} \end{aligned}$$

b

$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} \\ &= \frac{\frac{7}{30}}{\frac{2}{5}} \\ &= \frac{7}{12} \end{aligned}$$

c

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{7}{30}}{\frac{1}{3}} \\ &= \frac{7}{10} \end{aligned}$$

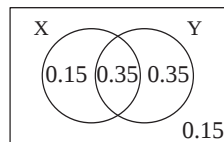
A and B are not independent as $P(A \cap B) = \frac{7}{30}$ whereas $P(A) \times P(B) = \frac{2}{15}$ i.e., $P(A \cap B) \neq P(A) \times P(B)$

3

$$\begin{aligned} \text{a As } X \text{ and } Y \text{ are independent} \\ P(X \cap Y) &= P(X) \times P(Y) \\ &= 0.5 \times 0.7 \\ &= 0.35 \\ \text{i.e., } P(\text{both } X \text{ and } Y) &= 0.35 \end{aligned}$$

b

$$\begin{aligned} P(X \text{ or } Y) &= P(X \cup Y) \\ &= P(X) + P(Y) - P(X \cap Y) \\ &= 0.5 + 0.7 - 0.35 \\ &= 0.85 \end{aligned}$$



$$\begin{aligned} \text{c } P(\text{neither } X \text{ nor } Y) &= 0.15 \end{aligned}$$

d

$$\begin{aligned} P(X \text{ but not } Y) &= 0.15 \end{aligned}$$

$$\text{e } P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{0.35}{0.70} = \frac{1}{2}$$

$$\begin{aligned}
 4 \quad & P(\text{at least one solves it}) \\
 &= 1 - P(\text{no-one solves it}) \\
 &= 1 - P(A' \text{ and } B' \text{ and } C') \\
 &= 1 - \frac{2}{5} \times \frac{1}{3} \times \frac{1}{2} \\
 &= 1 - \frac{1}{15} \\
 &= \frac{14}{15}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad a \quad & P(\text{at least one 6}) = 1 - P(\text{no 6s}) \\
 &= 1 - P(6' \text{ and } 6' \text{ and } 6') \\
 &= 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \\
 &= 1 - \frac{125}{216} \\
 &= \frac{91}{216}
 \end{aligned}$$

$$b \quad P(\text{at least one 6 in } n \text{ throws}) = 1 - \left(\frac{5}{6}\right)^n$$

$$\text{So we want } 1 - \left(\frac{5}{6}\right)^n > 0.99$$

$$\therefore -\left(\frac{5}{6}\right)^n > -0.01$$

$$\therefore \left(\frac{5}{6}\right)^n < 0.01$$

$$\therefore n \log\left(\frac{5}{6}\right) < \log(0.01)$$

$$\therefore n > \frac{\log(0.01)}{\log\left(\frac{5}{6}\right)}$$

$$\{\text{as } \log\left(\frac{5}{6}\right) < 0\}$$

$$\therefore n > 25.2585\dots$$

$$\text{i.e., } n = 26$$

$$6 \quad A \text{ and } B \text{ are independent} \Rightarrow P(A \cap B) = P(A)P(B) \dots\dots (1)$$

$$\text{Now } P(A \cap B') = P(A \cup B) - P(B) \quad \{\text{see diagram}\}$$

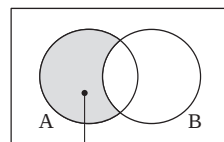
$$= P(A) + P(B) - P(A \cap B) - P(B)$$

$$= P(A) - P(A \cap B)$$

$$= P(A) - P(A)P(B) \quad \{\text{from (1)}\}$$

$$= P(A)[1 - P(B)]$$

$$= P(A) \times P(B')$$


 $A \cap B'$

$\therefore A$ and B' are also independent.

REVIEW SET 19A

- 1 ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA

- a There are 24 possible orderings.

$$\therefore P(A \text{ is next to } C)$$

$$= \frac{12}{24} \quad \{12 \text{ have } A \text{ next to } C\}$$

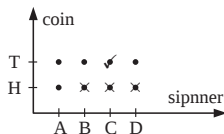
$$= \frac{1}{2}$$

- b $P(\text{exactly one person between } A \text{ and } C)$

$$= \frac{8}{24} \quad \{8 \text{ have one person between } A \text{ and } C\}$$

$$= \frac{1}{3}$$

2



- b $P(T \text{ and } C)$

$$= \frac{1}{8} \quad \{\text{those with a } \checkmark\}$$

- a Consonants are B, C and D

$$\therefore P(H \text{ and a consonant})$$

$$= \frac{3}{8} \quad \{\text{those with } \times\}$$

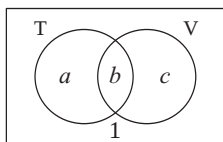
- c $P(T \text{ or vowel})$

$$= P(T \text{ or } A)$$

$$= P(T) + P(A) - P(T \text{ and } A)$$

$$= \frac{4}{8} + \frac{2}{8} - \frac{1}{8}$$

$$= \frac{5}{8}$$

3

$$a + b + c = 24$$

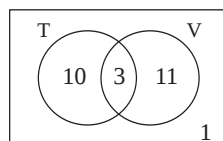
$$a + b = 13$$

$$b + c = 14$$

$$\begin{aligned}\text{Also } b &= 13 - a \\ &= 3\end{aligned}$$

$$\therefore 13 + c = 24 \quad \text{and} \quad a + 14 = 24$$

$$\therefore c = 11 \quad \text{and} \quad a = 10$$



a $P(T \text{ and } V)$

$$= \frac{3}{25}$$

b $P(\text{at least one})$

$$= 1 - P(\text{neither})$$

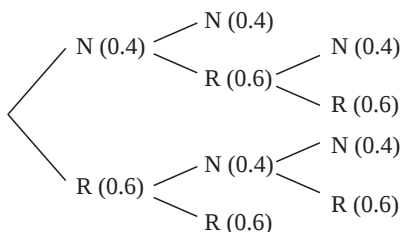
$$= 1 - \frac{1}{25}$$

$$= \frac{24}{25}$$

c $P(V | T')$

$$= \frac{11}{11 + 1}$$

$$= \frac{11}{12}$$

4

$$P(\text{Niklas wins})$$

$$= (0.4)(0.4) + (0.4)(0.6)(0.4) + (0.6)(0.4)(0.4)$$

$$= 0.352$$

5 $P(M) = \frac{3}{5}, \quad P(W) = \frac{2}{3}$

a $P(M \text{ and } W)$

$$\begin{aligned}&= \frac{3}{5} \times \frac{2}{3} \quad \{\text{assuming independence}\} \\ &= \frac{2}{5}\end{aligned}$$

b $P(\text{at least one})$

$$= P(M \text{ or } W)$$

$$= P(M) + P(W) - P(M \text{ and } W)$$

$$= \frac{3}{5} + \frac{2}{3} - \frac{2}{5}$$

$$= \frac{13}{15}$$

c $P(M' \text{ and } W)$

$$= (1 - \frac{3}{5}) \times \frac{2}{3}$$

$$= \frac{2}{5} \times \frac{2}{3}$$

$$= \frac{4}{15}$$

6 A repetition of 5 independent events $\therefore$ binomial model applies.

$$P(M) = \frac{4}{5}, \quad P(M') = \frac{1}{5}$$

a $P(M \text{ wins 3 games})$

$$= C_3^5 \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^2$$

$$= 0.2048$$

$$\div 0.205$$

b $P(M \text{ wins 4 or 5 games})$

$$= C_4^5 \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right)^1 + C_5^5 \left(\frac{4}{5}\right)^5$$

$$\div 0.737$$

7 a $P(\text{wins first 3 prizes})$

$$= P(WWW)$$

$$= \frac{4}{500} \times \frac{3}{499} \times \frac{2}{498}$$

$$\div 1.93 \times 10^{-7}$$

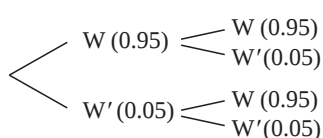
b $P(\text{wins at least one of the 3 prizes})$

$$= 1 - P(\text{wins none of them})$$

$$= 1 - P(W'W'W')$$

$$= 1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498}$$

$$\div 0.0239$$

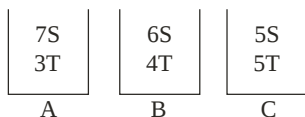
8

$$P(\text{works on at least one day})$$

$$= 0.95 \times 0.95 + 0.95 \times 0.05 + 0.05 \times 0.95$$

$$= 0.9975$$

9



$$P(A) = \frac{3}{6}, \quad P(B) = \frac{2}{6}, \quad P(C) = \frac{1}{6}$$

$$\text{a} \quad P(S) = P(A \text{ and } S \text{ or } B \text{ and } S \text{ or } C \text{ and } S)$$

$$\begin{aligned}
 &= \frac{3}{6} \times \frac{7}{10} + \frac{2}{6} \times \frac{6}{10} + \frac{1}{6} \times \frac{5}{10} \\
 &= \frac{38}{60} \\
 &= \frac{19}{30}
 \end{aligned}$$

$$\text{b} \quad P(B|S) = \frac{P(B \cap S)}{P(S)}$$

$$\begin{aligned}
 &= \frac{\frac{2}{6} \times \frac{6}{10}}{\frac{38}{60}} \\
 &= \frac{12}{38} \\
 &= \frac{6}{19}
 \end{aligned}$$

REVIEW SET 19B

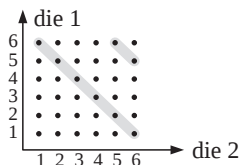
1

BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGBG, BGGB, GGBB, GBBG, GBGB, BGGG, GBGG, GGBG, GGGB, GGGG

$$P(2B \text{ and } 2G)$$

$$\begin{aligned}
 &= \frac{6}{16} \leftarrow 6 \text{ have } 2B \text{ and } 2G \\
 &= \frac{3}{8}
 \end{aligned}$$

2

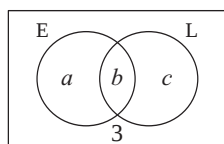


There are 36 possible outcomes.

$$\begin{aligned}
 \text{a} \quad &P(\text{sum of } 7 \text{ or } 11) \\
 &= \frac{8}{36} \quad \{\text{those shaded}\} \\
 &= \frac{2}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad &P(\text{sum of at least } 8) \\
 &= \frac{1+2+3+4+5}{36} \\
 &= \frac{15}{36} \\
 &= \frac{5}{12}
 \end{aligned}$$

3



$$a + b + c = 37$$

$$a + b = 22$$

$$b + c = 25$$

$$\therefore 22 + c = 37 \quad \text{and} \quad a + 25 = 37$$

$$\therefore c = 15 \quad \text{and} \quad a = 12$$

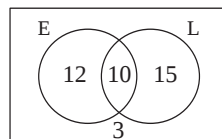
$$\text{Hence, } b = 22 - a = 10$$

$$\text{a} \quad P(E \text{ and } L)$$

$$\begin{aligned}
 &= \frac{10}{40} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\text{b} \quad P(\text{at least one})$$

$$\begin{aligned}
 &= \frac{12+10+15}{40} \\
 &= \frac{37}{40}
 \end{aligned}$$



$$\text{c} \quad P(E|L) = \frac{10}{15+10} = \frac{10}{25} = \frac{2}{5}$$

4

Multiples of 6 are: 6, 12, 18, 24, ..., 96 $\leftarrow 6 \times 16$ i.e., 16 of them

Multiples of 8 are: 8, 16, 24, 32, ..., 88, 96 $\leftarrow 8 \times 12$ i.e., 12 of them

Multiples of 6 and 8 are: 24, 48, 72, 96 i.e., 4 of them

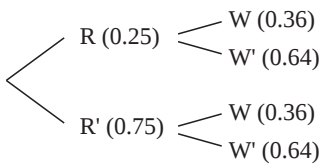
The integers could be 1, 2, 3, 4, ..., 99 {between 0 and 100} i.e., 99 of them

$$\therefore P(M_6 \text{ or } M_8) = P(M_6) + P(M_8) - P(M_6 \text{ and } M_8)$$

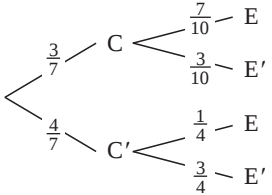
$$\begin{aligned}
 &= \frac{16}{99} + \frac{12}{99} - \frac{4}{99} \\
 &= \frac{24}{99} \\
 &= \frac{8}{33}
 \end{aligned}$$

- 5 a** $P(\text{both blue})$
 $= P(BB)$
 $= \frac{5}{12} \times \frac{4}{11}$
 $= \frac{5}{33}$
- b** $P(\text{both same colour})$
 $= P(BB \text{ or } RR \text{ or } YY)$
 $= \frac{5}{12} \times \frac{4}{11} + \frac{3}{12} \times \frac{2}{11} + \frac{4}{12} \times \frac{3}{11}$
 $= \frac{19}{66}$
- c** $P(\text{at least one R})$
 $= 1 - P(\text{no reds})$
 $= 1 - P(R'R')$
 $= 1 - \frac{9}{12} \times \frac{8}{11}$
 $= 1 - \frac{6}{11}$
 $= \frac{5}{11}$
- d** $P(\text{exactly one Y})$
 $= P(YY' \text{ or } Y'Y)$
 $= \frac{4}{12} \times \frac{8}{11} + \frac{8}{12} \times \frac{4}{11}$
 $= \frac{16}{33}$

- 6 a** Two events are independent if the occurrence of one does not influence the occurrence of the other. For A and B independent, $P(A) \times P(B) = P(A \text{ and } B)$
- b** Two events, A and B, are disjoint if they have no common outcomes, i.e., $P(A \text{ and } B) = 0$ and so $P(A \text{ or } B) = P(A) + P(B)$

- 7**
- 
- a** $P(W \text{ and } R)$
 $= 0.25 \times 0.36$
 $= 0.09$
- b** $P(W \text{ or } R)$
 $= P(W) + P(R) - P(W \text{ and } R)$
 $= 0.36 + 0.25 - 0.09$
 $= 0.52$
or $P(W \text{ or } R) = 1 - P(W'R')$
 $= 1 - 0.64 \times 0.75$
 $= 0.52$

- 8** $P(A) = 0.1$, $P(B) = 0.2$, $P(C) = 0.3$ $\therefore P(\text{group solves it}) = P(\text{at least one solves it})$
 $= 1 - P(\text{no-one solves it})$
 $= 1 - P(A' \text{ and } B' \text{ and } C')$
 $= 1 - (0.9 \times 0.8 \times 0.7)$
 $= 0.496$

- 9**
- 
- a** $P(E) = \frac{3}{7} \times \frac{7}{10} + \frac{4}{7} \times \frac{1}{4}$
 $= \frac{3}{10} + \frac{1}{7}$
 $= \frac{31}{70}$
- b** $P(C|E) = \frac{P(C \text{ and } E)}{P(E)}$
 $= \frac{\frac{3}{7} \times \frac{7}{10}}{\frac{31}{70}}$
 $= \frac{21}{31}$

- 10 a** $\left(\frac{3}{5} + \frac{2}{5}\right)^4 = \underbrace{\left(\frac{3}{5}\right)^4}_{4B} + 4 \underbrace{\left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)}_{\substack{3B \\ 1B'}} + 6 \underbrace{\left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2}_{\substack{2B \\ 2B'}} + 4 \underbrace{\left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^3}_{\substack{1B \\ 3B'}} + \underbrace{\left(\frac{2}{5}\right)^4}_{4B'}$ $P(B) = \frac{12}{20}$
 $= \frac{3}{5}$
 $\therefore P(B') = \frac{2}{5}$

- b i** $P(2 \text{ Blue inks})$
 $= P(2B \text{ and } 2B')$
 $= 6 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2$
 $= \frac{6 \times 9 \times 4}{5^4}$
 $= \frac{144}{625}$
- ii** $P(\text{at most 2 Blue inks})$
 $= P(2B \text{ and } 2B' \text{ or } 1B \text{ and } 3B' \text{ or } 4B')$
 $= 6 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2 + 4 \left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$
 $= \frac{6 \times 9 \times 4 + 4 \times 3 \times 8 + 16}{625}$
 $= \frac{328}{625}$

Chapter 20

INTRODUCTION TO CALCULUS

EXERCISE 20A.1

- 1 a** 67 beats/minute means that every minute Ozair's heart beats 67 times.
- b** Number of beats $= 67 \times 60 = 4020$ beats
- 2 a** Number of words $= 14 \times 380$
 $= 5320$ words
- $\therefore \text{rate} = \frac{8}{5320}$
 $\div 0.0015 \text{ errors/word}$
- b** errors/100 words $= \frac{8}{53.2}$
 $= 0.15$
- 3** Paul's hourly rate $= \frac{\$148.20}{12}$
 $= \$12.35/\text{hour}$
- Marita's hourly rate $= \frac{\$157.95}{13}$
 $= \$12.15/\text{hour}$
- $\therefore$ Paul's rate is better
- 4 a** Tyre wear $= 8.0 - 2.3$
 $= 5.7 \text{ mm}$
- $\therefore \text{wearing rate} = \frac{5.7}{32178} \text{ mm/km}$
 $= 1.77 \times 10^{-4} \text{ mm/km}$
- b** wearing rate $= \frac{5.7}{3.2178}$
 $= 1.77 \text{ mm} / 10\,000 \text{ km}$
- 5 a** The time for 11:43 am to 12:39 pm is 56 mins.
- $\therefore \text{speed} = \frac{71}{\frac{56}{60}} \text{ km/hour}$
 $\div 76.07 \text{ kmph}$
- b** speed $= 76.07 \times \left(\frac{1000}{3600}\right) \text{ m/s}$
 $= 21.1 \text{ m/s}$

EXERCISE 20A.2

- 1 a** Average speed from Tailem Bend to Nhill
- $$= \frac{\text{distance travelled}}{\text{time taken}}$$
- $$= \frac{(324 - 98) \text{ km}}{\left(\frac{204 - 63}{60}\right) \text{ h}}$$
- $$= \frac{226 \times 60}{141}$$
- $$= 96.17 \text{ kmph}$$
- b** Average speed from Horsham to Melbourne
- $$= \frac{729 - 431 \text{ km}}{\left(\frac{534 - 261}{60}\right) \text{ h}}$$
- $$= \frac{298 \times 60}{273}$$
- $$= 65.5 \text{ kmph}$$
- 2 a** 800 m to the newsagency
- b** $m = \frac{y\text{-step}}{x\text{-step}} = \frac{500 - 0}{4 - 0} = 125$

c Average speed = $\frac{(500 - 0) \text{ m}}{(4 - 0) \text{ min}} = 125 \text{ m/minute}$

d The *slope* represents the *average walking speed*.

e Paul stayed between the 12th and 20th minutes i.e., 8 minutes.

f Average speed = $\frac{(800 - 0) \text{ m}}{(32 - 20) \text{ min}}$ {return journey is from (20, 800) to (32, 0)}
= 66.7 m/minute

g Total distance travelled = 1600 m
= 1.6 km (to the shop and return)

3 Average water loss = $\frac{\text{amount of water lost}}{\text{time taken}}$
= $\frac{53.8 - 48.2}{11}$
= 0.509 million kL/day
= 509 000 kL/day

4 a First quarter: Used 106.8 kL Number of days = 31 + 28 + 31

$$\therefore \text{rate} = \frac{106.8}{90}$$
$$= 1.19 \text{ kL/day}$$

b First six months: Used $\begin{array}{r} 106.8 \\ + 79.4 \\ \hline 186.2 \end{array}$ kL Number of days = 90 + 30 + 31 + 30
= 181 days

$$\therefore \text{rate} = \frac{186.2}{181}$$
$$= 1.03 \text{ kL/day}$$

c The whole year: Used = 106.8 + 79.4 + 81.8 + 115.8
= 383.8 kL

$$\therefore \text{rate} = \frac{383.8}{365} \text{ kg}$$
$$= 1.05 \text{ kL/day}$$

EXERCISE 20A.3

1 a In the first 4 seconds: average speed = $\frac{\text{distance travelled}}{\text{time taken}}$
= $\frac{(2.4 - 2) \text{ m}}{(4 - 0) \text{ sec}}$
= 0.1 m/s

b In the last 4 seconds: average speed = $\frac{(6 - 2.4) \text{ m}}{(4 - 0) \text{ sec}}$
= 0.9 m/s

c In the 8 second period: average speed = $\frac{(6 - 2) \text{ m}}{(8 - 0) \text{ sec}}$
= 0.5 m/s

- 2 a i** Initially there are 35 beetles. After a 10 g dose there are 3 beetles.

$$\begin{aligned}\text{rate of change} &= \frac{\text{change in population}}{\text{change in dose}} \\ &= \frac{(35 - 3) \text{ beetles}}{(0 - 10) \text{ gm}} \\ &= -3.2 \text{ beetles/gm}\end{aligned}$$

i.e., rate of decrease = 3.2 beetles/gram

ii
$$\begin{aligned}\text{rate of change} &= \frac{(26 - 8) \text{ beetles}}{(4 - 8) \text{ beetles}} \\ &= -4.5 \text{ beetles/gm}\end{aligned}$$

i.e., rate of decrease = 4.5 beetles/gram

- b** A dose of 0 to 1 g has little or no effect.
A dose of 1 to 8 g has good or considerable effect (rapid decrease).
A dose of 8 to 14 g has an effect, but less rapid than the 1 to 8 rate.

EXERCISE 20B.1

1 a The tangent passes through (2, 3) and (0, 1) $\therefore$ slope of tangent $\div \frac{(3 - 1) \text{ m}}{(2 - 0) \text{ s}}$

$$\div \frac{2}{2} \text{ m/s}$$

$$\div 1 \text{ m/s}$$

b The tangent passes through (3.5, 6) and (2.2, 2) $\therefore$ slope of tangent $\div \frac{(6 - 2) \text{ km}}{(3 - 2.2) \text{ h}}$

$$\div \frac{4}{1.3} \text{ km/h}$$

$$\div 3 \text{ km/h}$$

c The tangent passes through (20, 2700) and (40, 3700) $\therefore$ slope of tangent

$$\div \frac{(3.7 - 2.7) \text{ thousands of \$}}{(40 - 20) \text{ items}}$$

$$\div 0.05 \text{ thousands of \$/item}$$

$$\div \$50/\text{item}$$

d The tangent passes through (0, 35) and (7, 0) $\therefore$ slope of tangent $\div \frac{(35 - 0) \text{ bats}}{(0 - 7) \text{ weeks}}$

$$\div -5 \text{ bats/week}$$

i.e., the population is decreasing at the rate of 5 bats/week after 5 weeks.

- 2 a** Originally (when $x = 0$) the volume was 8200 L.

- b** After 1 hour ($x = 1$) the volume was 3000 L.

- c** The tangent at (0, 8.2) passes through (1, 0)

$$\begin{aligned}\therefore \text{ slope of tangent} &= \frac{(8.2 - 0) \text{ kL}}{(0 - 1) \text{ h}} \\ &= -8.2 \text{ kL} \quad \text{i.e., loses 8200 L/hour}\end{aligned}$$

- d** After 1 hour, $x = 1$ and the tangent at (1, 3) passes through (0, 6)

$$\therefore \text{ slope of tangent} = \frac{(6 - 3) \text{ kL}}{(0 - 1) \text{ h}} = -3 \text{ kL/h} \quad \text{i.e., the rate of loss is 3000 L/hour}$$

EXERCISE 20B.2

- 1** Let M $(1.5 + h, (1.5 + h)^2)$ be a point on $y = x^2$ which is close to F $(1.5, 1.5^2)$.

$$\begin{aligned}\therefore \text{slope of MF} &= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{(1.5 + h)^2 - (1.5)^2}{(1.5 + h) - 1.5} \\ &= \frac{2.25 + h^2 + 3h - 2.25}{h} \\ &= \frac{h^2 + 3h}{h} \\ &= h + 3 \quad \{\text{as } h \neq 0\}\end{aligned}$$

As M approaches F, h approaches 0, $\therefore h + 3$ approaches 3, $\therefore$ slope of the tangent = 3

2 a

$$\begin{aligned}(x + h)^3 &= (x + h)^2(x + h) \\ &= (x^2 + 2xh + h^2)(x + h) \\ &= x^3 + 2x^2h + h^2x + hx^2 + 2xh^2 + h^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3\end{aligned}$$

- c** M $(1 + h, (1 + h)^3)$ is the point on $y = x^3$ which is close to F $(1, 1)$.

- e** As M approaches F, h approaches 0
 $\therefore$ slope MF approaches 3
 $\therefore$ slope of tangent = 3

b

$$\begin{aligned}(1 + h)^3 &= 1^3 + 3(1)^2h + 3(1)h^2 + h^3 \\ &= 1 + 3h + 3h^2 + h^3\end{aligned}$$

d slope of chord MF

$$\begin{aligned}&= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{(1 + h)^3 - 1}{(1 + h) - 1} \\ &= \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \\ &= \frac{3h + 3h^2 + h^3}{h} \\ &= 3 + 3h + h^2 \quad \{\text{as } h \neq 0\}\end{aligned}$$

3 b from **2 a** $(2 + h)^3 = 2^3 + 3(2)^2h + 3(2)h^2 + h^3$
 $= 8 + 12h + 6h^2 + h^3$

- c** M $(2 + h, (2 + h)^3)$ is the point on $y = x^3$ which is close to F $(2, 8)$.

d slope of chord MF

$$\begin{aligned}&= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{(2 + h)^3 - 2^3}{2 + h - 2} \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \frac{12h + 6h^2 + h^3}{h} \\ &= 12 + 6h + h^2 \quad \{\text{as } h \neq 0\}\end{aligned}$$

- e** as M approaches F,
 h approaches 0,
 slope approaches 12
 $\therefore$ slope of tangent = 12

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad \frac{1}{x+h} - \frac{1}{x} &= \frac{x(1) - 1(x+h)}{(x+h)x} \\
 &= \frac{x - x - h}{x(x+h)} \\
 &= -\frac{h}{x(x+h)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{M is } \left(2+h, \frac{1}{2+h}\right) \\
 \therefore y\text{-coordinate is } \frac{1}{2+h} \\
 \therefore \text{slope of MF} &= \frac{y\text{-step}}{x\text{-step}} \\
 &= \frac{\frac{1}{2+h} - \frac{1}{2}}{(2+h) - 2} \\
 &= \frac{-h}{2(2+h)} \quad \{\text{using } \mathbf{a} \text{ with } x = 2\} \\
 &= \frac{-h}{2h(2+h)} \\
 &= \frac{-1}{2(2+h)} \quad \{\text{as } h \neq 0\}
 \end{aligned}$$

c F is the point where $x = 2$. Now as h approaches 0, M approaches F, and the slope MF approximates the slope of the tangent at M.

But as h approaches 0, the slope MF approaches $-\frac{1}{4}$, $\therefore$ slope of tangent $= -\frac{1}{4}$

d At the point where $x = 3$, $y = \frac{1}{3}$.

Also, the point with x -coordinate $3+h$ has y -coordinate $\frac{1}{3+h}$.

Using the method in **b**, the slope of the line between these points is $-\frac{1}{3(3+h)}$, $h \neq 0$.

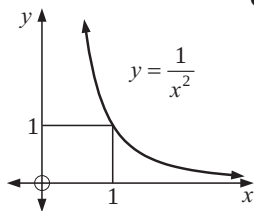
As h approaches 0, the slope of the line approximates the slope of the tangent when $x = 3$, and this is $-\frac{1}{9}$.

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \frac{1}{(x+h)^2} - \frac{1}{x^2} &= \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \\
 &= \frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2} \\
 &= \frac{-2xh - h^2}{x^2(x+h)^2}
 \end{aligned}$$

b When $x = 2$,

$$\frac{1}{(2+h)^2} - \frac{1}{4} = \frac{-4h - h^2}{4(2+h)^2}$$

c **d** Let M be close to F $\left(2, \frac{1}{2^2}\right)$ i.e., $\left(2+h, \frac{1}{(2+h)^2}\right)$



$$\begin{aligned}
 \therefore \text{slope MF} &= \frac{y\text{-step}}{x\text{-step}} = \frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{(2+h) - 2} \\
 &= \frac{-4h - h^2}{4(2+h)^2} \quad \{\text{using } \mathbf{b}\} \\
 &= \frac{-4h - h^2}{4h(2+h)^2} \\
 &= \frac{-4 - h}{4(2+h)^2} \quad \{\text{as } h \neq 0\}
 \end{aligned}$$

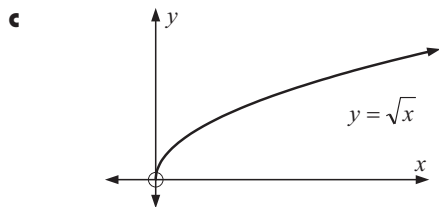
as h approaches 0, slope MF approaches $-\frac{4}{16}$

i.e., $-\frac{1}{4}$ $\therefore$ slope of tangent $= -\frac{1}{4}$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad & \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
 &= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
 &= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
 &= \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad h \neq 0
 \end{aligned}$$

b When $x = 9$,

$$\frac{\sqrt{9+h} - 3}{h} = \frac{1}{\sqrt{9+h} + 3} \quad \dots (*)$$



$$\begin{aligned}
 \mathbf{d} \quad \text{Slope MF} &= \frac{y\text{-step}}{x\text{-step}} \\
 &= \frac{\sqrt{9+h} - \sqrt{9}}{(9+h) - 9} \\
 &= \frac{\sqrt{9+h} - 3}{h} \\
 &= \frac{1}{\sqrt{9+h} + 3} \quad \{\text{using } *\}
 \end{aligned}$$

$\therefore$ as h approaches 0, slope MF approaches $\frac{1}{6}$, $\therefore$ slope of tangent $= \frac{1}{6}$

- e** $N = \sqrt{t}$ for $t \geq 4$ is the same graph as $y = \sqrt{x}$ for $x \geq 4$,
 $\therefore$ rate after 9 days is the same as the slope of the tangent in **d**, i.e., $\frac{1}{6}$
 $\therefore$ the population is increasing at a rate of $\frac{1}{6}$ (thousand insects/day) = 167 insects/day

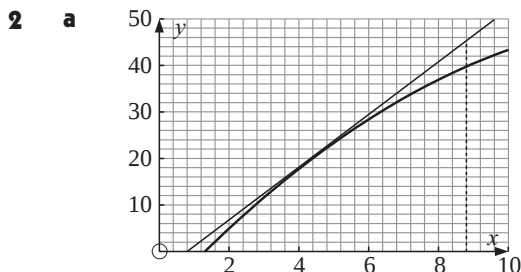
REVIEW SET 20

- 1 a** When $t = 1$, $d = 40$ and when $t = 4$, $d = 70$

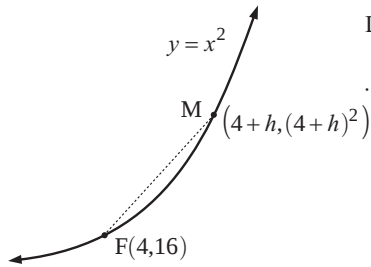
$$\begin{aligned}
 \therefore \text{average speed on } 1 \leq t \leq 4 \text{ is } & \frac{\text{distance travelled}}{\text{time taken}} \\
 &= \frac{70 - 40}{4 - 1} \\
 &= \frac{30}{3} \\
 &= 10 \text{ m/s}
 \end{aligned}$$

- b** When $t = 1$, $d = 40$ and when $t = 10$, $d = 81$

$$\therefore \text{average speed} = \frac{81 - 40}{10 - 1} = \frac{41}{9} \div 4.6 \text{ m/s}$$



$$\begin{aligned}
 \mathbf{b} \quad \text{slope} &= \frac{y\text{-step}}{x\text{-step}} \\
 &\div \frac{46 - 0}{8.8 - 0.8} \\
 &\div \frac{46}{8} \\
 &\div 5.8
 \end{aligned}$$

3Let $M(4+h, (4+h)^2)$ be a point close to $F(4, 16)$.

$$\begin{aligned}
 \therefore \text{slope of MF} &= \frac{(4+h)^2 - 16}{4+h-4} \\
 &= \frac{16+8h+h^2-16}{h} \\
 &= \frac{8h+h^2}{h} \\
 &= 8+h \quad (\text{as } h \neq 0)
 \end{aligned}$$

Now as M approaches F , $h \rightarrow 0$ and $8+h$ approaches 8 . $\therefore$ the tangent at F has slope 8 .**4**

$$\begin{aligned}
 \text{a} \quad f(x+h) &= (2(x+h)+3)^2 \\
 &= ((2x+2h)+3)^2 \\
 &= (2x+2h)^2 + 2(2x+2h)3 + 3^2 \\
 &= 4x^2 + 8xh + 4h^2 + 12x + 12h + 9
 \end{aligned}$$

b

$$\begin{aligned}
 \frac{f(x+h)-f(x)}{h} &= \frac{\cancel{4x^2} + 8xh + 4h^2 + \cancel{12x} + 12h + \cancel{9} - \cancel{4x^2} - \cancel{12x} - \cancel{9}}{h} \\
 &= \frac{8xh + 4h^2 + 12h}{h} \\
 &= 8x + 4h + 12 \quad \{\text{as } h \neq 0\}
 \end{aligned}$$

c $\frac{f(x+h)-f(x)}{h}$ is the slope of the secant (chord) AB .**d i** As $h \rightarrow 0$, $\frac{f(x+h)-f(x)}{h} = 8x + 4h + 12$ approaches $8x + 12$.**ii** As $h \rightarrow 0$, $\frac{f(1+h)-f(1)}{h} = 8 + 4h + 12$ approaches 20 .**e i** $8x + 12$ gives us the slope of the tangent at a point with x -coordinate x .**ii** 20 is the slope of the tangent at $x = 1$.**5**

$$\begin{aligned}
 \text{a Average speed on } 2 \leq t \leq 5 \text{ is } & \frac{s(5)-s(2)}{5-2} \text{ m/s} \\
 &= \frac{(25+20)-(4+8)}{3} \text{ m/s} \\
 &= \frac{33}{3} \text{ m/s} \\
 &= 11 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 \text{b Average speed on } 2 \leq t \leq 2+h \text{ is } & \frac{s(2+h)-s(2)}{2+h-2} \text{ m/s} \\
 &= \frac{(2+h)^2 + 4(2+h) - 12}{h} \text{ m/s} \\
 &= \frac{4 + 4h + h^2 + 8 + 4h - 12}{h} \text{ m/s} \\
 &= \frac{8h + h^2}{h} \text{ m/s} \\
 &= (8+h) \text{ m/s}
 \end{aligned}$$

c as $h \rightarrow 0$, $8+h \rightarrow 8$

$$\therefore \frac{s(2+h)-s(2)}{h} \rightarrow 8 \text{ m/s}$$

 8 m/s is the instantaneous velocity at $t = 2$ seconds.

Chapter 21

DIFFERENTIAL CALCULUS

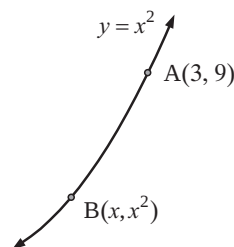
EXERCISE 21A

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \text{Slope AB} &= \frac{x^2 - 9}{x - 3} \\ &= \frac{(x+3)(\cancel{x-3})}{(\cancel{x-3})} \\ &= x+3 \quad (\text{provided } x \neq 3) \end{aligned}$$

$$\therefore \lim_{x \rightarrow 3} (\text{slope AB}) = 6 \quad \dots (1)$$

But, as $B \rightarrow A$, i.e., $x \rightarrow 3$ and $\lim_{x \rightarrow 3} (\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent = 6

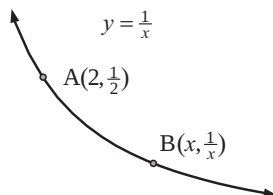


$$\begin{aligned} \mathbf{b} \quad \text{Slope AB} &= \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \\ &= \frac{2 - x}{2x(x - 2)} \\ &= \frac{-1(\cancel{x-2})}{2x(\cancel{x-2})} \\ &= -\frac{1}{2x} \quad (\text{provided } x \neq 2) \end{aligned}$$

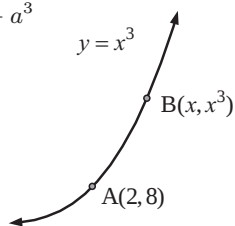
$$\therefore \lim_{x \rightarrow 2} (\text{slope AB}) = -\frac{1}{4}$$

But, as $B \rightarrow A$, i.e., $x \rightarrow 2$ and $\lim_{x \rightarrow 2} (\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent = $-\frac{1}{4}$



$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad (x-a)(x^2+ax+a^2) \\ = x^3+ax^2+a^2x-ax^2 \\ = x^3-a^3 \end{aligned}$$



$$\begin{aligned} \mathbf{b} \quad \text{Slope AB} &= \frac{x^3 - 8}{x - 2} \\ &= \frac{(x-2)(x^2+2x+4)}{(x-2)} \\ &= x^2+2x+4 \quad (\text{provided } x \neq 2) \end{aligned}$$

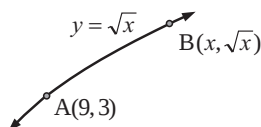
$$\therefore \lim_{x \rightarrow 2} (\text{slope of AB}) = 2^2 + 4 + 4 = 12$$

But, as $B \rightarrow A$, i.e., $x \rightarrow 2$,

$\lim_{x \rightarrow 2} (\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent = 12

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \frac{\sqrt{x} - \sqrt{a}}{x - a} \\ = \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a})} \\ = \frac{1}{\sqrt{x} + \sqrt{a}} \end{aligned}$$



$$\begin{aligned} \mathbf{b} \quad \text{Slope AB} &= \frac{\sqrt{x} - 3}{x - 9} = \frac{1}{\sqrt{x} + 3} \\ &\quad \{\text{from a, provided } x \neq 9\} \end{aligned}$$

$$\therefore \lim_{x \rightarrow 9} (\text{slope AB}) = \frac{1}{6}$$

But, as $B \rightarrow A$, i.e., $x \rightarrow 9$,

$\lim_{x \rightarrow 9} (\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent = $\frac{1}{6}$

EXERCISE 21B

1 a $f(x) = 1 - x^2$

$$\therefore f(2) = 1 - 2^2 = -3$$

$$\begin{aligned} f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(1 - x^2) - (-3)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{4 - x^2}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{-(x+2)(\cancel{x-2})}{\cancel{x-2}} \\ &= \lim_{x \rightarrow 2} -(x+2) \quad \{\text{as } x \neq 2\} \\ &= -4 \end{aligned}$$

c $f(x) = 5 - 2x^2$ at $x = 3$

$$\therefore f(3) = 5 - 2(3)^2 = -13$$

$$\begin{aligned} f'(3) &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(5 - 2x^2) - (-13)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{18 - 2x^2}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{-2(x^2 - 9)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{-2(x+3)(\cancel{x-3})}{\cancel{x-3}} \\ &= \lim_{x \rightarrow 3} -2(x+3) \quad \{\text{as } x \neq 3\} \\ &= -2(6) \\ &= -12 \end{aligned}$$

2 a $f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$

$$\begin{aligned} &= \lim_{x \rightarrow 2} \frac{\frac{4}{x} - \frac{4}{2}}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{8 - 4x}{2x(x - 2)} \\ &= \lim_{x \rightarrow 2} \frac{-4(\cancel{x-2})}{2x(\cancel{x-2})} \\ &= \lim_{x \rightarrow 2} \frac{-4}{2x} \quad \{\text{as } x \neq 2\} \\ &= -\frac{4}{4} \\ &= -1 \end{aligned}$$

b $f(x) = 2x^2 + 5x$ at $x = -1$

$$f(-1) = 2(-1)^2 + 5(-1) = -3$$

$$\begin{aligned} f'(-1) &= \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x \rightarrow -1} \frac{(2x^2 + 5x) - (-3)}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{2x^2 + 5x + 3}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{(2x+3)(\cancel{x+1})}{\cancel{x+1}} \\ &= \lim_{x \rightarrow -1} 2x + 3 \quad \{\text{as } x \neq -1\} \\ &= 1 \end{aligned}$$

d $f(x) = 3x + 5$ at $x = -2$

$$\therefore f(-2) = 3(-2) + 5 = -1$$

$$\begin{aligned} f'(-2) &= \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)} \quad \text{where} \\ &= \lim_{x \rightarrow -2} \frac{(3x + 5) - (-1)}{x + 2} \\ &= \lim_{x \rightarrow -2} \frac{3x + 6}{x + 2} \\ &= \lim_{x \rightarrow -2} \frac{3(\cancel{x+2})}{\cancel{x+2}} \\ &= \lim_{x \rightarrow -2} 3 \quad \{\text{as } x \neq -2\} \\ &= 3 \end{aligned}$$

b $f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)}$

$$\begin{aligned} &= \lim_{x \rightarrow -2} \frac{\frac{3}{x} - \frac{3}{2}}{x + 2} \\ &= \lim_{x \rightarrow -2} \frac{-6 - 3x}{2x(x + 2)} \\ &= \lim_{x \rightarrow -2} \frac{-3(\cancel{x+2})}{2x(\cancel{x+2})} \\ &= \lim_{x \rightarrow -2} \frac{-3}{2x} \quad \{\text{as } x \neq -2\} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f'(4) &= \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{\frac{1}{x^2} - \frac{1}{16}}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{16 - x^2}{16x^2(x - 4)} \\
 &= \lim_{x \rightarrow 4} \frac{-(x+4)\cancel{(x-4)}}{16x^2\cancel{(x-4)}} \\
 &= \lim_{x \rightarrow 4} \frac{-(x+4)}{16x^2} \quad \{\text{as } x \neq 4\} \\
 &= \frac{-8}{256} \\
 &= -\frac{1}{32}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{\frac{4x}{x-3} - \left(-\frac{8}{1}\right)}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{4x + 8(x-3)}{1(x-2)(x-3)} \\
 &= \lim_{x \rightarrow 2} \frac{12\cancel{(x-2)}}{\cancel{(x-2)}(x-3)} \\
 &= \lim_{x \rightarrow 2} \frac{12}{x-3} \quad \{\text{as } x \neq 2\} \\
 &= \frac{12}{-1} \\
 &= -12
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad f'(5) &= \lim_{x \rightarrow 5} \frac{f(x) - f(5)}{(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{\frac{4x+1}{x-2} - \frac{7}{1}}{x-5} \\
 &= \lim_{x \rightarrow 5} \frac{4x+1-7(x-2)}{(x-2)(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{4x+1-7x+14}{(x-2)(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{-3x+15}{(x-2)(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{-3\cancel{(x-5)}}{(x-2)\cancel{(x-5)}} \\
 &= \lim_{x \rightarrow 5} \frac{-3}{x-2} \quad \{\text{as } x \neq 5\} \\
 &= -\frac{3}{3} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad f'(-4) &= \lim_{x \rightarrow -4} \frac{f(x) - f(-4)}{x - (-4)} \\
 &= \lim_{x \rightarrow -4} \frac{\frac{3x}{x^2+1} - \left(-\frac{12}{17}\right)}{x+4} \\
 &= \lim_{x \rightarrow -4} \frac{51x+12(x^2+1)}{17(x^2+1)(x+4)} \\
 &= \lim_{x \rightarrow -4} \frac{12x^2+51x+12}{17(x+4)(x^2+1)} \\
 &= \lim_{x \rightarrow -4} \frac{\cancel{(x+4)}(12x+3)}{17(x^2+1)\cancel{(x+4)}} \\
 &= \lim_{x \rightarrow -4} \frac{12x+3}{17(x^2+1)} \quad \{x \neq -4\} \\
 &= -\frac{45}{17 \times 17} \\
 &= -\frac{45}{289}
 \end{aligned}$$

$$3 \quad \text{a} \quad f(x) = \sqrt{x} \quad \text{and} \quad f(4) = \sqrt{4} = 2$$

$$\begin{aligned}
 f'(4) &= \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{\cancel{\sqrt{x}-2}}{(\sqrt{x}+2)\cancel{(\sqrt{x}-2)}} \\
 &= \lim_{x \rightarrow 4} \frac{1}{(\sqrt{x}+2)} \quad \{\text{as } x \neq 4\} \\
 &= \frac{1}{2+2} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\text{b} \quad f(x) = \sqrt{x} \quad \text{and} \quad f\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\begin{aligned}
 f'\left(\frac{1}{4}\right) &= \lim_{x \rightarrow \frac{1}{4}} \frac{f(x) - f\left(\frac{1}{4}\right)}{x - \frac{1}{4}} \\
 &= \lim_{x \rightarrow \frac{1}{4}} \frac{\sqrt{x} - \frac{1}{2}}{x - \frac{1}{4}} \\
 &= \lim_{x \rightarrow \frac{1}{4}} \frac{\cancel{\sqrt{x}-\frac{1}{2}}}{\left(\sqrt{x}+\frac{1}{2}\right)\cancel{\left(\sqrt{x}-\frac{1}{2}\right)}} \\
 &= \lim_{x \rightarrow \frac{1}{4}} \frac{1}{\sqrt{x}+\frac{1}{2}} \quad \{\text{as } x \neq \frac{1}{4}\} \\
 &= \frac{1}{\frac{1}{2}+\frac{1}{2}} \\
 &= 1
 \end{aligned}$$

$$\text{c} \quad f(x) = \frac{2}{\sqrt{x}} \quad \text{and} \quad f(9) = \frac{2}{3}$$

$$\begin{aligned} f'(9) &= \lim_{x \rightarrow 9} \frac{f(x) - f(9)}{x - 9} \\ &= \lim_{x \rightarrow 9} \frac{\frac{2}{\sqrt{x}} - \frac{2}{3}}{x - 9} \\ &= \lim_{x \rightarrow 9} \frac{2(3 - \sqrt{x})}{3\sqrt{x}(x - 9)} \\ &= \lim_{x \rightarrow 9} \frac{-2(\cancel{\sqrt{x} - 3})}{3\sqrt{x}(\sqrt{x} + 3)(\cancel{\sqrt{x} - 3})} \\ &= \lim_{x \rightarrow 9} \frac{-2}{3\sqrt{x}(\sqrt{x} + 3)} \quad \{x \neq 9\} \\ &= \frac{-2}{3(3)(6)} \\ &= -\frac{1}{27} \end{aligned}$$

$$\text{d} \quad f(x) = \sqrt{x-6} \quad \text{and} \quad f(10) = 2$$

$$\begin{aligned} f'(10) &= \lim_{x \rightarrow 10} \frac{f(x) - f(10)}{x - 10} \\ &= \lim_{x \rightarrow 10} \frac{\sqrt{x-6} - 2}{x - 10} \\ &= \lim_{x \rightarrow 10} \frac{(\sqrt{x-6} - 2)(\sqrt{x-6} + 2)}{(x - 10)(\sqrt{x-6} + 2)} \\ &= \lim_{x \rightarrow 10} \frac{x - 6 - 4}{(x - 10)(\sqrt{x-6} + 2)} \\ &= \lim_{x \rightarrow 10} \frac{\cancel{1} \cancel{x} \cancel{10}}{(\cancel{x} - 10)(\sqrt{x-6} + 2)} \\ &= \lim_{x \rightarrow 10} \frac{1}{\sqrt{x-6} + 2} \quad \{\text{as } x \neq 10\} \\ &= \frac{1}{2 + 2} \\ &= \frac{1}{4} \end{aligned}$$

$$\text{4 a} \quad f(x) = x^2 + 3x - 4 \quad \text{at} \quad x = 3$$

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \quad \text{where} \quad f(3) = 3^2 + 3(3) - 4 = 14 \\ &= \lim_{h \rightarrow 0} \frac{[(3+h)^2 + 3(3+h) - 4] - 14}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 + 9 + 3h - 4 - 14}{h} \\ &= \lim_{h \rightarrow 0} \frac{9h + h^2}{h} \\ &= \lim_{h \rightarrow 0} 9 + h \quad \{\text{as } h \neq 0\} \\ &= 9 \end{aligned}$$

$$\text{b} \quad f(x) = 5 - 2x - 3x^2 \quad \text{at} \quad x = -2$$

$$\begin{aligned} f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} \quad \text{where} \quad f(-2) = 5 - 2(-2) - 3(4) = -3 \\ &= \lim_{h \rightarrow 0} \frac{[5 - 2(-2+h) - 3(-2+h)^2] - (-3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5 + 4 - 2h - 12 + 12h - 3h^2 + 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{10h - 3h^2}{h} \\ &= \lim_{h \rightarrow 0} 10 - 3h \quad \{\text{as } h \neq 0\} \\ &= 10 \end{aligned}$$

$$\mathbf{c} \quad f(x) = \frac{1}{2x-1}$$

$$\therefore f(-2) = \frac{1}{2(-2)-1} = -\frac{1}{5}$$

$$\begin{aligned} f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{2(-2+h)-1} - \left(-\frac{1}{5}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{2h-5} + \frac{1}{5}}{h} \\ &= \lim_{h \rightarrow 0} \frac{5 + 1(2h-5)}{5h(2h-5)} \\ &= \lim_{h \rightarrow 0} \frac{2h}{5h(2h-5)} \\ &= \lim_{h \rightarrow 0} \frac{2}{5(2h-5)} \quad \{\text{as } h \neq 0\} \\ &= -\frac{2}{25} \end{aligned}$$

$$\mathbf{d} \quad f(x) = \frac{1}{x^2}$$

$$\therefore f(3) = \frac{1}{3^2} = \frac{1}{9}$$

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(3+h)^2} - \frac{1}{9}}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 - (3+h)^2}{9h(3+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{9 - 9 - 6h - h^2}{9h(3+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-h(6+h)}{9h(3+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-(6+h)}{9(3+h)^2} \quad \{\text{as } h \neq 0\} \\ &= \frac{-6}{81} \\ &= -\frac{2}{27} \end{aligned}$$

$$\mathbf{e} \quad f(x) = \sqrt{x}$$

$$\therefore f(4) = \sqrt{4} = 2$$

$$\begin{aligned} f'(4) &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \left(\frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right) \\ &= \lim_{h \rightarrow 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\ &= \lim_{h \rightarrow 0} \frac{1\mathcal{K}}{\mathcal{K}(\sqrt{4+h}+2)} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h}+2} \quad \{\text{as } h \neq 0\} \\ &= \frac{1}{4} \end{aligned}$$

$$\mathbf{f} \quad f(x) = \frac{1}{\sqrt{x}}$$

$$\therefore f(1) = \frac{1}{\sqrt{1}} = 1$$

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{1+h}} - \frac{1}{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - \sqrt{1+h}}{h\sqrt{1+h}} \\ &= \lim_{h \rightarrow 0} \frac{(1 - \sqrt{1+h})}{h\sqrt{1+h}} \left(\frac{1 + \sqrt{1+h}}{1 + \sqrt{1+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1 - (1+h)}{h(\sqrt{1+h})(1 + \sqrt{1+h})} \\ &= \lim_{h \rightarrow 0} \frac{-\mathcal{K}^1}{\mathcal{K}\sqrt{1+h}(1 + \sqrt{1+h})} \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{1+h}(1 + \sqrt{1+h})} \quad \{h \neq 0\} \\ &= \frac{-1}{1(1+1)} \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad f(x) = x^3, \quad \therefore f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \quad \text{where } f(2) = 2^3 = 8 \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} \\
 &= \lim_{h \rightarrow 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} 12 + 6h + h^2 \quad \{\text{as } h \neq 0\} \\
 &= 12
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) = x^4, \quad \therefore f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \quad \text{where } f(3) = 3^4 = 81 \\
 &= \lim_{h \rightarrow 0} \frac{(3+h)^4 - 3^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{81 + 108h + 54h^2 + 12h^3 + h^4 - 81}{h} \\
 &= \lim_{h \rightarrow 0} \frac{108h + 54h^2 + 12h^3 + h^4}{h} \\
 &= \lim_{h \rightarrow 0} 108 + 54h + 12h^2 + h^3 \quad \{\text{as } h \neq 0\} \\
 &= 108
 \end{aligned}$$

EXERCISE 21C

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad f(x) &= x \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h} \\
 &= \lim_{h \rightarrow 0} 1 \quad \{\text{as } h \neq 0\} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) &= 5 \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5 - 5}{h} \\
 &= \lim_{h \rightarrow 0} \frac{0}{h} \\
 &= \lim_{h \rightarrow 0} 0 \quad \{\text{as } h \neq 0\} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad f(x) = x^3, \quad \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 \quad \{\text{as } h \neq 0\} \\
 &= 3x^2
 \end{aligned}$$

$$\mathbf{d} \quad f(x) = x^4$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \rightarrow 0} 4x^3 + 6x^2h + 4xh^2 + h^3 \quad (\text{as } h \neq 0) \\ &= 4x^3 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad f(x) = 2x + 5$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2(x+h) + 5) - (2x + 5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x + 2h + 5 - 2x - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h} \\ &= \lim_{h \rightarrow 0} 2 \quad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

$$\mathbf{b} \quad f(x) = x^2 - 3x$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} \\ &= \lim_{h \rightarrow 0} 2x + h - 3 \quad \{\text{as } h \neq 0\} \\ &= 2x - 3 \end{aligned}$$

$$\mathbf{c} \quad f(x) = x^3 - 2x^2 + 3$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 2(x+h)^2 + 3] - [x^3 - 2x^2 + 3]}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 + 3) - (x^3 - 2x^2 + 3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h} \\ &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 - 4x - 2h \quad \{\text{as } h \neq 0\} \\ &= 3x^2 - 4x \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad f(x) = \frac{1}{x+2},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)+2} - \frac{1}{x+2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+2) - (x+h+2)}{h(x+h+2)(x+2)} \\ &= \lim_{h \rightarrow 0} \frac{-\cancel{h}^1}{\cancel{h}(x+2)(x+h+2)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+2)(x+h+2)} \quad \{\text{as } h \neq 0\} \\ &= \frac{-1}{(x+2)^2} \end{aligned}$$

$$\mathbf{b} \quad f(x) = \frac{1}{2x-1},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{2(x+h)-1} - \frac{1}{2x-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2x-1) - [2(x+h)-1]}{h[2(x+h)-1](2x-1)} \\ &= \lim_{h \rightarrow 0} \frac{2x-1-2x-2h+1}{h[2(x+h)-1](2x-1)} \\ &= \lim_{h \rightarrow 0} \frac{-2\cancel{h}}{\cancel{h}[2(x+h)-1](2x-1)} \\ &= \lim_{h \rightarrow 0} \frac{-2}{[2(x+h)-1](2x-1)} \quad \{\text{as } h \neq 0\} \\ &= \frac{-2}{(2x-1)^2} \end{aligned}$$

$$\mathbf{c} \quad f(x) = \frac{1}{x^2},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{[x+(x+h)][x-(x+h)]}{hx^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{(2x+h)(-\cancel{h})}{\cancel{h}x^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-(2x+h)}{x^2(x+h)^2} \quad \{\text{as } h \neq 0\} \\ &= \frac{-2x}{x^4} \\ &= -\frac{2}{x^3} \end{aligned}$$

$$\mathbf{d} \quad f(x) = \frac{1}{x^3},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 - (x+h)^3}{hx^3(x+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{x^3 - x^3 - 3x^2h - 3xh^2 - h^3}{hx^3(x+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{-3x^2h - 3xh^2 - h^3}{hx^3(x+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{-3x^2 - 3xh - h^2}{x^3(x+h)^3} \quad \{\text{as } h \neq 0\} \\ &= \frac{-3x^2}{x^3 \times x^3} \\ &= -\frac{3}{x^4} \end{aligned}$$

$$\begin{aligned}
\mathbf{4} \quad \mathbf{a} \quad f(x) &= \sqrt{x+2}, \quad \therefore f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \\
&= \lim_{h \rightarrow 0} \frac{(\sqrt{x+2+h} - \sqrt{x+2})(\sqrt{x+2+h} + \sqrt{x+2})}{h(\sqrt{x+2+h} + \sqrt{x+2})} \\
&= \lim_{h \rightarrow 0} \frac{(x+2+h) - (x+2)}{h(\sqrt{x+2+h} + \sqrt{x+2})} \\
&= \lim_{h \rightarrow 0} \frac{\mathcal{K}^1}{h(\sqrt{x+2+h} + \sqrt{x+2})} \\
&= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+2+h} + \sqrt{x+2}} \quad \{\text{as } h \neq 0\} \\
&= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad f(x) &= \frac{1}{\sqrt{x}}, \quad \therefore f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h} \times \sqrt{x}} \\
&= \lim_{h \rightarrow 0} \frac{(\sqrt{x} - \sqrt{x+h})}{h\sqrt{x+h} \times \sqrt{x}} \left(\frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right) \\
&= \lim_{h \rightarrow 0} \frac{x - (x+h)}{h\sqrt{x+h} \times \sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\
&= \lim_{h \rightarrow 0} \frac{-\mathcal{K}^1}{h\sqrt{x+h} \times \sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\
&= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{x+h} \times \sqrt{x}(\sqrt{x} + \sqrt{x+h})} \quad \{\text{as } h \neq 0\} \\
&= \frac{-1}{\sqrt{x} \times \sqrt{x} \times 2\sqrt{x}} \\
&= \frac{-1}{2x\sqrt{x}}
\end{aligned}$$

c $f(x) = \sqrt{2x+1}$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \left(\frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)+1] - [2x+1]}{h(\sqrt{2(x+h)+1} + \sqrt{2x+1})} \\
 &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2(x+h)+1} + \sqrt{2x+1})} \\
 &= \lim_{h \rightarrow 0} \frac{2}{(\sqrt{2(x+h)+1} + \sqrt{2x+1})} \quad \{\text{as } h \neq 0\} \\
 &= \frac{2}{2\sqrt{2x+1}} \\
 &= \frac{1}{\sqrt{2x+1}}
 \end{aligned}$$

5

Function	Derivative	Function	Derivative	Function	Derivative
x^1	$1x^0 = 1$	x^{-1}	$-1x^{-2}$	$x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$
x^2	$2x^1 = 2x$	x^{-2}	$-2x^{-3}$	$x^{-\frac{1}{2}}$	$-\frac{1}{2}x^{-\frac{3}{2}}$
x^3	$3x^2$	x^{-3}	$-3x^{-4}$		
x^4	$4x^3$				

If $f(x) = x^n$, $f'(x) = nx^{n-1}$

EXERCISE 21D

1 a $f(x) = x^3$,

$\therefore f'(x) = 3x^2$

b $f(x) = 2x^3$,

$\therefore f'(x) = 3 \times 2x^2$
 $= 6x^2$

c $f(x) = 7x^2$,

$\therefore f'(x) = 2 \times 7x$
 $= 14x$

d $f(x) = x^2 + x$,

$\therefore f'(x) = 2x + 1$

e $f(x) = 4 - 2x^2$,

$\therefore f'(x) = 0 - 2 \times 2x$
 $= -4x$

f $f(x) = x^2 + 3x - 5$,

$\therefore f'(x) = 2x + 3 - 0$
 $= 2x + 3$

g $f(x) = x^3 + 3x^2 + 4x - 1$

$\therefore f'(x) = 3x^2 + 2(3x) + 4 - 0$
 $= 3x^2 + 6x + 4$

h $f(x) = 5x^4 - 6x^2$

$\therefore f'(x) = 4(5x^3) - 2(6x)$
 $= 20x^3 - 12x$

i $f(x) = \frac{3x-6}{x} = 3 - 6x^{-1}$

$\therefore f'(x) = 0 - (-1) \times 6x^{-2}$
 $= \frac{6}{x^2}$

j $f(x) = \frac{2x-3}{x^2} = \frac{2x}{x^2} - \frac{3}{x^2}$

$= 2x^{-1} - 3x^{-2}$
 $\therefore f'(x) = -2x^{-2} + 6x^{-3}$
 $= \frac{-2}{x^2} + \frac{6}{x^3}$

$$\mathbf{k} \quad f(x) = \frac{x^3 + 5}{x} = x^2 + 5x^{-1}$$

$$\begin{aligned}\therefore f'(x) &= 2x - 5x^{-2} \\ &= 2x - \frac{5}{x^2}\end{aligned}$$

$$\mathbf{l} \quad f(x) = \frac{x^3 + x - 3}{x} = x^2 + 1 - 3x^{-1}$$

$$\begin{aligned}\therefore f'(x) &= 2x + 0 + 3x^{-2} \\ &= 2x + \frac{3}{x^2}\end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad f(x) = \frac{1}{4}x^4$$

$$\begin{aligned}\therefore f'(x) &= \frac{1}{4} \times 4x^3 \\ &= x^3\end{aligned}$$

$$\mathbf{b} \quad f(x) = x + \frac{1}{x} = x + x^{-1}$$

$$\begin{aligned}\therefore f'(x) &= 1 - x^{-2} \\ &= 1 - \frac{1}{x^2}\end{aligned}$$

$$\mathbf{c} \quad f(x) = \frac{x+1}{x} = 1 + x^{-1}$$

$$\begin{aligned}\therefore f'(x) &= 0 - x^{-2} \\ &= -\frac{1}{x^2}\end{aligned}$$

$$\mathbf{d} \quad f(x) = \frac{x^2 + 5}{x^3} = \frac{x^2}{x^3} + \frac{5}{x^3} = x^{-1} + 5x^{-3}$$

$$\begin{aligned}\therefore f'(x) &= -x^{-2} - 15x^{-4} \\ &= -\frac{1}{x^2} - \frac{15}{x^4}\end{aligned}$$

$$\mathbf{e} \quad f(x) = (x+1)(x-2)$$

$$= x^2 - x - 2$$

$$\therefore f'(x) = 2x - 1$$

$$\mathbf{f} \quad f(x) = \frac{1}{x^2} + 6\sqrt{x} = x^{-2} + 6x^{\frac{1}{2}}$$

$$\begin{aligned}\therefore f'(x) &= -2x^{-3} + 3x^{-\frac{1}{2}} \\ &= -\frac{2}{x^3} + \frac{3}{\sqrt{x}}\end{aligned}$$

$$\mathbf{g} \quad f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$\therefore f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = \frac{-1}{2x\sqrt{x}}$$

$$\mathbf{h} \quad f(x) = (2x-1)^2 = 4x^2 - 4x + 1$$

$$\therefore f'(x) = 8x - 4$$

$$\mathbf{i} \quad f(x) = (x+2)^3$$

$$= x^3 + 3x^2(2) + 3x(2^2) + 2^3$$

$$= x^3 + 6x^2 + 12x + 8$$

$$\therefore f'(x) = 3x^2 + 12x + 12$$

$$\mathbf{3} \quad \mathbf{a} \quad y = 2x^3 - 7x^2 - 1$$

$$\therefore \frac{dy}{dx} = 6x^2 - 14x$$

$$\mathbf{b} \quad y = \pi x^2$$

$$\therefore \frac{dy}{dx} = 2\pi x$$

$$\mathbf{c} \quad y = \frac{1}{5x^2} = \frac{1}{5}x^{-2}$$

$$\therefore \frac{dy}{dx} = -\frac{2}{5}x^{-3} = \frac{-2}{5x^3}$$

$$\mathbf{d} \quad y = 100x$$

$$\therefore \frac{dy}{dx} = 100$$

$$\mathbf{e} \quad y = 10(x+1)$$

$$= 10x + 10$$

$$\therefore \frac{dy}{dx} = 10$$

$$\mathbf{f} \quad y = 4\pi x^3$$

$$\therefore \frac{dy}{dx} = 12\pi x^2$$

$$\mathbf{4} \quad \mathbf{a} \quad \frac{d}{dx}(6x+2)$$

$$= 6$$

$$\mathbf{b} \quad \frac{d}{dx}(x\sqrt{x})$$

$$= \frac{d}{dx}(x^{\frac{3}{2}})$$

$$= \frac{3}{2}x^{\frac{1}{2}}$$

$$= \frac{3}{2}\sqrt{x}$$

$$\mathbf{c} \quad \frac{d}{dx}(5-x)^2$$

$$= \frac{d}{dx}(25 - 10x + x^2)$$

$$= -10 + 2x$$

$$= 2x - 10$$

$$\mathbf{d} \quad \frac{d}{dx} \left(\frac{6x^2 - 9x^4}{3x} \right)$$

$$= \frac{d}{dx} (2x - 3x^3)$$

$$= 2 - 9x^2$$

$$\mathbf{e} \quad \frac{d}{dx} \left(4x - \frac{1}{4x} \right)$$

$$= \frac{d}{dx} \left(4x - \frac{1}{4}x^{-1} \right)$$

$$= 4 + \frac{1}{4}x^{-2}$$

$$= 4 + \frac{1}{4x^2}$$

$$\mathbf{f} \quad \frac{d}{dx} (x(x+1)(2x-5))$$

$$= \frac{d}{dx} (x(2x^2 - 3x - 5))$$

$$= \frac{d}{dx} (2x^3 - 3x^2 - 5x)$$

$$= 6x^2 - 6x - 5$$

$$\mathbf{5} \quad \mathbf{a} \quad y = x^2 \quad \text{at} \quad x = 2$$

$$f(x) = x^2$$

$$\therefore f'(x) = 2x$$

$$\text{and so } f'(2) = 2(2)$$

$$\therefore \text{tangent has slope of } 4$$

$$\mathbf{b} \quad y = \frac{8}{x^2} \quad \text{at} \quad x = 9$$

$$f(x) = 8x^{-2}$$

$$\therefore f'(x) = -16x^{-3}$$

$$= -\frac{16}{x^3}$$

$$\text{and so } f'(9) = -\frac{16}{729}$$

$$\therefore \text{tangent has slope of } -\frac{16}{729}$$

$$\mathbf{c} \quad y = 2x^2 - 3x + 7 \quad \text{at} \quad x = -1$$

$$f(x) = 2x^2 - 3x + 7$$

$$\therefore f'(x) = 4x - 3 + 0$$

$$\text{and so } f'(-1) = 4(-1) - 3$$

$$= -7$$

$$\therefore \text{tangent has slope of } -7$$

$$\mathbf{d} \quad y = \frac{2x^2 - 5}{x} \quad \text{at} \quad x = 2$$

$$f(x) = 2x - 5x^{-1}$$

$$\therefore f'(x) = 2 + 5x^{-2}$$

$$= 2 + \frac{5}{x^2}$$

$$\text{and so } f'(2) = 2 + \frac{5}{4} = \frac{13}{4}$$

$$\therefore \text{tangent has slope of } \frac{13}{4}$$

$$\mathbf{e} \quad y = \frac{x^2 - 4}{x^2} \quad \text{at} \quad x = 4$$

$$f(x) = 1 - 4x^{-2}$$

$$\therefore f'(x) = 0 + 8x^{-3}$$

$$= \frac{8}{x^3}$$

$$\text{and so } f'(4) = \frac{8}{4^3} = \frac{1}{8}$$

$$\therefore \text{tangent has slope of } \frac{1}{8}$$

$$\mathbf{f} \quad y = \frac{x^3 - 4x - 8}{x^2} \quad \text{at} \quad x = -1$$

$$f(x) = x - 4x^{-1} - 8x^{-2}$$

$$\therefore f'(x) = 1 + 4x^{-2} + 16x^{-3}$$

$$= 1 + \frac{4}{x^2} + \frac{16}{x^3}$$

$$\text{and so } f'(-1) = 1 + 4 - 16$$

$$= -11$$

$$\therefore \text{tangent has slope of } -11$$

$$\mathbf{6} \quad \mathbf{a} \quad f(x) = 4\sqrt{x} + x = 4x^{\frac{1}{2}} + x$$

$$\therefore f'(x) = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 1$$

$$= \frac{2}{\sqrt{x}} + 1$$

$$\mathbf{b} \quad f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\therefore f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$= \frac{1}{3\sqrt[3]{x^2}}$$

$$\mathbf{c} \quad f(x) = -\frac{2}{\sqrt{x}} = -2x^{-\frac{1}{2}}$$

$$\therefore f'(x) = -2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$= x^{-\frac{3}{2}}$$

$$= \frac{1}{x\sqrt{x}}$$

$$\mathbf{d} \quad f(x) = 2x - \sqrt{x} = 2x - x^{\frac{1}{2}}$$

$$\therefore f'(x) = 2 - \frac{1}{2}x^{-\frac{1}{2}}$$

$$= 2 - \frac{1}{2\sqrt{x}}$$

$$\mathbf{e} \quad f(x) = \frac{4}{\sqrt{x}} - 5 = 4x^{-\frac{1}{2}} - 5$$

$$\begin{aligned} \therefore f'(x) &= 4\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} - 0 \\ &= -2x^{-\frac{3}{2}} \quad \text{or} \quad \frac{-2}{x\sqrt{x}} \end{aligned}$$

$$\mathbf{f} \quad f(x) = 3x^2 - x\sqrt{x} = 3x^2 - x^{\frac{3}{2}}$$

$$\begin{aligned} \therefore f'(x) &= 6x - \frac{3}{2}x^{\frac{1}{2}} \\ &= 6x - \frac{3}{2}\sqrt{x} \end{aligned}$$

$$\mathbf{g} \quad f(x) = \frac{5}{x^2\sqrt{x}} = 5x^{-\frac{5}{2}}$$

$$\begin{aligned} \therefore f'(x) &= 5\left(-\frac{5}{2}\right)x^{-\frac{7}{2}} \\ &= -\frac{25}{2}x^{-\frac{7}{2}} \\ &= \frac{-25}{2x^3\sqrt{x}} \end{aligned}$$

$$\mathbf{h} \quad f(x) = 2x - \frac{3}{x\sqrt{x}} = 2x - 3x^{-\frac{3}{2}}$$

$$\begin{aligned} \therefore f'(x) &= 2 - 3\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} \\ &= 2 + \frac{9}{2}x^{-\frac{5}{2}} \\ &= 2 + \frac{9}{2x^2\sqrt{x}} \end{aligned}$$

$$\mathbf{7} \quad \mathbf{a} \quad y = 4x - \frac{3}{x} = 4x - 3x^{-1} \quad \therefore \quad \frac{dy}{dx} = 4 + 3x^{-2} = 4 + \frac{3}{x^2}$$

$\frac{dy}{dx}$ is the slope function of $y = 4x - \frac{3}{x}$ from which the slope at any point can be found.

$$\mathbf{b} \quad S = 2t^2 + 4t \text{ m} \quad \therefore \quad \frac{dS}{dt} = 4t + 4 \text{ ms}^{-1}$$

$\frac{dS}{dt}$ is the instantaneous rate of change in position at time t , i.e., it is the velocity function.

$$\mathbf{c} \quad C = 1785 + 3x + 0.002x^2 \text{ dollars.}$$

$$\frac{dC}{dx} = 3 + 2(0.002)x = 3 + 0.004x \text{ dollars/toaster}$$

$\frac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 21E.1

$$\mathbf{1} \quad \mathbf{a} \quad f(x) = x^2, \quad g(x) = 2x + 7,$$

$$\begin{aligned} \therefore f(g(x)) &= f(2x + 7) \\ &= (2x + 7)^2 \end{aligned}$$

$$\mathbf{b} \quad f(x) = 2x + 7, \quad g(x) = x^2,$$

$$\begin{aligned} f(g(x)) &= f(x^2) \\ &= 2x^2 + 7 \end{aligned}$$

$$\mathbf{c} \quad f(x) = \sqrt{x}, \quad g(x) = 3 - 4x,$$

$$f(g(x)) = f(3 - 4x) = \sqrt{3 - 4x}$$

$$\mathbf{d} \quad f(x) = 3 - 4x, \quad g(x) = \sqrt{x},$$

$$f(g(x)) = f(\sqrt{x}) = 3 - 4\sqrt{x}$$

$$\mathbf{e} \quad f(x) = \frac{2}{x}, \quad g(x) = x^2 + 3,$$

$$\begin{aligned} f(g(x)) &= f(x^2 + 3) \\ &= \frac{2}{x^2 + 3} \end{aligned}$$

$$\mathbf{f} \quad f(x) = x^2 + 3, \quad g(x) = \frac{2}{x},$$

$$\begin{aligned} f(g(x)) &= f\left(\frac{2}{x}\right) = \left(\frac{2}{x}\right)^2 + 3 \\ &= \frac{4}{x^2} + 3 \end{aligned}$$

$$\mathbf{g} \quad f(x) = 2^x, \quad g(x) = 3x + 4,$$

$$f(g(x)) = f(3x + 4) = 2^{3x+4}$$

$$\mathbf{h} \quad f(x) = 3x + 4, \quad g(x) = 2^x,$$

$$f(g(x)) = f(2^x) = 3(2^x) + 4$$

$$\mathbf{2} \quad \mathbf{a} \quad f(g(x)) = (3x + 10)^3 \quad \therefore \quad f(x) = x^3, \quad g(x) = 3x + 10$$

$$\mathbf{b} \quad f(g(x)) = \frac{1}{2x + 4} \quad \therefore \quad f(x) = \frac{1}{x}, \quad g(x) = 2x + 4$$

$$\mathbf{c} \quad f(g(x)) = \sqrt{x^2 - 3x} \quad \therefore \quad f(x) = \sqrt{x}, \quad g(x) = x^2 - 3x$$

$$\mathbf{d} \quad f(g(x)) = \frac{1}{\sqrt{5-2x}} \quad \therefore \quad f(x) = \frac{1}{\sqrt{x}}, \quad g(x) = 5-2x$$

$$\mathbf{e} \quad f(g(x)) = (x^2 + 5x - 1)^4 \quad \therefore \quad f(x) = x^4, \quad g(x) = x^2 + 5x - 1$$

$$\mathbf{f} \quad f(g(x)) = \frac{10}{(3x - x^2)^3} \quad \therefore \quad f(x) = \frac{10}{x^3}, \quad g(x) = 3x - x^2$$

EXERCISE 21E.2

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \frac{1}{(2x-1)^2} \\ &= (2x-1)^{-2} \\ &= u^{-2}, \quad \text{where } u = 2x-1 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \frac{2}{\sqrt{2-x^2}} \\ &= 2(2-x^2)^{-\frac{1}{2}} \\ &= 2u^{-\frac{1}{2}}, \quad \text{where } u = 2-x^2 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \frac{4}{(3-x)^3} \\ &= 4(3-x)^{-3} \\ &= 4u^{-3}, \quad \text{where } u = 3-x \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & y = (4x-5)^2 \\ \therefore & y = u^2 \quad \text{where } u = 4x-5 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= 2u(4) \\ &= 8u \\ &= 8(4x-5) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & y = \sqrt{3x-x^2} \\ \therefore & y = u^{\frac{1}{2}} \quad \text{where } u = 3x-x^2 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}}(3-2x) \\ &= \frac{(3-2x)}{2\sqrt{u}} \\ &= \frac{3-2x}{2\sqrt{3x-x^2}} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & y = 6(5-x)^3 \\ \therefore & y = 6u^3 \quad \text{where } u = 5-x \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= 18u^2(-1) \\ &= -18u^2 \\ &= -18(5-x)^2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \sqrt{x^2-3x} \\ &= (x^2-3x)^{\frac{1}{2}} \\ &= u^{\frac{1}{2}}, \quad \text{where } u = x^2-3x \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \sqrt[3]{x^3-x^2} \\ &= (x^3-x^2)^{\frac{1}{3}} \\ &= u^{\frac{1}{3}}, \quad \text{where } u = x^3-x^2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \frac{10}{x^2-3} \\ &= 10(x^2-3)^{-1} \\ &= 10u^{-1}, \quad \text{where } u = x^2-3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & y = \frac{1}{5-2x} = u^{-1} \quad \text{where } u = 5-2x \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = -u^{-2}(-2) \\ &= \frac{2}{u^2} \\ &= \frac{2}{(5-2x)^2} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & y = (1-3x)^4 \\ \therefore & y = u^4 \quad \text{where } u = 1-3x \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= 4u^3(-3) \\ &= -12u^3 \\ &= -12(1-3x)^3 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & y = \sqrt[3]{2x^3-x^2} \\ \therefore & y = u^{\frac{1}{3}} \quad \text{where } u = 2x^3-x^2 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \frac{1}{3}u^{-\frac{2}{3}}(6x^2-3x) \\ &= \frac{6x^2-3x}{3\sqrt[3]{(2x^3-x^2)^2}} \end{aligned}$$

$$\mathbf{g} \quad y = \frac{6}{(5x-4)^2}$$

$$\therefore y = 6u^{-2} \text{ where } u = 5x - 4$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= -12u^{-3}(5) \\ &= -\frac{60}{u^3} \\ &= \frac{-60}{(5x-4)^3} \end{aligned}$$

$$\mathbf{i} \quad y = 2 \left(x^2 - \frac{2}{x} \right)^3$$

$$\therefore y = 2u^3 \text{ where } u = x^2 - 2x^{-1}$$

$$\mathbf{h} \quad y = \frac{4}{3x-x^2}$$

$$\therefore y = 4u^{-1} \text{ where } u = 3x - x^2$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= -4u^{-2}(3-2x) \\ &= \frac{-4(3-2x)}{u^2} \\ &= \frac{-4(3-2x)}{(3x-x^2)^2} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = 6u^2(2x+2x^{-2}) \\ &= 6 \left(2x + \frac{2}{x^2} \right) \left(x^2 - \frac{2}{x} \right)^2 \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad y = \sqrt{1-x^2} \text{ at } x = \frac{1}{2}$$

$$\therefore y = \sqrt{u} \text{ where } u = 1 - x^2$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-2x) \\ &= \frac{-x}{\sqrt{u}} \\ &= \frac{-x}{\sqrt{1-x^2}} \end{aligned}$$

$$\begin{aligned} \text{at } x = \frac{1}{2}, \quad \frac{dy}{dx} &= \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} \\ &= -\frac{1}{2} \left(\frac{2}{\sqrt{3}} \right) \end{aligned}$$

$$\therefore \text{ slope of tangent} = -\frac{1}{\sqrt{3}}$$

$$\mathbf{c} \quad y = \frac{1}{(2x-1)^4} \text{ at } x = 1$$

$$\therefore y = u^{-4} \text{ where } u = 2x - 1$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= -4u^{-5}(2) \\ &= \frac{-8}{u^5} \\ &= \frac{-8}{(2x-1)^5} \end{aligned}$$

$$\text{at } x = 1, \quad \frac{dy}{dx} = \frac{-8}{1^5}$$

$$\therefore \text{ slope of tangent} = -8$$

$$\mathbf{b} \quad y = (3x+2)^6 \text{ at } x = -1$$

$$\therefore y = u^6 \text{ where } u = 3x + 2$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= 6u^5(3) \\ &= 18u^5 \\ &= 18(3x+2)^5 \end{aligned}$$

$$\text{at } x = -1, \quad \frac{dy}{dx} = 18(-1)^5$$

$$\therefore \text{ slope of tangent} = -18$$

$$\mathbf{d} \quad y = 6 \times \sqrt[3]{1-2x} \text{ at } x = 0$$

$$\therefore y = 6u^{\frac{1}{3}} \text{ where } u = 1 - 2x$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= 6\left(\frac{1}{3}\right)u^{-\frac{2}{3}}(-2) \\ &= 2u^{-\frac{2}{3}}(-2) \\ &= \frac{-4}{\sqrt[3]{u^2}} \\ &= \frac{-4}{\sqrt[3]{(1-2x)^2}} \end{aligned}$$

$$\text{at } x = 0, \quad \frac{dy}{dx} = \frac{-4}{\sqrt[3]{1^2}}$$

$$\therefore \text{ slope of tangent} = -4$$

$$\text{e } y = \frac{4}{x + 2\sqrt{x}} \quad \text{at } x = 4$$

$$\therefore y = 4u^{-1} \quad \text{where } u = x + 2x^{\frac{1}{2}}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= -4u^{-2}(1 + x^{-\frac{1}{2}})$$

$$= -\frac{4}{u^2} \left(1 + \frac{1}{\sqrt{x}}\right)$$

$$= \frac{-4}{(x + 2\sqrt{x})^2} \left(1 + \frac{1}{\sqrt{x}}\right)$$

$$\text{at } x = 4, \quad \frac{dy}{dx} = \frac{-4}{(4 + 4)^2} \left(1 + \frac{1}{2}\right)$$

$$= -\frac{6}{64}$$

$$\therefore \text{ slope of tangent} = -\frac{3}{32}$$

$$\text{f } y = \left(x + \frac{1}{x}\right)^3 \quad \text{at } x = 1$$

$$\therefore y = u^3 \quad \text{where } u = x + x^{-1}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= 3u^2(1 - u^{-2})$$

$$= 3 \left(x + \frac{1}{x}\right)^2 \left(1 - \frac{1}{x^2}\right)$$

$$\text{at } x = 1, \quad \frac{dy}{dx} = 3(1 + 1)^2(1 - 1)$$

$$\therefore \text{ slope of tangent} = 0$$

$$4 \quad \text{a } y = x^3 \quad \therefore \frac{dy}{dx} = 3x^2$$

$$x = y^{\frac{1}{3}} \quad \therefore \frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$$

$$\frac{dy}{dx} \frac{dx}{dy} = 3x^2 \left(\frac{1}{3}\right) y^{-\frac{2}{3}}$$

$$= x^2(y)^{-\frac{2}{3}}$$

$$= x^2(x^3)^{-\frac{2}{3}} \quad \{\text{substituting } y = x^3\}$$

$$= x^2(x^{-2})$$

$$= x^0$$

$$= 1 \quad \text{as required.}$$

$$\text{b } \text{We know that } \frac{dy}{du} \frac{du}{dx} = \frac{dy}{dx} \quad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} \frac{dx}{dy} = \frac{dy}{dy} = 1$$

EXERCISE 21F.1

$$1 \quad \text{a } y = x^2(2x - 1) \quad \text{is the product of } u = x^2 \quad \text{and } v = 2x - 1$$

$$\therefore u' = 2x \quad \text{and } v' = 2$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 2x(2x - 1) + x^2(2)$$

$$= 4x^2 - 2x + 2x^2$$

$$= 6x^2 - 2x$$

$$\text{b } y = 4x(2x + 1)^3 \quad \text{is the product of } u = 4x \quad \text{and } v = (2x + 1)^3$$

$$\therefore u' = 4 \quad \text{and } v' = 6(2x + 1)^2$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 4(2x + 1)^3 + 24x(2x + 1)^2$$

$$= [4(2x + 1) + 24x](2x + 1)^2$$

$$= [32x + 4](2x + 1)^2$$

$$\text{c } y = x^2\sqrt{3-x} \text{ is the product of } u = x^2 \text{ and } v = (3-x)^{\frac{1}{2}}$$

$$\therefore u' = 2x \text{ and } v' = -\frac{1}{2}(3-x)^{-\frac{1}{2}}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 2x(3-x)^{\frac{1}{2}} + x^2 \left[-\frac{1}{2}(3-x)^{-\frac{1}{2}} \right]$$

$$= 2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}}$$

$$\text{d } y = \sqrt{x}(x-3)^2 \text{ is the product of } y = x^{\frac{1}{2}} \text{ and } v = (x-3)^2$$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 2(x-3)^1$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$$

$$\text{e } y = 5x^2(3x^2-1)^2 \text{ is the product of } u = 5x^2 \text{ and } v = (3x^2-1)^2$$

$$\therefore u' = 10x \text{ and } v' = 12x(3x^2-1)$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 10x(3x^2-1)^2 + 5x^2(12x)(3x^2-1)$$

$$= 10x(3x^2-1)^2 + 60x^3(3x^2-1)$$

$$\text{f } y = \sqrt{x}(x-x^2)^3 \text{ is the product of } u = x^{\frac{1}{2}} \text{ and } v = (x-x^2)^3$$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 3(1-2x)(x-x^2)^2$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(1-2x)(x-x^2)^2$$

$$\text{2 a } y = x^4(1-2x)^2 \text{ is the product of } u = x^4 \text{ and } v = (1-2x)^2$$

$$u' = 4x^3 \text{ and } v' = 2(1-2x)^1(-2)$$

$$= -4(1-2x)$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 4x^3(1-2x)^2 - 4x^4(1-2x)$$

$$\text{at } x = -1, \frac{dy}{dx} = -4(3^2) - 4(3) = -48 \quad \therefore \text{slope of tangent} = -48$$

$$\text{b } y = \sqrt{x}(x^2-x+1)^2 \text{ is the product of } u = x^{\frac{1}{2}} \text{ and } v = (x^2-x+1)^2$$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 2(x^2-x+1)(2x-1)$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x^2-x+1)^2 + 2\sqrt{x}(2x-1)(x^2-x+1)$$

$$\text{at } x = 4, \frac{dy}{dx} = \frac{1}{4}(13)^2 + 4(7)(13) = 406\frac{1}{4}$$

$$\therefore \text{slope of tangent} = 406\frac{1}{4}$$

$$\begin{aligned} \text{c } y = x\sqrt{1-2x} \text{ is the product of } u = x \text{ and } v = (1-2x)^{\frac{1}{2}} \\ \therefore u' = 1 \text{ and } v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) \\ = -(1-2x)^{-\frac{1}{2}} \end{aligned}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = \sqrt{1-2x} - \frac{x}{\sqrt{1-2x}}$$

$$\text{at } x = -4, \frac{dy}{dx} = \sqrt{9} - \frac{(-4)}{\sqrt{9}} = 3 + \frac{4}{3} = \frac{13}{3}$$

$$\therefore \text{ slope of tangent} = \frac{13}{3}$$

$$\begin{aligned} \text{d } y = x^3\sqrt{5-x^2} \text{ is the product of } u = x^3 \text{ and } v = (5-x^2)^{\frac{1}{2}} \\ \therefore u' = 3x^2 \text{ and } v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x) \\ = -x(5-x^2)^{-\frac{1}{2}} \end{aligned}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 3x^2\sqrt{5-x^2} - \frac{x^4}{\sqrt{5-x^2}}$$

$$\text{at } x = 1, \frac{dy}{dx} = 3(1)^2\sqrt{4} - \frac{1}{\sqrt{4}} = 6 - \frac{1}{2} = \frac{11}{2} \quad \therefore \text{ slope of tangent} = \frac{11}{2}$$

$$\begin{aligned} \text{3 } y = \sqrt{x}(3-x)^2 \text{ is the product of } u = x^{\frac{1}{2}} \text{ and } v = (3-x)^2 \\ \therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 2(3-x)^1(-1) = -2(3-x) \end{aligned}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2\sqrt{x}}(3-x)^2 - 2\sqrt{x}(3-x) \\ &= \frac{(3-x)^2 - 4x(3-x)}{2\sqrt{x}} \\ &= \frac{(3-x)[(3-x) - 4x]}{2\sqrt{x}} \\ &= \frac{(3-x)(3-5x)}{2\sqrt{x}} \text{ as required} \end{aligned}$$

Tangents are horizontal when their slopes are 0.

$$\begin{aligned} \therefore \frac{dy}{dx} = 0 \text{ when } (3-x)(3-5x) = 0 \\ \text{i.e., } 3-x = 0 \text{ or } 3-5x = 0 \\ \text{i.e., } x = 3 \text{ or } x = \frac{3}{5} \end{aligned}$$

EXERCISE 21F.2

$$\text{1 a } y = \frac{1+3x}{2-x} \text{ is a quotient where } u = 1+3x \text{ and } v = 2-x$$

$$\therefore u' = 3 \text{ and } v' = -1$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{3(2-x) - (-1)(1+3x)}{(2-x)^2} = \frac{7}{(2-x)^2}$$

b $y = \frac{x^2}{2x+1}$ is a quotient where $u = x^2$ and $v = 2x+1$

$\therefore u' = 2x$ and $v' = 2$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \frac{dy}{dx} = \frac{2x(2x+1) - x^2(2)}{(2x+1)^2} = \frac{2x^2 + 2x}{(2x+1)^2}$

c $y = \frac{x}{x^2-3}$ is a quotient where $u = x$ and $v = x^2-3$

$\therefore u' = 1$ and $v' = 2x$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \frac{dy}{dx} = \frac{1(x^2-3) - x(2x)}{(x^2-3)^2} = \frac{-3-x^2}{(x^2-3)^2}$

d $y = \frac{\sqrt{x}}{1-2x}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = 1-2x$

$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = -2$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \frac{dy}{dx} = \frac{\frac{1-2x}{2\sqrt{x}} - (-2)\sqrt{x}}{(1-2x)^2} = \frac{1-2x+4x}{2\sqrt{x}(1-2x)^2} = \frac{1+2x}{2\sqrt{x}(1-2x)^2}$

e $y = \frac{x^2-3}{3x-x^2}$ is a quotient where $u = x^2-3$ and $v = 3x-x^2$

$\therefore u' = 2x$ and $v' = 3-2x$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \frac{dy}{dx} = \frac{2x(3x-x^2) - (x^2-3)(3-2x)}{(3x-x^2)^2} = \frac{6x^2-2x^3-3x^2+2x^3+9-6x}{(3x-x^2)^2}$
 $= \frac{3x^2-6x+9}{(3x-x^2)^2}$

f $y = \frac{x}{\sqrt{1-3x}}$ is a quotient where $u = x$ and $v = (1-3x)^{\frac{1}{2}}$

$\therefore u' = 1$ and $v' = -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \frac{dy}{dx} = \frac{\sqrt{1-3x} - x\left(\frac{-3}{2\sqrt{1-3x}}\right)}{(1-3x)} = \frac{2(1-3x) + 3x}{2(1-3x)^{\frac{3}{2}}} = \frac{2-3x}{2(1-3x)^{\frac{3}{2}}}$

2 a $y = \frac{x}{1-2x}$ is a quotient where $u = x$ and $v = 1-2x$

$\therefore u' = 1$ and $v' = -2$ Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \frac{dy}{dx} = \frac{1(1-2x) - x(-2)}{(1-2x)^2} = \frac{1}{(1-2x)^2}$

at $x = 1$ $\frac{dy}{dx} = \frac{1}{(1-2)^2} = \frac{1}{(-1)^2} = 1 \therefore$ slope of tangent = 1

b $y = \frac{x^3}{x^2+1}$ is a quotient where $u = x^3$ and $v = x^2 + 1$

$$\therefore u' = 3x^2 \text{ and } v' = 2x$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2} = \frac{x^4 + 3x^2}{(x^2+1)^2}$$

$$\therefore \text{ at } x = -1 \quad \frac{dy}{dx} = \frac{1+3}{(1+1)^2} = \frac{4}{4} = 1$$

and so, the slope of tangent = 1

c $y = \frac{\sqrt{x}}{2x+1}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = 2x + 1$

$$u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 2$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\frac{1}{2\sqrt{x}}(2x+1) - \sqrt{x}(2)}{(2x+1)^2}$$

$$\text{at } x = 4 \quad \frac{dy}{dx} = \frac{\frac{9}{4} - 4}{81} = \frac{(\frac{9}{4} - 4)}{81} \times \frac{4}{4} = \frac{9-16}{324}$$

$$\therefore \text{ slope of tangent} = -\frac{7}{324}$$

d $y = \frac{x^2}{\sqrt{x^2+5}}$ is a quotient where $u = x^2$ and $v = (x^2+5)^{\frac{1}{2}}$

$$\therefore u' = 2x \text{ and } v' = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x) \\ = x(x^2+5)^{-\frac{1}{2}}$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{2x\sqrt{x^2+5} - x^2\left(\frac{x}{\sqrt{x^2+5}}\right)}{(x^2+5)}$$

$$\text{at } x = -2 \quad \frac{dy}{dx} = \frac{-4(3) - \left(\frac{-8}{3}\right)}{9} = \frac{(-12 + \frac{8}{3})}{9} \times \frac{3}{3} = \frac{-36+8}{27}$$

$$\therefore \text{ slope of tangent} = -\frac{28}{27}$$

3 a $y = \frac{2\sqrt{x}}{1-x}$ is a quotient where $u = 2x^{\frac{1}{2}}$ and $v = 1 - x$

$$\therefore u' = 2x^{-\frac{1}{2}} \text{ and } v' = -1$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$\therefore \frac{dy}{dx} = \left(\frac{\frac{1}{\sqrt{x}}(1-x) - 2\sqrt{x}(-1)}{(1-x)^2} \right) \frac{\sqrt{x}}{\sqrt{x}} = \frac{(1-x) + 2x}{\sqrt{x}(1-x)^2} = \frac{x+1}{\sqrt{x}(1-x)^2} \text{ as required.}$$

i $\frac{dy}{dx} = 0$ when $x+1=0$ i.e., $x=-1$. However $\frac{dy}{dx}$ is not defined for $x \leq 0$ because of the $\sqrt{x}$ term. Hence $\frac{dy}{dx}$ never equals 0.

ii $\frac{dy}{dx}$ is undefined when $x \leq 0$ and when $x=1$

b $y = \frac{x^2 - 3x + 1}{x + 2}$ is a quotient where $y = x^2 - 3x + 1$ and $v = x + 2$

$\therefore u' = 2x - 3$ and $v' = 1$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{(2x - 3)(x + 2) - (x^2 - 3x + 1)(1)}{(x + 2)^2} \\ &= \frac{2x^2 + 4x - 3x - 6 - x^2 + 3x - 1}{(x + 2)^2} \\ &= \frac{x^2 + 4x - 7}{(x + 2)^2} \text{ as required.}\end{aligned}$$

i $\frac{dy}{dx} = 0$ when $x^2 + 4x - 7 = 0$
i.e., $x = \frac{-4 \pm \sqrt{44}}{2} = -2 \pm \sqrt{11}$

ii $\frac{dy}{dx}$ is undefined when $(x + 2)^2 = 0$ i.e., $x = -2$

EXERCISE 21G

1 a $y = x - 2x^2 + 3$ at $x = 2$

Since when $x = 2$, $y = 2 - 2(2)^2 + 3 = -3$, the point of contact is $(2, -3)$.

Now $\frac{dy}{dx} = 1 - 4x$

$\therefore$ at $x = 2$, $\frac{dy}{dx} = 1 - 8 = -7$

$\therefore$ the tangent has equation $\frac{y - (-3)}{x - 2} = -7$ i.e., $y + 3 = -7(x - 2)$
i.e., $y = -7x + 14 - 3$
i.e., $y = -7x + 11$

b $y = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$ at $x = 4$

Since when $x = 4$, $y = \sqrt{4} + 1 = 3$, the point of contact is $(4, 3)$.

Now $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

at $x = 4$, $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$\therefore$ the tangent has equation $\frac{y - 3}{x - 4} = \frac{1}{4}$ i.e., $4y - 12 = x - 4$
i.e., $4y = x + 8$

c $y = x^3 - 5x$ at $x = 1$

Since when $x = 1$, $y = 1^3 - 5(1) = -4$ the point of contact is $(1, -4)$.

Now $\frac{dy}{dx} = 3x^2 - 5$

at $x = 1$, $\frac{dy}{dx} = 3 - 5 = -2$

$\therefore$ tangent has equation $\frac{y - (-4)}{x - 1} = -2$ i.e., $y + 4 = -2x + 2$
i.e., $y = -2x - 2$

d $y = \frac{4}{\sqrt{x}}$ at $(1, 4)$. Now $y = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = -2x^{-\frac{3}{2}}$$

$$\text{at } x = 1, \frac{dy}{dx} = -2 \left(1^{-\frac{3}{2}} \right) = -2$$

$$\therefore \text{ the tangent has equation } \frac{y-4}{x-1} = -2 \quad \text{i.e., } y-4 = -2x+2$$

$$\text{i.e., } y = -2x+6$$

e $y = \frac{3}{x} - \frac{1}{x^2}$ at $(-1, -4)$. Now $y = \frac{3}{x} - \frac{1}{x^2} = 3x^{-1} - x^{-2}$

$$\frac{dy}{dx} = -3x^{-2} + 2x^{-3}$$

$$\text{at } x = -1, \frac{dy}{dx} = \frac{-3}{(-1)^2} + \frac{2}{(-1)^3} = -5$$

$$\therefore \text{ tangent has equation } \frac{y-(-4)}{x-(-1)} = -5 \quad \text{i.e., } y+4 = -5(x+1)$$

$$\text{i.e., } y = -5x-5-4$$

$$\text{i.e., } y = -5x-9$$

f $y = 3x^2 - \frac{1}{x} = 3x^2 - x^{-1}$ at $x = -1$

$$\text{Since when } x = -1, y = 3(-1)^2 - \frac{1}{(-1)} = 4 \quad \text{the point of contact is } (1, 4).$$

$$\text{Now } \frac{dy}{dx} = 6x - (-1x^{-2}) = 6x + \frac{1}{x^2}$$

$$\text{at } x = -1, \frac{dy}{dx} = 6(-1) + \frac{1}{(-1)^2} = -5$$

$$\therefore \text{ the tangent has equation } \frac{y-4}{x-(-1)} = -5 \quad \text{i.e., } y-4 = -5x-5$$

$$\text{i.e., } y = -5x-1$$

2 a $y = x^2$ at $(3, 9)$

$$\frac{dy}{dx} = 2x$$

$$\text{at } x = 3, \frac{dy}{dx} = 2(3) = 6 = \frac{6}{1}$$

$$\therefore \text{ the normal at } (3, 9) \text{ has slope } -\frac{1}{6}, \text{ so the equation of the normal is}$$

$$\frac{y-9}{x-3} = -\frac{1}{6} \quad \text{i.e., } 6y-54 = -x+3$$

$$\text{i.e., } 6y = -x+57$$

b $y = x^3 - 5x + 2$ at $x = -2$

$$\text{Since when } x = -2, y = (-2)^3 - 5(-2) + 2 = 4$$

$$\text{and so the point of contact is } (-2, 4)$$

$$\text{Now } \frac{dy}{dx} = 3x^2 - 5$$

$$\therefore \text{ at } x = -2, \frac{dy}{dx} = 3(-2)^2 - 5 = 7$$

$$\therefore \text{ the normal at } (-2, 4) \text{ has slope } -\frac{1}{7}, \text{ so the equation of the normal is}$$

$$\frac{y-4}{x-(-2)} = -\frac{1}{7} \quad \text{i.e., } 7y-28 = -(x+2)$$

$$\text{i.e., } 7y = -x+26$$

c $y = 2\sqrt{x} + 3$ at $x = 1$

Since when $x = 1$, $y = 2\sqrt{1} + 3 = 5$ the point of contact is $(1, 5)$

$$\text{Now } y = 2\sqrt{x} + 3 = 2x^{\frac{1}{2}} + 3$$

$$\therefore \frac{dy}{dx} = x^{-\frac{1}{2}}$$

and so, at $x = 1$, $\frac{dy}{dx} = \frac{1}{\sqrt{1}} = 1$

$\therefore$ the normal at $(1, 5)$ has slope -1 , and so the equation of the normal is

$$\begin{aligned} \frac{y-5}{x-1} &= -1 & \text{i.e., } y-5 &= -x+1 \\ & & \text{i.e., } y &= -x+6 \end{aligned}$$

d $y = \frac{3}{\sqrt{x}}$ and so, when $x = 9$, $y = \frac{3}{\sqrt{9}} = 1$ $\therefore$ the point of contact is $(9, 1)$

$$y = \frac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}}$$

$$\therefore \text{ at } x = 9, \frac{dy}{dx} = -\frac{3}{2}\left(9^{-\frac{3}{2}}\right) = -\frac{3}{2}\left(3^{-3}\right) = -\frac{1}{18}$$

$\therefore$ the normal at $(9, 1)$ has slope 18 , so the equation of the normal is

$$\begin{aligned} \frac{y-1}{x-9} &= 18 & \text{i.e., } y-1 &= 18x-162 \\ & & \text{i.e., } y &= 18x-161 \end{aligned}$$

e $y = \frac{5}{\sqrt{x}} - \sqrt{x}$ at $(1, 4)$

$$\text{Now } y = 5x^{-\frac{1}{2}} - x^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$$

$$\therefore \text{ at } x = 1, \frac{dy}{dx} = -\frac{5}{2}\left(1^{-\frac{3}{2}}\right) - \frac{1}{2}\left(1^{-\frac{1}{2}}\right) = -\frac{5}{2} - \frac{1}{2} = -3$$

$\therefore$ the normal at $(1, 4)$ has slope $\frac{1}{3}$, so the equation of the normal is

$$\begin{aligned} \frac{y-4}{x-1} &= \frac{1}{3} & \text{i.e., } 3y-12 &= x-1 \\ & & \text{i.e., } 3y &= x+11 \end{aligned}$$

f $y = 8\sqrt{x} - \frac{1}{x^2}$ at $x = 1$

Since when $x = 1$, $y = 8\sqrt{1} - \frac{1}{1^2} = 7$ the point of contact is $(1, 7)$

$$\text{Now } y = 8\sqrt{x} - \frac{1}{x^2} = 8x^{\frac{1}{2}} - x^{-2}$$

$$\therefore \frac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$$

$$\therefore \text{ at } x = 1, \frac{dy}{dx} = 4 + 2 = 6$$

$\therefore$ the normal at $(1, 7)$ has slope $-\frac{1}{6}$, so the equation of the normal is

$$\begin{aligned} \frac{y-7}{x-1} &= -\frac{1}{6} & \text{i.e., } 6y-42 &= -x+1 \\ & & \text{i.e., } 6y &= -x+43 \end{aligned}$$

3 a $y = 2x^3 + 3x^2 - 12x + 1 \quad \therefore \frac{dy}{dx} = 6x^2 + 6x - 12$

Horizontal tangents have slope = 0 so $6x^2 + 6x - 12 = 0$
 $x^2 + x - 2 = 0$
 $(x+2)(x-1) = 0$
 $\therefore x = -2 \text{ or } x = 1$

Now at $x = -2$, $y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 21$

and at $x = 1$, $y = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$

i.e., points of contact are $(-2, 21)$ and $(1, -6)$

$\therefore$ tangents are $y = -6$ and $y = 21$

b Now $y = \frac{2x+1}{\sqrt{x}} = 2\sqrt{x} + \frac{1}{\sqrt{x}} = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

Horizontal tangents have slope = 0 $\therefore \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}} = 0$
 $\therefore \frac{2x-1}{2x\sqrt{x}} = 0$
 $\therefore 2x-1 = 0$
 $\therefore x = \frac{1}{2}$

Now at $x = \frac{1}{2}$, $y = \frac{2(\frac{1}{2})+1}{\sqrt{\frac{1}{2}}} = \frac{2}{\frac{1}{2}\sqrt{2}} = 2\sqrt{2}$

$\therefore$ the only horizontal tangent touches at $(\frac{1}{2}, 2\sqrt{2})$

c Now $y = 2x^3 + kx^2 - 3$

$\therefore \frac{dy}{dx} = 6x^2 + 2kx$

and so when $x = 2$, $\frac{dy}{dx} = 4 \quad \therefore 6(2)^2 + 2k(2) = 4$
 $\therefore 24 + 4k = 4$
 $\therefore 4k = -20$
 $\therefore k = -5$

d Now $y = 1 - 3x + 12x^2 - 8x^3 \quad \therefore \frac{dy}{dx} = -3 + 24x - 24x^2$

when $x = 1$, $\frac{dy}{dx} = -3 + 24 - 24 = -3$

i.e., the tangent at $(1, 2)$ has slope -3

$\therefore -3 + 24x - 24x^2 = -3$ for all points x where the tangent has slope -3

$\therefore 24x^2 - 24x = 0$

$\therefore 24x(x-1) = 0$

i.e., when $x = 0$ or $x = 1$

When $x = 0$, $y = 1 - 0 + 0 - 0 = 1$

and when $x = 1$, $y = 1 - 3 + 12 - 8 = 2$

$\therefore$ the tangents are $\frac{y-1}{x-0} = -3$ and $\frac{y-2}{x-1} = -3$

i.e., $y = -3x + 1$ and $y = -3x + 5$

4 a Now $y = x^2 + ax + b$ $\therefore \frac{dy}{dx} = 2x + a$

$\therefore$ at $x = 1$, $\frac{dy}{dx} = 2 + a$

$\therefore$ the slope of the tangent to the curve at $x = 1$ will be $2 + a$

However the equation of the tangent is $y + 2x = 6$ i.e., $y = -2x + 6$

and so the slope of the tangent is -2 . $\therefore 2 + a = -2$

$$a = -4$$

$$\therefore y = x^2 - 4x + b$$

Also, at $x = 1$, the tangent line contacts the curve

i.e., $1^2 - 4(1) + b = -2(1) + 6$

$$1 - 4 + b = 4$$

$$b = 7$$

$$\therefore a = -4, b = 7$$

b

Now $y = a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$

$$\therefore \text{ at } x = 4, \frac{dy}{dx} = \frac{a}{2}\left(4^{-\frac{1}{2}}\right) - \frac{b}{2}\left(4^{-\frac{3}{2}}\right) = \frac{a}{2}\left(\frac{1}{2}\right) - \frac{b}{2}\left(\frac{1}{8}\right) = \frac{a}{4} - \frac{b}{16}$$

$$\therefore \text{ the slope of the tangent to the curve at } x = 4 \text{ will be } \frac{a}{4} - \frac{b}{16} = \frac{4a - b}{16}$$

However the equation of the *normal* is $4x + y = 22$, i.e., $y = -4x + 22$

$\therefore$ the normal has slope -4

$$\therefore \text{ the tangent has slope } \frac{1}{4} \quad \text{and so, } \frac{4a - b}{16} = \frac{1}{4}$$

$$4a - b = 4$$

$$b = 4a - 4 \quad \dots\dots (1)$$

Also, at $x = 4$ the normal line intersects the curve.

$$\text{i.e., } a\sqrt{4} + \frac{b}{\sqrt{4}} = -4(4) + 22$$

$$\therefore 2a + \frac{b}{2} = 6$$

Consequently, $2a + \frac{4a - 4}{2} = 6$ using (1)

$$\therefore 2a + 2a - 2 = 6$$

$$\therefore 4a = 8$$

$$\therefore a = 2 \quad \text{and so } b = 4(2) - 4 = 4 \quad \text{from (1)}$$

5 a $y = \sqrt{2x+1}$ when $x = 4$, $y = \sqrt{2(4)+1} = 3$ $\therefore$ the point of contact is $(4, 3)$

Since $y = (2x+1)^{\frac{1}{2}}$ then $\frac{dy}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{2x+1}}$

$$\therefore \text{ at } x = 4, \frac{dy}{dx} = \frac{1}{\sqrt{2(4)+1}} = \frac{1}{3}$$

so the tangent has equation $\frac{y-3}{x-4} = \frac{1}{3}$ i.e., $3y = x + 5$

$$\mathbf{b} \quad y = \frac{1}{2-x} = (2-x)^{-1} \quad \text{at } x = -1, \quad y = \frac{1}{2-(-1)} = \frac{1}{3}$$

$\therefore$ the point of contact is $(-1, \frac{1}{3})$

$$\text{Now } \frac{dy}{dx} = -1(2-x)^{-2}(-1) = \frac{1}{(2-x)^2}$$

$$\therefore \text{ at } x = -1, \quad \frac{dy}{dx} = \frac{1}{(2-(-1))^2} = \frac{1}{9}$$

$$\therefore \text{ the tangent has equation } \frac{y - \frac{1}{3}}{x - (-1)} = \frac{1}{9} \quad \text{i.e., } 9y - 3 = x + 1$$

$$\text{i.e., } 9y = x + 4$$

$$\mathbf{c} \quad y = \frac{1}{(x^2+1)^2} \quad \text{at } (1, \frac{1}{4})$$

$$\text{as } y = (x^2+1)^{-2}$$

$$\frac{dy}{dx} = -2(x^2+1)^{-3}(2x) = \frac{-4x}{(x^2+1)^3}$$

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{-4}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2}$$

$\therefore$ normal at $(1, \frac{1}{4})$ has slope 2.

$$\text{So the equation of the normal is } \frac{y - \frac{1}{4}}{x - 1} = 2 \quad \text{i.e., } y - \frac{1}{4} = 2x - 2$$

$$2x - y = \frac{7}{4}$$

$$\text{i.e., } y = 2x - \frac{7}{4}$$

$$\mathbf{d} \quad y = \frac{1}{\sqrt{3-2x}} \quad \text{at } x = -3, \quad y = \frac{1}{\sqrt{3-2(-3)}} = \frac{1}{3}$$

$\therefore$ the point of contact is $(-3, \frac{1}{3})$

$$\text{Now } y = (3-2x)^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}(3-2x)^{-\frac{3}{2}}(-2) = (3-2x)^{-\frac{3}{2}}$$

$$\therefore \text{ at } x = -3, \quad \frac{dy}{dx} = (3-2(-3))^{-\frac{3}{2}} = 9^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27}$$

$\therefore$ normal at $(-3, \frac{1}{3})$ has slope -27 ,

$$\text{So the equation of the normal is } \frac{y - \frac{1}{3}}{x - (-3)} = -27 \quad \text{i.e., } y - \frac{1}{3} = -27(x+3)$$

$$\text{i.e., } y = -27x - \frac{242}{3}$$

6 The tangent has equation $3x + y = 5$, i.e., $y = -3x + 5$

$\therefore$ tangent has slope -3 (1)

$$\text{Also, at } x = -1, \quad y = -3(-1) + 5 = 8$$

$\therefore$ the tangent contacts the curve at $(-1, 8)$ (2)

$$\text{Now } y = a(1-bx)^{\frac{1}{2}}, \quad \therefore \frac{dy}{dx} = \frac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b)$$

$$\therefore -3 = \frac{1}{2}a(1+b)^{-\frac{1}{2}}(-b) \quad \text{using (1)}$$

$$6 = \frac{ab}{\sqrt{1+b}} \quad \text{..... (3)}$$

and $(-1, 8)$ must lie on the curve $y = a\sqrt{1-bx}$

$$\text{and so } 8 = a\sqrt{1+b} \quad \dots\dots (4)$$

$$\therefore \frac{6\sqrt{1+b}}{b} = \frac{8}{\sqrt{1+b}} \quad \{\text{equating } a\text{'s in (3) and (4)}\}$$

$$\therefore 6(1+b) = 8b$$

$$\therefore 6 + 6b = 8b$$

$$\therefore 6 = 2b$$

$$\therefore b = 3 \quad \text{and} \quad a = \frac{8}{\sqrt{4}} = 4$$

7 a $f(x) = \frac{x}{1-3x}$ at $(-1, -\frac{1}{4})$

$f(x)$ is a quotient where $u = x$ and $v = 1 - 3x$

$$\therefore u' = 1 \quad \text{and} \quad v' = -3$$

$$\text{Now } f'(x) = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$\therefore f'(x) = \frac{1(1-3x) - x(-3)}{(1-3x)^2} = \frac{1}{(1-3x)^2}$$

$$\therefore f'(-1) = \frac{1}{(1-3(-1))^2} = \frac{1}{16}$$

$$\text{So the tangent has equation } \frac{y - (-\frac{1}{4})}{x - (-1)} = \frac{1}{16} \quad \text{i.e., } 16y + 4 = x + 1$$

$$\text{i.e., } 16y = x - 3$$

b $f(x) = \sqrt{x}(1-x)^2$

since $f(4) = \sqrt{4}(1-4)^2 = 18$, the point of contact is $(4, 18)$

Now $f(x)$ is a product where $u = x^{\frac{1}{2}}$ and $v = (1-x)^2$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(1-x)(-1) = -2(1-x)$$

$$\text{Now } f'(x) = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^2 - x^{\frac{1}{2}}2(1-x)$$

$$\therefore f'(4) = \frac{1}{2\sqrt{4}}(1-4)^2 - \sqrt{4}(2)(1-4) = \frac{1}{4}(9) - 2(2)(-3) = \frac{57}{4}$$

$$\therefore \text{the normal at } (4, 18) \text{ has slope } -\frac{4}{57}.$$

$$\text{So the equation of the normal is } \frac{y - 18}{x - 4} = -\frac{4}{57} \quad \text{i.e., } 57(y - 18) = -4(x - 4)$$

$$\text{i.e., } 57y = -4x + 1042$$

c $f(x) = \frac{x^2}{1-x}$ at $(2, -4)$

$f(x)$ is a quotient where $u = x^2$ and $v = 1 - x$

$$u' = 2x \quad \text{and} \quad v' = -1$$

$$\text{Now } f'(x) = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$\therefore f'(x) = \frac{2x(1-x) - x^2(-1)}{(1-x)^2} = \frac{2x - 2x^2 + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}$$

$$\therefore f'(2) = \frac{2(2) - 2^2}{(1-2)^2} = \frac{4-4}{1} = 0$$

As the tangent has slope 0, i.e., is horizontal, it has equation $y = c$ and as the contact point is $(2, -4)$, the tangent has equation $y = -4$

d
$$f(x) = \frac{x^2 - 1}{2x + 3}$$

Since $f(-1) = \frac{(-1)^2 - 1}{2(-1) + 3} = \frac{0}{1} = 0$ the point of contact is $(-1, 0)$

Now $f(x)$ is a quotient where $u = x^2 - 1$ and $v = 2x + 3$

$\therefore u' = 2x$ and $v' = 2$

Now
$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{2x(2x + 3) - (x^2 - 1)(2)}{(2x + 3)^2}$$

$$\therefore f'(-1) = \frac{2(-1)(-2 + 3) - ((-1)^2 - 1)(2)}{(2(-1) + 3)^2} = \frac{-2(-1) - (0)(2)}{(1)^2} = -2$$

$\therefore$ the normal at $(-1, 0)$ has slope $\frac{1}{2}$.

So the equation of the normal is $\frac{y - 0}{x - (-1)} = \frac{1}{2}$ i.e., $2y = x + 1$

8 a $y = x^3$ at $x = 2$

Since when $x = 2$, $y = 2^3 = 8$ the point of contact is $(2, 8)$

Now $\frac{dy}{dx} = 3x^2$ and so at $x = 2$, $\frac{dy}{dx} = 3(2)^2 = 12$

$\therefore$ tangent at $(2, 8)$ has slope 12 and its equation is $\frac{y - 8}{x - 2} = 12$ i.e., $y - 8 = 12x - 24$
i.e., $y = 12x - 16$

$\therefore$ the tangent meets the curve where $12x - 16 = x^3$
i.e., $x^3 - 12x + 16 = 0$

Because the tangent touches the curve at $x = 2$, there must be a repeated solution at this point. $\therefore (x - 2)^2$ must be a factor of this cubic

$$\therefore (x - 2)^2(x + 4) = 0$$

$\therefore$ tangent meets curve again when $x = -4$

and when $x = -4$, $y = (-4)^3 = -64$

$\therefore$ tangent meets curve again at $(-4, -64)$

b $y = -x^3 + 2x^2 + 1$ at $x = -1$

Since when $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1 = 4$

and so the point of contact is $(-1, 4)$

Now $\frac{dy}{dx} = -3x^2 + 4x$ $\therefore$ at $x = -1$, $\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$

$\therefore$ tangent at $(-1, 4)$ has slope -7

$\therefore$ its equation is $\frac{y - 4}{x - (-1)} = -7$ $\therefore y - 4 = -7(x + 1)$
i.e., $y = -7x - 3$

Now the tangent meets the curve where $-7x - 3 = -x^3 + 2x^2 + 1$
i.e., $x^3 - 2x^2 - 7x - 4 = 0$

Because the tangent touches the curve at $x = -1$, there must be a repeated solution at this point. $\therefore (x + 1)^2$ must be a factor of this cubic

$$\therefore (x + 1)^2(x - 4) = 0$$

$\therefore$ tangent meets curve again when $x = 4$

and when $x = 4$, $y = -(4)^3 + 2(4)^2 + 1 = -64 + 32 + 1 = -31$

$\therefore$ tangent meets curve again at $(4, -31)$

c $y = x^2 - \frac{3}{x}$ at $x = 3$

Since when $x = 3$, $y = 3^2 - \frac{3}{3} = 8$, the point of contact is $(3, 8)$

Now $\frac{dy}{dx} = 2x + \frac{3}{x^2}$

$\therefore$ at $x = 3$, $\frac{dy}{dx} = 2(3) + \frac{3}{3^2} = 6 + \frac{1}{3} = \frac{19}{3}$

$\therefore$ tangent at $(3, 8)$ has slope $\frac{19}{3}$ and therefore its equation is

$$\frac{y-8}{x-3} = \frac{19}{3} \quad \text{i.e., } 3(y-8) = 19(x-3)$$

$$3y = 19x - 33$$

$$\text{i.e., } y = \frac{19}{3}x - 11$$

Now the tangent meets the curve where $\frac{19}{3}x - 11 = x^2 - \frac{3}{x}$

$$\therefore 19x^2 - 33x = 3x^3 - 9$$

$$\text{i.e., } 3x^3 - 19x^2 + 33x - 9 = 0$$

Because the tangent touches the curve at $x = 3$, there must be a repeated solution at this point.

Hence, $3x^3 - 19x^2 + 33x - 9 = (x-3)^2(3x-1) = 0$

{since coefficient of x^3 is 3 and constant term is -9 }

$\therefore$ the tangent meets the curve again at $x = \frac{1}{3}$ where $y = \left(\frac{1}{3}\right)^2 - \frac{3}{\left(\frac{1}{3}\right)} = \frac{1}{9} - 9 = -\frac{80}{9}$

$\therefore$ tangent meets curve again at $\left(\frac{1}{3}, -\frac{80}{9}\right)$

d $y = x^3 + \frac{4}{x}$ at $x = 1$

Since when $x = 1$, $y = 1^3 + \frac{4}{1} = 5$ and so the point of contact is $(1, 5)$

Now $\frac{dy}{dx} = 3x^2 - \frac{4}{x^2}$ and at $x = 1$, $\frac{dy}{dx} = 3 - 4 = -1$

$\therefore$ tangent at $(1, 5)$ has slope -1 and therefore its equation is

$$\frac{y-5}{x-1} = -1 \quad \text{i.e., } y-5 = -x+1$$

$$\text{i.e., } y = -x+6$$

Now the tangent meets the curve where $-x+6 = x^3 + \frac{4}{x}$

$$\therefore x^3 + x - 6 + \frac{4}{x} = 0$$

$$\therefore x^4 + x^2 - 6x + 4 = 0$$

Using a graphics calculator, this quartic has a graph which touches the x -axis at $x = 1$ and has no other x -intercepts. i.e., the tangent *never* meets the curve again.

9 a $y = x^2 - x + 9$ at $x = a$

Since when $x = a$, $y = a^2 - a + 9$ the point of contact is $(a, a^2 - a + 9)$

Now $\frac{dy}{dx} = 2x - 1$ $\therefore$ at $x = a$, $\frac{dy}{dx} = 2a - 1$

$\therefore$ the slope of the tangent at $(a, a^2 - a + 9)$ is $2a - 1$

$\therefore$ the equation of the tangent is $\frac{y - (a^2 - a + 9)}{x - a} = 2a - 1$

$$\text{i.e., } y - (a^2 - a + 9) = (2a - 1)(x - a)$$

$$\text{i.e., } y = (2a - 1)x - 2a^2 + a + a^2 - a + 9$$

$$\text{i.e., } y = (2a - 1)x - a^2 - 9$$

But this tangent passes through (0, 0) $\therefore 0 = a^2 - 9$

$$\therefore (a+3)(a-3) = 0$$

$$\therefore a = \pm 3$$

$\therefore$ tangents are: at $a = 3$: $y - (9 - 3 + 9) = 5(x - 3)$

$$\therefore y = 5x, \text{ with contact at } (3, 15)$$

at $a = -3$: $y - (9 + 3 + 9) = -7(x + 3)$

$$\therefore y = -7x, \text{ with contact at } (-3, 21)$$

points of contact are (3, 15) and (-3, 21)

b Let (a, a^3) lie on $y = x^3$

$$\text{Now } \frac{dy}{dx} = 3x^2, \therefore \text{ at } x = a, \frac{dy}{dx} = 3a^2$$

$\therefore$ the slope of the tangent at (a, a^3) is $3a^2$

$$\therefore \text{ the equation of the tangent is } \frac{y - a^3}{x - a} = 3a^2 \quad \text{i.e., } y - a^3 = (3a^2)(x - a)$$

But this tangent passes through (-2, 0) $\therefore a - a^3 = 3a^2(-2 - a)$

$$\therefore -a^3 = -6a^2 - 3a^3$$

$$\therefore 2a^3 + 6a^2 = 0$$

$$\therefore 2a^2(a + 3) = 0$$

$$\therefore a = 0 \text{ or } -3$$

If $a = 0$, tangent equation is $y = 0$, with contact point (0, 0)

If $a = -3$, tangent equation is $y - (-27) = 27(x + 3)$

$$\text{i.e., } y = 27x + 54, \text{ with contact point } (-3, -27)$$

c Let $(a, \sqrt{a})$ lie on $y = \sqrt{x}$.

$$\text{Now } \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \therefore \text{ at } x = a, \frac{dy}{dx} = \frac{1}{2\sqrt{a}}$$

$\therefore$ the slope of the tangent at $(a, \sqrt{a})$ is $\frac{1}{2\sqrt{a}}$

$\therefore$ the slope of the normal at $(a, \sqrt{a})$ is $-2\sqrt{a}$

$\therefore$ the normal has equation $\frac{y - \sqrt{a}}{x - a} = -2\sqrt{a}$ i.e., $y - \sqrt{a} = -2\sqrt{a}(x - a)$

But this normal passes through (4, 0) $\therefore 0 - \sqrt{a} = -2\sqrt{a}(4 - a)$

$$\therefore 2\sqrt{a}(4 - a) - \sqrt{a} = 0$$

$$\therefore \sqrt{a}(8 - 2a - 1) = 0$$

$$\therefore \sqrt{a}(7 - 2a) = 0$$

$$\therefore a = 0 \text{ or } \frac{7}{2}$$

when $a = 0$, normal equation is $y = 0$, and contact point is (0, 0)

when $a = \frac{7}{2}$, $y - \sqrt{\frac{7}{2}} = -2\sqrt{\frac{7}{2}}\left(x - \frac{7}{2}\right)$

$$\text{i.e., } \sqrt{2}y - \sqrt{7} = -2\sqrt{7}\left(x - \frac{7}{2}\right)$$

$$\text{i.e., } \sqrt{2}y + 2\sqrt{7}x = 7\sqrt{7} + \sqrt{7}$$

$$\text{i.e., } \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$$

$$\text{i.e., } y = -\sqrt{14}x + 4\sqrt{14} \quad \text{with contact point } \left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$$

EXERCISE 21H

1 a $f(x) = 3x^2 - 6x + 2$

$$\therefore f'(x) = 6x - 6$$

$$\therefore f''(x) = 6$$

b $f(x) = 2x^3 - 3x^2 - x + 5$

$$\therefore f'(x) = 6x^2 - 6x - 1 + 0$$

$$\therefore f''(x) = 12x - 6$$

c $f(x) = \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1$

$$\therefore f'(x) = -x^{-\frac{3}{2}}$$

$$\therefore f''(x) = \frac{3}{2}x^{-\frac{5}{2}}$$

d $f(x) = \frac{2-3x}{x^2} = 2x^{-2} - 3x^{-1}$

$$\therefore f'(x) = -4x^{-3} + 3x^{-2}$$

$$\begin{aligned}\therefore f''(x) &= 12x^{-4} - 6x^{-3} \\ &= \frac{12-6x}{x^4}\end{aligned}$$

e $f(x) = (1-2x)^3$

$$\begin{aligned}\therefore f'(x) &= 3(1-2x)^2(-2) \\ &= -6(1-2x)^2\end{aligned}$$

$$\therefore f''(x) = -12(1-2x)^1(-2) = 24(1-2x)$$

f $f(x) = \frac{x+2}{2x-1}$ is a quotient where $u = x+2$ and $v = 2x-1$

$$\therefore u' = 1 \quad \text{and} \quad v' = 2$$

$$\therefore f'(x) = \frac{1(2x-1) - 2(x+2)}{(2x-1)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{-5}{(2x-1)^2}$$

$$= -5(2x-1)^{-2}$$

$$\text{and } f''(x) = 10(2x-1)^{-3}(2)$$

$$= \frac{20}{(2x-1)^3}$$

2 a $y = x - x^3$

$$\therefore \frac{dy}{dx} = 1 - 3x^2$$

$$\therefore \frac{d^2y}{dx^2} = -6x$$

b $y = x^2 - \frac{5}{x^2}$

$$= x^2 - 5x^{-2}$$

$$\therefore \frac{dy}{dx} = 2x + 10x^{-3}$$

$$\therefore \frac{d^2y}{dx^2} = 2 - 30x^{-4} = 2 - \frac{30}{x^4}$$

c $y = 2 - \frac{3}{\sqrt{x}}$

$$= 2 - 3x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}x^{-\frac{3}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{9}{4}x^{-\frac{5}{2}}$$

d $y = \frac{4-x}{x} = 4x^{-1} - 1$

$$\therefore \frac{dy}{dx} = -4x^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 8x^{-3} = \frac{8}{x^3}$$

e $y = (x^2 - 3x)^3$

$$\therefore \frac{dy}{dx} = 3(x^2 - 3x)^2(2x - 3) = (6x - 9)(x^2 - 3x)^2$$

which is a product where $u = 6x - 9$ and $v = (x^2 - 3x)^2$

$$\therefore u' = 6 \quad \text{and} \quad v' = 2(x^2 - 3x)^1(2x - 3)$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= 6(x^2 - 3x)^2 + (6x - 9)(2)(x^2 - 3x)(2x - 3) \\ &= 6(x^2 - 3x) [(x^2 - 3x) + (2x - 3)^2] \\ &= 6(x^2 - 3x)(x^2 - 3x + 4x^2 - 12x + 9) \\ &= 6(x^2 - 3x)(5x^2 - 15x + 9)\end{aligned}$$

f $y = x^2 - x + \frac{1}{1-x} = x^2 - x + (1-x)^{-1}$

$$\therefore \frac{dy}{dx} = 2x - 1 + (-1)(1-x)^{-2}(-1) = 2x - 1 + (1-x)^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 2 - 2(1-x)^{-3}(-1) = 2 + \frac{2}{(1-x)^3}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & f(x) = 2x^3 - 6x^2 + 5x + 1 \\
 & \therefore f'(x) = 6x^2 - 12x + 5 \\
 & \therefore f''(x) = 12x - 12 \quad \text{and so } f''(x) = 0 \quad \text{when } 12x - 12 = 0 \\
 & \qquad \qquad \qquad \therefore 12x = 12 \\
 & \qquad \qquad \qquad \text{i.e., when } x = 1
 \end{aligned}$$

$$\mathbf{b} \quad f(x) = \frac{x}{x^2 + 2} \quad \text{is a quotient where } u = x \quad \text{and } v = x^2 + 2$$

$$\therefore u' = 1, \quad v' = 2x$$

$$\therefore f'(x) = \frac{1(x^2 + 2) - 2x^2}{(x^2 + 2)^2} \quad \{\text{using the quotient rule}\}$$

$$= \frac{2 - x^2}{(x^2 + 2)^2} \quad \text{which is another quotient}$$

$$\text{with } u = 2 - x^2 \quad \text{and } v = (x^2 + 2)^2$$

$$\therefore u' = -2x, \quad v' = 2(x^2 + 2)(2x)$$

$$f''(x) = \frac{-2x(x^2 + 2)^2 - 4x(x^2 + 2)(2 - x^2)}{(x^2 + 2)^4}$$

$$= \frac{-2x(x^2 + 2)[x^2 + 2 + 2(2 - x^2)]}{(x^2 + 2)^4}$$

$$= \frac{-2x[-x^2 + 6]}{(x^2 + 2)^3}$$

$$= \frac{2x[x^2 - 6]}{(x^2 + 2)^3} \quad \text{and so } f''(x) = 0 \quad \text{when } 2x[x^2 - 6] = 0$$

$$\text{i.e., } x = 0 \quad \text{or } x^2 - 6 = 0$$

$$\text{i.e., when } x = 0 \quad \text{or } x = \pm\sqrt{6}$$

REVIEW SET 21A

$$\mathbf{1} \quad y = -2x^2. \quad \text{Since when } x = -1, \quad y = -2(-1)^2 = -2, \quad \text{the point of contact is } (-1, -2).$$

$$\text{Now } \frac{dy}{dx} = -4x$$

$$\therefore \text{ at } x = -1, \quad \frac{dy}{dx} = -4(-1) = 4$$

$$\therefore \text{ tangent has equation } \frac{y - (-2)}{x + 1} = 4 \quad \text{i.e., } y + 2 = 4x + 4$$

$$\text{i.e., } y = 4x + 2$$

$$\mathbf{2} \quad \mathbf{a} \quad y = 3x^2 - x^4$$

$$\therefore \frac{dy}{dx} = 6x - 4x^3$$

$$\mathbf{b} \quad y = \frac{x^3 - x}{x^2} = x - x^{-1}$$

$$\therefore \frac{dy}{dx} = 1 + x^{-2} = 1 + \frac{1}{x^2}$$

$$\mathbf{3} \quad f(x) = x^2 + 2x, \quad \therefore f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2h}{h}$$

$$= \lim_{h \rightarrow 0} (2x + 2 + h) \quad \{\text{as } h \neq 0\}$$

$$= 2x + 2 \quad \text{Checking: } f(x) = x^2 + 2x, \quad \therefore f'(x) = 2x + 2 \quad \checkmark$$

4 $y = \frac{1-2x}{x^2}$. Since when $x = 1$, $y = \frac{1-2(1)}{1^2} = -1$ the point of contact is $(1, -1)$

Since $y = \frac{1}{x^2} - \frac{2}{x}$ then $\frac{dy}{dx} = -2x^{-3} + 2x^{-2} = -\frac{2}{x^3} + \frac{2}{x^2}$

$\therefore$ at $x = 1$, $\frac{dy}{dx} = -2 + 2 = 0$

i.e., the tangent is a horizontal line. $\therefore$ the normal is a vertical line, of the form $x = k$.

As the normal passes through $(1, -1)$ its equation must be $x = 1$

5 $y = 2x^3 + 4x - 1$ at $(1, 5)$

Now $\frac{dy}{dx} = 6x^2 + 4$ $\therefore$ at $x = 1$, $\frac{dy}{dx} = 6(1)^2 + 4 = 10$

$\therefore$ the tangent has equation $\frac{y-5}{x-1} = 10$ i.e., $y = 10x - 5$

Now the tangent meets the curve again where $10x - 5 = 2x^3 + 4x - 1$

$$2x^3 - 6x + 4 = 0$$

$$x^3 - 3x + 2 = 0$$

We know that $(x-1)^2$ is a factor since the line is tangent to the curve (i.e., touches) at $x = 1$

Consequently, $x^3 - 3x + 2 = (x-1)^2(x+2) = 0$ {since the constant term is 2}

Thus $x = -2$ is the other solution and when $x = -2$, $y = 2(-2)^3 + 4(-2) - 1 = -25$

$\therefore$ tangent meets curve again at $(-2, -25)$

6 $y = \frac{ax+b}{\sqrt{x}} = a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}} = \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}$

The equation of the tangent at $x = 1$ is $2x - y = 1$

i.e., $y = 2x - 1$ $\therefore$ the slope is 2

$\therefore$ at $x = 1$, $\frac{dy}{dx} = \frac{a}{2} - \frac{b}{2} = 2$

$\therefore a - b = 4$

$\therefore a = b + 4$ (1)

Also, at $x = 1$ the tangent touches the curve i.e., $\frac{a(1)+b}{\sqrt{1}} = 2(1) - 1$

i.e., $a + b = 1$

$\therefore b + 4 + b = 1$ {using (1)}

$\therefore 2b = -3$

$\therefore b = -\frac{3}{2}$

$\therefore a = 4 - \frac{3}{2} = \frac{5}{2}$

i.e., $a = \frac{5}{2}$, $b = -\frac{3}{2}$

7 $y = 4(ax+1)^{-2}$

Since when $x = 0$, $y = 4(0+1)^{-2} = 4$, the point of contact is $(0, 4)$

Now $\frac{dy}{dx} = -8(ax+1)^{-3}(a) = \frac{-8a}{(ax+1)^3}$ $\therefore$ at $x = 0$, $\frac{dy}{dx} = -8a$

$\therefore$ the tangent has equation $\frac{y-4}{x-0} = -8a$ i.e., $y - 4 = -8ax$

and this passes through $(1, 0)$ $\therefore 0 - 4 = -8a(1)$ $\therefore a = \frac{1}{2}$

8 $y = \frac{1}{\sqrt{x}}$ at $x = 4$

Since when $x = 4$ $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$ the point of contact is $(4, \frac{1}{2})$

Now $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$ $\therefore$ at $x = 4$, $\frac{dy}{dx} = -\frac{1}{2}\left(4^{-\frac{3}{2}}\right) = -\frac{1}{2}\left(\frac{1}{8}\right) = -\frac{1}{16}$

$\therefore$ the normal at $(4, \frac{1}{2})$ has slope 16.

So the equation is $\frac{y - \frac{1}{2}}{x - 4} = 16$ i.e., $y - \frac{1}{2} = 16x - 64$ i.e., $y = 16x - \frac{127}{2}$

9 a $M = (t^2 + 3)^4$
 $\frac{dM}{dt} = 4(t^2 + 3)^3(2t)$
 $\frac{dM}{dt} = 8t(t^2 + 3)^3$

b $A = \frac{\sqrt{t+5}}{t^2}$ is a quotient with
 $u = (t+5)^{\frac{1}{2}}$ and $v = t^2$
 $\therefore u' = \frac{1}{2}(t+5)^{-\frac{1}{2}}, v' = 2t$
 $\frac{dA}{dt} = \frac{\left(\frac{1}{2\sqrt{t+5}}\right)t^2 - \sqrt{t+5}(2t)}{t^4}$
 $= \frac{t^2 - 2(2t)(t+5)}{2t^4\sqrt{t+5}}$
 $= \frac{t - 4(t+5)}{2t^3\sqrt{t+5}}$
 $= \frac{-3t - 20}{2t^3\sqrt{t+5}}$

10 a $y = \frac{4}{\sqrt{x}} - 3x = 4x^{-\frac{1}{2}} - 3x$
 $\therefore \frac{dy}{dt} = -2x^{-\frac{3}{2}} - 3 = \frac{-2}{x\sqrt{x}} - 3$

b $y = \left(x - \frac{1}{x}\right)^4 = (x - x^{-1})^4$
 $\therefore \frac{dy}{dx} = 4(x - x^{-1})^3(1 + x^{-2})$
 $= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right)$

c $y = \sqrt{x^2 - 3x} = (x^2 - 3x)^{\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = \frac{1}{2}(x^2 - 3x)^{-\frac{1}{2}}(2x - 3)$
 $= \frac{2x - 3}{2\sqrt{x^2 - 3x}}$

REVIEW SET 21B

1 a $y = 5x - 3x^{-1}$
 $\frac{dy}{dx} = 5 + 3x^{-2} = 5 + \frac{3}{x^2}$

b $y = (3x^2 + x)^4$
 $\frac{dy}{dx} = 4(3x^2 + x)^3(6x + 1)$

c $y = (x^2 + 1)(1 - x^2)^3$ is a product with $u = x^2 + 1$ and $v = (1 - x^2)^3$
 $\therefore u' = 2x$ and $v' = 3(1 - x^2)^2(-2x)$
 $\frac{dy}{dx} = 2x(1 - x^2)^3 - 6x(x^2 + 1)(1 - x^2)^2$ {using the product rule}

2 $y = x^3 - 3x^2 - 9x + 2$, $\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$

horizontal tangents occur when $\frac{dy}{dx} = 0$ i.e., when $3x^2 - 6x - 9 = 0$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$\therefore x = 3 \text{ or } x = -1$$

when $x = 3$, horizontal tangent has equation $y = -25$

when $x = -1$, horizontal tangent has equation $y = 7$

3 $y = \frac{x+1}{x^2-2}$ and since when $x = 1$, $y = \frac{1+1}{1^2-2} = -2$, the point of contact is $(1, -2)$

Now $y = \frac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2-2$

$\therefore u' = 1$ and $v' = 2x$

$$\frac{dy}{dx} = \frac{1(x^2-2) - (x+1)(2x)}{(x^2-2)^2} \quad \{\text{using the quotient rule}\}$$

$\therefore$ at $x = 1$, $\frac{dy}{dx} = \frac{(1^2-2) - (1+1)2(1)}{(1^2-2)^2} = \frac{(-1)-4}{1} = -5$

$\therefore$ the normal at $(1, -2)$ has slope $\frac{1}{5}$

$\therefore$ the equation of the normal is $\frac{y - (-2)}{x - 1} = \frac{1}{5}$ i.e., $5y = x - 11$

4 a $f(x) = \frac{(x+3)^3}{\sqrt{x}} = (x+3)^3 x^{-\frac{1}{2}}$ which is a product with $u = (x+3)^3$ and $v = x^{-\frac{1}{2}}$

$\therefore u' = 3(x+3)^2$ and $v' = -\frac{1}{2}x^{-\frac{3}{2}}$

$\therefore f'(x) = \frac{3(x+3)^2}{\sqrt{x}} - \frac{(x+3)^3}{2x\sqrt{x}} \quad \{\text{using the product rule}\}$

b $f(x) = x^4 \sqrt{x^2+3}$ which is a product with $u = x^4$ and $v = (x^2+3)^{\frac{1}{2}}$

$\therefore u' = 4x^3$ and $v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x) = x(x^2+3)^{-\frac{1}{2}}$

$\therefore f'(x) = 4x^3 \sqrt{x^2+3} + \frac{x^5}{\sqrt{x^2+3}} \quad \{\text{using the product rule}\}$

5 a $f(x) = \sqrt{x}(1-x)^2$ is a product with $u = x^{\frac{1}{2}}$ and $v = (1-x)^2$

$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-x)(-1)$

$\therefore f'(x) = \frac{(1-x)^2}{2\sqrt{x}} - 2\sqrt{x}(1-x) \quad \{\text{product rule}\}$

b $f(x) = \sqrt{3x-x^2} = (3x-x^2)^{\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2}(3x-x^2)^{-\frac{1}{2}}(3-2x)$

c $f(x) = \frac{1}{2-x} = (2-x)^{-1}$

$\therefore f'(x) = -1(2-x)^{-2}(-1)$

$= \frac{1}{(2-x)^2}$

6 a $f(x) = 3x^2 - \frac{1}{x} = 3x^2 - x^{-1}$

$\therefore f'(x) = 6x + x^{-2}$

and $f''(x) = 6 - 2x^{-3} = 6 - \frac{2}{x^3}$

b $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

7 $y = x^2 \sqrt{1-x}$ at $x = -3$

Since when $x = -3$, $y = (-3)^2 \sqrt{1-(-3)} = 9\sqrt{4} = 18$,

the point of contact is $(-3, 18)$

Also, $y = x^2 \sqrt{1-x}$ is a product, with $u = x^2$ and $v = (1-x)^{\frac{1}{2}}$

$\therefore u' = 2x$ and $v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$$\therefore \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} - x^2 \left(\frac{1}{2}(1-x)^{-\frac{1}{2}} \right)$$

$$\therefore \text{ at } x = -3, \frac{dy}{dx} = 2(-3)(1-(-3))^{\frac{1}{2}} - (-3)^2 \left(\frac{1}{2} \right) (1-(-3))^{-\frac{1}{2}} = -6(2) - 9\left(\frac{1}{2}\right)^2 = -\frac{57}{4}$$

$$\therefore \text{ the tangent at } (-3, 18) \text{ has equation } \frac{y-18}{x-(-3)} = -\frac{57}{4} \quad \text{i.e., } 4y-72 = -57x-171$$

$$\text{i.e., } 4y = -57x-99$$

$$\text{Now when } x = 0, y = -\frac{99}{4} \text{ and when } y = 0, x = -\frac{99}{57}$$

$$\therefore \text{ area of } \triangle OAB = \frac{1}{2} \left(\frac{99}{4} \right) \left(\frac{99}{57} \right) \div 21.5 \text{ units}^2$$

$$\mathbf{8} \quad y = x^3 + ax + b \quad \therefore \frac{dy}{dx} = 3x^2 + a \quad \text{and so at } x = 1, \frac{dy}{dx} = 3 + a$$

Now the equation of the tangent at $x = 1$ is $y = 2x$

$$\therefore \text{ the slope is } 2 \quad \therefore 3 + a = 2 \quad \text{and so } a = -1$$

Also, at $x = 1$ the tangent touches the curve.

$$\text{i.e., } x^3 + ax + b = 2x \quad \text{when } x = 1$$

$$\therefore (1)^3 + (-1)(1) + b = 2(1)$$

$$\therefore 1 - 1 + b = 2$$

$$\therefore b = 2 \quad \text{and so } a = -1.$$

$$\mathbf{9} \quad y = x^3 + ax^2 - 4x + 3 \quad \therefore \frac{dy}{dx} = 3x^2 + 2ax - 4$$

The tangent at $x = 1$ is parallel to $y = 3x$, so at $x = 1$, $\frac{dy}{dx} = 3$

$$\text{i.e., } 3 = 3(1)^2 + 2a(1) - 4$$

$$\therefore 2a = 4$$

$$\therefore a = 2$$

$$\text{Since when } x = 1, y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$$

The contact point is (1, 2) and since the slope is 3, the tangent at (1, 2) has equation

$$\frac{y-2}{x-1} = 3 \quad \text{i.e., } y-2 = 3x-3$$

$$\text{i.e., } y = 3x-1$$

$$\text{The tangent meets the curve where } x^3 + 2x^2 - 4x + 3 = 3x - 1$$

$$\text{i.e., } x^3 + 2x^2 - 7x + 4 = 0$$

Since the line touches the curve (i.e., is tangent to it) at $x = 1$, $(x-1)^2$ must be a factor.

Consequently, $x^3 + 2x^2 - 7x + 4 = (x-1)^2(x+4) = 0$ {since the constant term is 4}

$$\therefore \text{ the curve cuts the tangent at } x = -4, y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$$

$$\text{i.e., at } (-4, -13)$$

$$\mathbf{10} \quad f(x) = 2x^3 + Ax + B \quad \therefore f'(x) = 6x^2 + A$$

Now as the slope at $(-2, 33)$ is 10,

$$\therefore f'(-2) = 10$$

$$\therefore 10 = 6(-2)^2 + A$$

$$\therefore A = -14$$

$$\therefore f(x) = 2x^3 - 14x + B$$

and as $(-2, 33)$ lies on the curve

$$f(-2) = 33$$

$$\therefore 2(-2)^3 - 14(-2) + B = 33$$

$$\therefore -16 + 28 + B = 33$$

$$\therefore B = 49 - 28$$

$$\therefore B = 21$$

REVIEW SET 21C

1 a $y = x^3\sqrt{1-x^2}$ is a product where $y = x^3$ and $v = (1-x^2)^{\frac{1}{2}}$

$$\therefore u' = 3x^2 \quad \text{and} \quad v' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -x(1-x^2)^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = 3x^2\sqrt{1-x^2} - \frac{x^4}{\sqrt{1-x^2}} \quad \{\text{using the product rule}\}$$

b $y = \frac{x^2-3x}{\sqrt{x+1}}$ is a quotient where $u = x^2-3x$ and $v = (x+1)^{\frac{1}{2}}$

$$u' = 2x-3 \quad \text{and} \quad v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1} \quad \{\text{using the quotient rule}\}$$

2 $y = \frac{x+1}{x^2-2}$ at $x = 1$

Since when $x = 1$, $y = \frac{1+1}{1^2-2} = -2$, the point of contact is $(1, -2)$

$y = \frac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2-2$

$$\therefore u' = 1 \quad \text{and} \quad v' = 2x$$

$$\therefore \frac{dy}{dx} = \frac{1(x^2-2) - (x+1)2x}{(x^2-2)^2} \quad \{\text{using the quotient rule}\}$$

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{(1-2) - 2(1+1)}{(1-2)^2} = \frac{-1-4}{1} = -5$$

$\therefore$ the normal at $(1, -2)$ has slope $\frac{1}{5}$.

So the normal has equation: $\frac{y - (-2)}{x - 1} = \frac{1}{5}$ i.e., $5y + 10 = x - 1$
i.e., $5y = x - 11$

3 $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$

$$\therefore f'(x) = 8x^3 - 12x^2 - 18x + 4$$

$$\therefore f''(x) = 24x^2 - 24x - 18, \quad \therefore f''(x) = 0 \quad \text{where} \quad 24x^2 - 24x - 18 = 0$$

$$4x^2 - 4x - 3 = 0$$

$$(2x+1)(2x-3) = 0$$

$$\therefore x = -\frac{1}{2} \quad \text{or} \quad x = \frac{3}{2}$$

4 $f(x) = \frac{3x}{1+x}$ at $(2, 2)$

$f(x)$ is the product of u and v where $u = 3x$ and $v = 1+x$ $\therefore u' = 3$ and $v' = 1$

$$\therefore \text{ by the quotient rule } f'(x) = \frac{3(1+x) - 1(3x)}{(1+x)^2} = \frac{3}{(1+x)^2} \quad \therefore f'(2) = \frac{3}{9} = \frac{1}{3}$$

$\therefore$ the normal at $(2, 2)$ has slope -3

So the equation of the normal is $\frac{y-2}{x-2} = -3$ $\therefore y-2 = -3(x-2)$

$$\text{i.e., } y-2 = -3x+6$$

$$\text{i.e., } y = -3x+8$$

$\therefore$ when $x = 0$, $y = 8$ and when $y = 0$, $x = \frac{8}{3}$

$$\begin{aligned}
 \therefore \text{ B and C are at } (0, 8) \text{ and } \left(\frac{8}{3}, 0\right), \text{ and the distance BC} &= \sqrt{\left(0 - \frac{8}{3}\right)^2 + (8 - 0)^2} \\
 &= \sqrt{\frac{64}{9} + 64} \\
 &= \sqrt{\frac{640}{9}} \\
 &= \frac{8\sqrt{10}}{3} \text{ units}
 \end{aligned}$$

5 a $y = \frac{x^2}{3-2x}$ is a quotient with $u = x^2$ and $v = 3-2x$

$$\therefore u' = 2x \text{ and } v' = -2$$

$$\frac{dy}{dx} = \frac{2x(3-2x) - x^2(-2)}{(3-2x)^2} = \frac{6x-2x^2}{(3-2x)^2} \quad \{\text{using the quotient rule}\}$$

b $y = \sqrt{x}(x^2-x)^3$ is a product with $u = x^{\frac{1}{2}}$ and $v = (x^2-x)^3$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 3(x^2-x)^2(2x-1)$$

$$\therefore \frac{dy}{dx} = \frac{(x^2-x)^3}{2\sqrt{x}} + 3\sqrt{x}(2x-1)(x^2-x)^2 \quad \{\text{using the product rule}\}$$

6 a $y = 3x^4 - \frac{2}{x} = 3x^4 - 2x^{-1}$

$$\therefore \frac{dy}{dx} = 12x^3 + 2x^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 36x^2 - 4x^{-3} = 36x^2 - \frac{4}{x^3}$$

b $y = x^3 - x + \frac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = 3x^2 - 1 - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = 6x + \frac{3}{4}x^{-\frac{5}{2}}$$

7 $y = \frac{x}{\sqrt{1-x}}$ at $x = -3$.

Since when $x = -3$, $y = \frac{-3}{\sqrt{1-(-3)}} = -\frac{3}{2}$ the point of contact is $(-3, -\frac{3}{2})$

Now $y = \frac{x}{\sqrt{1-x}}$ is a quotient with $u = x$ and $v = (1-x)^{\frac{1}{2}}$

$$\therefore u' = 1, \quad v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$\therefore \frac{dy}{dx} = \frac{(1)\sqrt{1-x} - x(\frac{1}{2})(1-x)^{-\frac{1}{2}}(-1)}{1-x} \quad \{\text{quotient rule}\}$$

$$\text{at } x = -3, \quad \frac{dy}{dx} = \frac{\sqrt{4} - (-3)(\frac{1}{2})(4)^{-\frac{1}{2}}(-1)}{4} = \frac{2 - 3(\frac{1}{2})(\frac{1}{2})}{4} = \frac{2 - \frac{3}{4}}{4} = \frac{5}{16}$$

$$\therefore \text{ the tangent at } (-3, -\frac{3}{2}) \text{ has equation } \frac{y - (-\frac{3}{2})}{x - (-3)} = \frac{5}{16} \quad \text{i.e., } 16(y + \frac{3}{2}) = 5(x + 3)$$

$$16y + 24 = 5x + 15$$

$$\text{i.e., } 5x - 16y = 9$$

$$\therefore b = -16 \text{ and } a = 9$$

8 $f(x) = 3x^3 + Ax^2 + B$

As the point $(-2, 14)$ lies on the curve, $14 = -24 + 4A + B$

$$4A + B = 38 \quad \dots\dots (1)$$

Now $f'(x) = 9x^2 + 2Ax$ and as $f'(-2) = 0$

$$36 - 4A = 0$$

$$\therefore A = 9$$

$$\therefore 4(9) + B = 38 \quad \{\text{using (1)}\}$$

$$\therefore B = 38 - 36 = 2$$

That is, $A = 9$ and $B = 2$

$$\therefore f'(x) = 9x^2 + 18x$$

$$\therefore f''(x) = 18x + 18 \quad \text{and so} \quad f''(-2) = -36 + 18 = -18$$

$$9 \quad y = \frac{a}{(x+2)^2} = a(x+2)^{-2}$$

The slope of the line AB is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 4}{0 - 2} = \frac{4}{-2} = -2$

$\therefore$ the equation of the tangent is $\frac{y - 8}{x - 0} = -2$ i.e., $y = -2x + 8$

$$\frac{dy}{dx} = -2(a)(x+2)^{-3} = -2$$

$$\therefore \frac{a}{(x+2)^3} = 1 \quad \therefore a = (x+2)^3 \quad \dots\dots (1)$$

The line AB meets the curve where $-2x + 8 = \frac{a}{(x+2)^2}$

$$\therefore -2x + 8 = \frac{(x+2)^3}{(x+2)^2} \quad \{\text{using (1)}\}$$

$$\therefore -2x + 8 = x + 2$$

$$\therefore -3x = -6$$

$$\therefore x = 2$$

$$\text{and so } a = (2+2)^3 = 64$$

10 The curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x-x^2}$ meet when $\sqrt{(3x+1)} = \sqrt{5x-x^2}$

Squaring both sides, $3x+1 = 5x-x^2$

$$\therefore x^2 - 2x + 1 = 0$$

$$\therefore (x-1)^2 = 0$$

$$\therefore x = 1$$

and when $x = 1$, $y = \sqrt{3+1} = 2$ $\therefore$ the curves meet at $(1, 2)$

Now for $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$$

$$\therefore \text{ at } (1, 2) \quad \frac{dy}{dx} = \frac{3}{2(3+1)^{\frac{1}{2}}} = \frac{3}{4}$$

and for $y = \sqrt{5x-x^2} = (5x-x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(5x-x^2)^{-\frac{1}{2}}(5-2x) = \frac{5-2x}{2\sqrt{5x-x^2}}$$

$$\therefore \text{ at } (1, 2) \quad \frac{dy}{dx} = \frac{5-2}{2\sqrt{5-1}} = \frac{3}{4}$$

i.e., the curves have the same slope of $\frac{3}{4}$ at their point of intersection.

Now the equation of the common tangent at $(1, 2)$ is $\frac{y-2}{x-1} = \frac{3}{4}$ i.e., $4(y-2) = 3(x-1)$

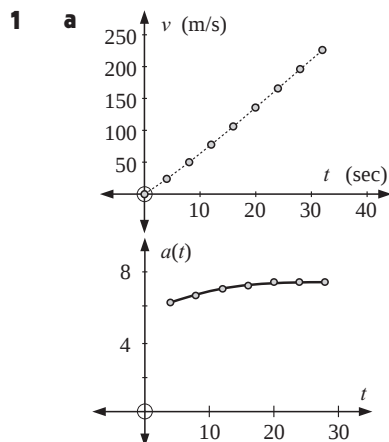
$$4y - 8 = 3x - 3$$

$$\text{i.e., } 4y = 3x + 5$$

Chapter 22

APPLICATIONS OF DIFFERENTIAL CALCULUS

EXERCISE 22A



c For $v(t) = 5.242 \times t^{1.0865}$
 $\therefore v'(t) = 5.242 \times 1.0865 t^{0.0865}$
 $\therefore v'(t) = 5.695 t^{0.0865}$

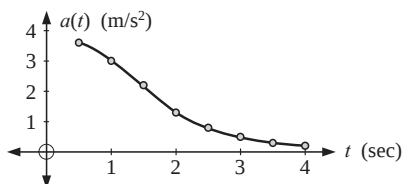
b $v'(4) \doteq \frac{v(8) - v(0)}{8} \doteq \frac{50}{8} \doteq 6.25$
 $v'(8) \doteq \frac{v(12) - v(4)}{8} \doteq \frac{54}{8} \doteq 6.75$
 $v'(12) \doteq \frac{v(16) - v(8)}{8} \doteq \frac{56}{8} \doteq 7$
 $v'(16) \doteq \frac{v(20) - v(12)}{8} \doteq \frac{58}{8} \doteq 7.25$
 $v'(20) \doteq \frac{v(24) - v(16)}{8} \doteq \frac{60}{8} \doteq 7.5$
 $v'(24) \doteq \frac{v(28) - v(20)}{8} \doteq \frac{60}{8} \doteq 7.5$
 $v'(28) \doteq \frac{v(32) - v(24)}{8} \doteq \frac{60}{8} \doteq 7.5$

$v'(4) \doteq 6.42$ $v'(20) \doteq 7.38$
 $v'(8) \doteq 6.82$ $v'(24) \doteq 7.50$
 $v'(12) \doteq 7.06$ $v'(28) \doteq 7.60$
 $v'(16) \doteq 7.24$ The fit is excellent.

2 a i $v'(1) \doteq \frac{v(1.5) - v(0.5)}{1.5 - 0.5}$
 $\doteq \frac{5.0 - 2.0}{1}$
 $\doteq 3 \text{ m/s}^2$

ii $v'(2.5) \doteq \frac{v(3) - v(2)}{3 - 2}$
 $\doteq \frac{6.6 - 5.8}{1}$
 $\doteq 0.8 \text{ m/s}^2$

b $v'(0.5) \doteq 3.6 - 0.0 \doteq 3.6$
 $v'(1.5) \doteq 5.8 - 3.6 \doteq 2.2$
 $v'(2) \doteq 6.3 - 5 \doteq 1.3$
 $v'(3) \doteq 6.8 - 6.3 \doteq 0.5$
 $v'(3.5) \doteq 6.9 - 6.6 \doteq 0.3$



EXERCISE 22B

1 $P(t) = 2t^2 - 12t + 118$ thousand dollars, $t \geq 0$

a $P(0) = \$118\,000$ is the current annual profit

b $\frac{dP}{dt} = 4t - 12$ thousand dollars/year

c $\frac{dP}{dt}$ is the rate of change in profit with time

d i Profit decreases when $\frac{dP}{dt} \leq 0$ i.e., $4t - 12 \leq 0$
 $\therefore 4t \leq 12$
i.e., $t \leq 3$

But $t \geq 0 \therefore 0 \leq t \leq 3$ years

ii Profit increases when $\frac{dP}{dt} \geq 0$ i.e., $t \geq 3$ years

- e** The profit function is a quadratic with $a > 0$ $\therefore$ shape is 
- So, a minimum profit occurs when $\frac{dP}{dt} = 0$ i.e., at $t = 3$ years
 and $P(3) = 18 - 36 + 118 = 100$ thousand dollars i.e., \$100 000.
- f** **i** When $t = 4$, $\frac{dP}{dt} = 4$ thousand dollars per year.
 So, the profit is increasing at \$4000/year after 4 years.
- ii** When $t = 10$, $\frac{dP}{dt} = 28$ thousand dollars per year.
 So, the profit is increasing at \$28 000/year after 10 years.
- iii** When $t = 25$, $\frac{dP}{dt} = 88$ thousand dollars per year.
 So, the profit is increasing at \$88 000/year after 25 years.

- 2**  Area, $A = s^2$ **a** average rate of change in A for $4 \leq s \leq 4.5$
- $$= \frac{A(4.5) - A(4)}{4.5 - 4}$$
- $$= \frac{4.5^2 - 4^2}{0.5}$$
- $$= 8.5 \text{ cm}^2 / \text{cm}$$
- b** $\frac{dA}{dt} = 2s$ So, when $s = 4$, $\frac{dA}{dt} = 8 \text{ cm}^2 / \text{cm}$

- 3** $V = 200(50 - t)^2 \text{ m}^3$
- a** average rate on $0 \leq t \leq 5$
- $$= \frac{V(5) - V(0)}{5 - 0}$$
- $$= \frac{200(45)^2 - 200(50)^2}{5}$$
- $$= -19\,000 \text{ m}^3/\text{min}$$
- i.e., leaving at $19\,000 \text{ m}^3/\text{min}$
- b** $V'(t) = 400(50 - t)^1 \times (-1)$
 $\therefore V'(5) = 400 \times 45 \times -1$
 $= -18\,000 \text{ m}^3/\text{min}$
 i.e., leaving at $18\,000 \text{ m}^3/\text{min}$

- 4** **a** $\frac{dV}{dt} = 1.2 \text{ m}^3/\text{min}$ **b** $V = \frac{4}{3}\pi r^3$
- $\therefore \frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$
- c** $\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$ {chain rule}
- $\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ **d** when $r = 3.6$
- $$1.2 = 4\pi(3.6)^2 \frac{dr}{dt}$$
- $$\therefore \frac{1.2}{4\pi(3.6)^2} = \frac{dr}{dt}$$
- $$\therefore \frac{dr}{dt} \doteq 0.007\,37 \text{ m/min}$$

- 5** $s(t) = 1.2 + 28.1t - 4.9t^2$ metres
- a** When released, $t = 0$ and $s(0) = 1.2 \text{ m}$ $\therefore$ it is released 1.2 m above the ground.
- b** $s'(t) = 28.1 - 9.8t \text{ m/s}$ is the instantaneous velocity of the ball at the time t seconds after release.
- c** When $s'(t) = 0$, $28.1 - 9.8t = 0$ $\therefore t = \frac{28.1}{9.8} \doteq 2.87 \text{ sec.}$
 So, after 2.87 sec the ball has reached its maximum height.

$$\mathbf{d} \quad s(2.867) = 1.2 + 28.1 \times 2.867 - 4.9 \times 2.867^2 \div 41.5 \text{ m}$$

So, the maximum height reached is about 41.5 m.

$$\mathbf{e} \quad \mathbf{i} \quad s'(0) = 28.1 \text{ m/s} \quad \mathbf{ii} \quad s'(2) = 28.1 - 19.6 = 8.5 \text{ m/s} \quad \mathbf{iii} \quad s'(5) = 28.1 - 49 = -20.9 \text{ m/s}$$

If $s'(t) \geq 0$, the ball is travelling upwards.

If $s'(t) \leq 0$, the ball is travelling downwards.

$$\mathbf{f} \quad s(t) = 0 \quad \text{when} \quad 1.2 + 28.1t - 4.9t^2 = 0$$

$$\text{i.e., } 4.9t^2 - 28.1t - 1.2 = 0$$

$$\therefore t = \frac{28.1 \pm \sqrt{28.1^2 - 4(4.9)(-1.2)}}{9.8}$$

$$\div -0.0424 \quad \text{or} \quad 5.777$$

$\therefore$ hits the ground after 5.78 sec.

$$\mathbf{g} \quad \frac{d^2s}{dt^2} = -9.8 \text{ m/s}^2 \quad \text{and is the constant rate of change in } \frac{ds}{dt}$$

i.e., the instantaneous acceleration is constant at -9.8 m/s^2 for the entire motion.

$$\mathbf{6} \quad \mathbf{a} \quad s(t) = bt - 4.9t^2$$

$$s'(t) = b - 9.8t$$

$$\therefore s'(0) = b$$

i.e., the initial velocity is $b \text{ m/s}$.

$$\mathbf{b} \quad \text{since } s(14.2) = 0$$

$$b(14.2) - 4.9(14.2)^2 = 0$$

$$\therefore 14.2[b - 4.9 \times 14.2] = 0$$

$$\therefore b = 4.9 \times 14.2$$

$$\therefore b \div 69.6$$

$\therefore$ the initial velocity is 69.6 m/s

EXERCISE 22C.1

$$\mathbf{1} \quad Q = 100 - 10\sqrt{t}, \quad t \geq 0$$

$$\mathbf{a} \quad \mathbf{i} \quad \text{At } t = 0, \quad Q = 100 \text{ units}$$

$$\mathbf{ii} \quad \text{At } t = 25, \quad Q = 50 \text{ units}$$

$$\mathbf{iii} \quad \text{At } t = 100, \quad Q = 0 \text{ units}$$

$$\mathbf{b} \quad \frac{dQ}{dt} = -5t^{-\frac{1}{2}} = -\frac{5}{\sqrt{t}}$$

$$\mathbf{i} \quad \text{At } t = 25, \quad \frac{dQ}{dt} = -1 \text{ units/year}$$

i.e., decreasing at 1 unit/year

$$\mathbf{ii} \quad \text{At } t = 50, \quad \frac{dQ}{dt} = -\frac{5}{\sqrt{50}}$$

$$= -\frac{1}{\sqrt{2}} \text{ units/year}$$

i.e., decreasing at $\frac{1}{\sqrt{2}}$ units/year

$$\mathbf{c} \quad \frac{dQ}{dt} = -\frac{5}{\sqrt{t}}$$

$$\therefore \frac{dQ}{dt} < 0, \quad \text{for all } t > 0, \quad \therefore \frac{dQ}{dt} \text{ is decreasing for all } t > 0$$

$$\mathbf{2} \quad H = 20 - \frac{9}{t+5} \text{ m}, \quad t \geq 0$$

$$\mathbf{a} \quad \text{At planting, } t = 0 \quad \therefore H(0) = 20 - \frac{9}{0+5} = 18.2 \text{ m}$$

$$\mathbf{b} \quad \mathbf{i} \quad \text{At } t = 4, \\ H(4) = 20 - \frac{9}{4+5} \\ = 19 \text{ m}$$

$$\mathbf{ii} \quad \text{At } t = 8, \\ H(8) = 20 - \frac{9}{8+5} \\ \div 19.3 \text{ m}$$

$$\mathbf{iii} \quad \text{At } t = 12, \\ H(12) = 20 - \frac{9}{12+5} \\ \div 19.5 \text{ m}$$

$$\mathbf{c} \quad \text{Now } \frac{dH}{dt} = 9(t+5)^{-2} = \frac{9}{(t+5)^2}$$

$$\mathbf{i} \quad \text{When } t = 0,$$

$$\frac{dH}{dt} = \frac{9}{25} = 0.36 \text{ m/yr}$$

$$\mathbf{ii} \quad \text{When } t = 5,$$

$$\frac{dH}{dt} = \frac{9}{100} = 0.09 \text{ m/yr}$$

$$\mathbf{iii} \quad \text{When } t = 10,$$

$$\frac{dH}{dt} = \frac{9}{225} = 0.04 \text{ m/yr}$$

6 a $V = x^3 \quad \therefore \quad \frac{dV}{dx} = 3x^2 \quad \text{mm}^3/\text{mm}$

This is the rate at which the volume increases as the length of side increases.

b When $x = 2$, $\frac{dV}{dx} = 3(2)^2 = 12 \text{ mm}^3/\text{mm}$

i.e., for every mm increase in length of side the volume increases by 12 mm^3 .

c When $V = x^3$, $\frac{dV}{dx} = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}A$, where A is the surface area of the cube

$$\therefore \frac{dV}{dx} \propto A$$

As the salt crystal grows, its additional volume is related exactly to the outer surface area of the crystal because as it grows it deposits on the outer surface.

7 a $V = 50\,000 \left(1 - \frac{t}{80}\right)^2, \quad 0 \leq t \leq 80$

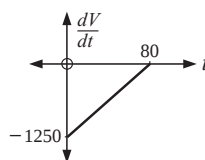
$$\frac{dV}{dt} = 2 \times 50\,000 \left(1 - \frac{t}{80}\right)^1 \times \left(-\frac{1}{80}\right) = -1250 \left(1 - \frac{t}{80}\right)$$

b Outflow was fastest when $t = 0$ (when the tap was first opened).

c As $\frac{dV}{dt} = -1250 + \frac{1250}{80}t$, then $\frac{d^2V}{dt^2} = \frac{1250}{80} = \frac{125}{8}$

and since $\frac{d^2V}{dt^2}$ is constant and positive, it shows that $\frac{dV}{dt}$ is constantly increasing

i.e., the outflow is decreasing at a constant rate.



8 a $\frac{dP}{dt} = aP \left(1 - \frac{P}{b}\right) - \left(\frac{c}{100}\right)P$ and when $\frac{dP}{dt} = 0$, the rate of change of population is zero, $\therefore$ the population is not changing and is stable.

b If $a = 0.06$, $b = 24\,000$, $c = 5$ then

$$\begin{aligned} \frac{dP}{dt} &= 0.06P \left(1 - \frac{P}{24\,000}\right) - \frac{5}{100}P \\ &= 0.06P - 0.05P - \frac{0.06P^2}{24\,000} \\ &= P \left(0.01 - \frac{P}{400\,000}\right) \end{aligned}$$

Now for a stable population

$$\frac{dP}{dt} = 0$$

$$\therefore P = 0 \quad \text{or} \quad \frac{P}{400\,000} = 0.01$$

i.e., $P = 0$ or 4000

i.e., the stable population is 4000 fish

c If the harvest rate is 4%, then $\frac{dP}{dt} = 0.06P \left(1 - \frac{P}{24\,000}\right) - \frac{4}{100}P$

$$= P \left(0.02 - \frac{0.06P}{24\,000}\right)$$

For stable population $\frac{dP}{dt} = 0 \quad \therefore \quad 0 = P \left(0.02 - \frac{0.06P}{24\,000}\right)$

$$\therefore P = 0 \quad \text{or} \quad \frac{0.06P}{24\,000} = 0.02$$

$$\therefore P = 0 \quad \text{or} \quad \frac{0.02 \times 24\,000}{0.06}$$

$$\therefore P = 0 \quad \text{or} \quad 8000$$

i.e., the stable population is 8000 fish.

EXERCISE 22C.2

1 a $C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250$

$$\therefore C'(x) = 0.0009x^2 + 0.04x + 4 \quad (\text{dollars/pair})$$

b $C'(220) = 0.0009(220)^2 + 0.04(220) + 4$
 $= \$56.36 \text{ per pair}$

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $C(221) - C(220) = \$7348.98 - \7292.40
 $= \$56.58$

This is the actual cost to make the extra pair of jeans (221 instead of 220).

d $C''(x) = 0.0018x + 0.04$

$$C''(x) = 0 \quad \text{when} \quad 0.0018x + 0.04 = 0 \quad \text{i.e.,} \quad x = -\frac{0.04}{0.0018}$$

$$\div -22.2 \quad \text{but} \quad x \geq 0$$

$\therefore$ when the rate of change is a minimum it is out of the bounds of our model (we cannot make < 0 jeans!)

2 a $C(x) = 0.000072x^3 - 0.00061x^2 + 0.19x + 893 \text{ dollars}$

$$\therefore C'(x) = 0.000216x^2 - 0.00122x + 0.19 \quad (\text{dollars/item})$$

$C'(x)$ represents the cost/item for the x items of production.

b $C'(300) = 0.000216(300)^2 - 0.00122(300) + 0.19 = \19.26

This estimates the cost of making the 301st item if 300 items are currently being made.

c Actual cost $= C(301) - C(300)$
 $= \$2858.43 - \2839.10
 $= \$19.33 \quad \text{i.e., the actual cost of the 301st item is } \19.33

d $C''(x) = 0.000432x - 0.00122 \quad \text{which is } 0 \quad \text{when} \quad 0.000432x = 0.00122$

$$x = \frac{0.00122}{0.000432}$$

$$x \div 2.82$$

$\therefore$ the rate of change in price is a minimum when $x \div 2.8$

EXERCISE 22D.1

1 a $s(t) = t^2 + 3t - 2 \quad t \geq 0$

$$\text{Average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(3) - s(1)}{3 - 1} = \frac{16 - 2}{2} = 7 \text{ ms}^{-1}$$

b Average velocity

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(1+h) - s(1)}{(1+h) - 1}$$

$$= \frac{(1+h)^2 + 3(1+h) - 2 - 2}{h}$$

$$= \frac{2h + h^2 + 3h}{h}$$

$$= (5+h) \text{ ms}^{-1}$$

c $\lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h}$

$$= \lim_{h \rightarrow 0} 5 + h$$

$$= 5 \text{ ms}^{-1}$$

This is the *instantaneous velocity* at $t = 1$ second.

$$\begin{aligned}
 \text{d Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\
 &= \frac{s(t+h) - s(t)}{(t+h) - t} \\
 &= \frac{[(t+h)^2 + 3(t+h) - 2] - [t^2 + 3t - 2]}{h} \\
 &= \frac{2ht + h^2 + 3h}{h} \\
 &= (2t + 3 + h) \text{ ms}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} &= \lim_{h \rightarrow 0} (2t + 3 + h) \\
 &= (2t + 3) \text{ ms}^{-1}
 \end{aligned}$$

This is the *instantaneous velocity* at t seconds.

2 a $s(t) = 5 - 2t^2 \text{ cm}$

$$\begin{aligned}
 \text{Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\
 &= \frac{s(5) - s(2)}{5 - 2} \\
 &= \frac{(-45) - (-3)}{3} \\
 &= -\frac{42}{3} \\
 &= -14 \text{ cm s}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{b Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\
 &= \frac{s(2+h) - s(2)}{(2+h) - 2} \\
 &= \frac{5 - 2(2+h)^2 + 3}{h} \\
 &= \frac{-8h - 2h^2}{h} \\
 &= (-8 - 2h) \text{ cm s}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h} &= \lim_{h \rightarrow 0} (-8 - 2h) \\
 &= -8 \text{ cm s}^{-1}
 \end{aligned}$$

This is the *instantaneous velocity* when $t = 2$ seconds.

$$\begin{aligned}
 \text{d } \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} &= \lim_{h \rightarrow 0} \frac{[5 - 2(t+h)^2] - [5 - 2t^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-4th - 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} (-4t - 2h) \\
 &= -4t \text{ cm s}^{-1} \\
 &\quad -4t \text{ cm s}^{-1} \text{ is the } \textit{instantaneous velocity} \text{ at } t \text{ seconds.}
 \end{aligned}$$

3 $v(t) = 2\sqrt{t} + 3 \text{ cm s}^{-1}, \quad t \geq 0$

$$\begin{aligned}
 \text{a Average acceleration} &= \frac{v(t_2) - v(t_1)}{t_2 - t_1} \\
 &= \frac{v(4) - v(1)}{4 - 1} \\
 &= \frac{7 - 5}{3} \\
 &= \frac{2}{3} \text{ cm s}^{-2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b Average acceleration} &= \frac{v(t_2) - v(t_1)}{t_2 - t_1} \\
 &= \frac{v(1+h) - v(1)}{(1+h) - 1} \\
 &= \frac{[2\sqrt{1+h} + 3] - [2\sqrt{1} + 3]}{h} \\
 &= \frac{2\sqrt{1+h} - 2}{h} \text{ cm s}^{-2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \lim_{h \rightarrow 0} \frac{v(1+h) - v(1)}{(1+h) - 1} &= \lim_{h \rightarrow 0} \frac{[2\sqrt{1+h} + 3] - [2 + 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2[\sqrt{1+h} - 1]}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\
 &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{1+h} + 1)} \\
 &= \frac{2}{2} \\
 &= 1 \text{ cm s}^{-2} \quad \text{This is the instantaneous acceleration when } t = 1 \text{ second.}
 \end{aligned}$$

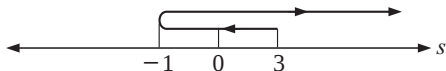
$$\begin{aligned}
 \text{d } \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h} &= \lim_{h \rightarrow 0} \frac{2\sqrt{t+h} - 2\sqrt{t}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(\sqrt{t+h} - \sqrt{t})}{h} \times \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}} \\
 &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{t+h} + \sqrt{t})} \\
 &= \frac{2}{2\sqrt{t}} \\
 &= \frac{1}{\sqrt{t}} \text{ cm s}^{-2} \quad \text{This is the instantaneous acceleration at } t \text{ seconds.}
 \end{aligned}$$

- 4 a This is the *instantaneous velocity* at $t = 3$ seconds.
 b This is the *instantaneous acceleration* at $t = 5$ seconds.
 c This is the *instantaneous velocity* at t seconds.
 d This is the *instantaneous acceleration* at t seconds.

EXERCISE 22D.2

- 1 a $s(t) = t^2 - 4t + 3 \text{ cm}, t \geq 0 \quad \therefore v(t) = 2t - 4 \text{ cm s}^{-1}$
 and $a(t) = 2 \text{ cm s}^{-2}$
- b When $t = 0, s(0) = 3 \text{ cm}$
 $v(0) = -4 \text{ cm s}^{-1}$
 $a(0) = 2 \text{ cm s}^{-2} \quad \therefore$ the object is 3 cm right of O and is moving to the left with a velocity of 4 cm s^{-1} and slowing down, its acceleration being 2 cm s^{-2} to the right.
- c When $t = 2, s(2) = -1 \text{ cm}$
 $v(2) = 0 \text{ cm s}^{-1}$
 $a(2) = 2 \text{ cm s}^{-2} \quad \therefore$ the object is 1 cm left of O, momentarily at rest, but with acceleration 2 cm s^{-2} to the right.
- d The object reverses direction when $v(t) = 0$ i.e., at $t = 2$ seconds.
 At $t = 2$, the particle is 1 cm left of O.

e



- f Speed decreases when $v(t)$ and $a(t)$ have opposite signs, i.e., when $0 \leq t \leq 2$.

2 $s(t) = 98t - 4.9t^2 \text{ m } t \geq 0$

a $v(t) = 98 - 9.8t \text{ ms}^{-1}$
 $a(t) = -9.8 \text{ ms}^{-2}$

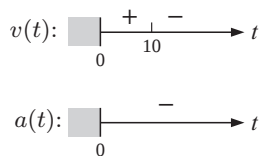
b When $t = 0, s(0) = 0 \text{ m}, v(0) = 98 \text{ ms}^{-1}$

c When $t = 5, s(5) = 367.5 \text{ m}$
 $v(5) = 49 \text{ ms}^{-1}$
 $a(5) = -9.8 \text{ ms}^{-2}$

When $t = 12, s(12) = 470.4 \text{ m}$
 $v(12) = -19.6 \text{ ms}^{-1}$
 $a(12) = -9.8 \text{ ms}^{-2}$

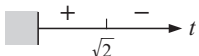
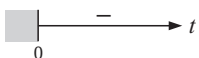
The stone is 367.5 m above the ground, travelling upwards at 49 ms^{-1} and slowing down.

The stone is 470.4 m above the ground and travelling downwards at 19.6 ms^{-1} , increasing in speed.



d Maximum height is reached when $v(t) = 0 \text{ ms}^{-1}$ $\therefore$ the maximum height is
 $\therefore 98 - 9.8t = 0$ $s(10) = 9.8(10) - 4.9(100)$
 $9.8t = 98$ $= 980 - 490$
 $t = 10 \text{ seconds}$ $= 490 \text{ m}$

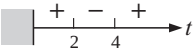
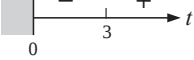
e The stone is at ground level when $s(t) = 0$ i.e., when $98t - 4.9t^2 = 0$
 $\therefore 4.9t(20 - t) = 0$
 $\therefore t = 0$ or 20 seconds
 i.e., it hits the ground after 20 seconds .

3 a $s(t) = 12t - 2t^3 - 1 \text{ cm}, t \geq 0$ $v(t)$: 
 $\therefore v(t) = 12 - 6t^2 \text{ cm s}^{-1}$
 and $a(t) = -12t \text{ cm s}^{-2}$ $a(t)$: 

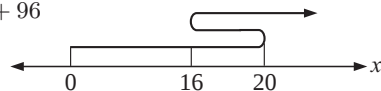
b When $t = 0$, $s(0) = -1 \text{ cm}$
 $v(0) = 12 \text{ cm s}^{-1}$ The particle is 1 cm left of O, moving right at
 $a(0) = 0 \text{ cm s}^{-2}$ 12 cm s^{-1} with constant speed.

c The particle reverses direction when $v(t) = 0$ i.e., at $t = \sqrt{2} \text{ seconds}$.
 When $t = \sqrt{2}$, $s(\sqrt{2}) = 12\sqrt{2} - 2(2\sqrt{2}) - 1$
 $= 8\sqrt{2} - 1$ i.e., the particle is $(8\sqrt{2} - 1) \text{ cm}$ to the right of O.

d i From the sign diagrams in **a**, the speed increases for $t \geq \sqrt{2} \text{ seconds}$.
ii The velocity of the particle never increases $\{a(t) \leq 0\}$.

4 a $x(t) = t^3 - 9t^2 + 24t \text{ m}, t \geq 0$ $v(t)$: 
 $v(t) = 3t^2 - 18t + 24$ and $a(t) = 6t - 18$
 $= 3(t^2 - 6t + 8)$ $= 6(t - 3) \text{ ms}^{-2}$ $a(t)$: 
 $= 3(t - 4)(t - 2) \text{ ms}^{-1}$

b Reverses direction when $v(t) = 0$, i.e., at $t = 2 \text{ seconds}$ and $t = 4 \text{ seconds}$.
 $x(2) = 8 - 36 + 48 \text{ m}$ and $x(4) = 64 - 144 + 96$
 $= 20 \text{ m}$ $= 16 \text{ m}$



c i The speed decreases when $v(t)$ and $a(t)$ have the same sign, i.e., when $0 \leq t \leq 2$ and $3 \leq t \leq 4$.
ii The velocity decreases when $a(t) < 0$, i.e., when $0 \leq t \leq 3$.

d When $t = 5$, $s(5) = 5^3 - 9.5^2 + 24.5$ $\therefore$ distance travelled $= 20 + 4 + 4 \text{ m}$
 $= 125 - 225 + 120$ $= 28 \text{ m}$
 $= 20 \text{ m}$

5 a Let the equation be $s(t) = at^2 + bt + c$
 $\therefore v(t) = 2at + b$
 and $a(t) = 2a = g$ {gravitational acceleration}
 $\therefore a = \frac{1}{2}g$

Also $v(t) = gt + b$

But, when $t = 0$, $v(0) = g \times 0 + b$

$\therefore v(0) = b$ i.e., initial velocity is b

$\therefore v(t) = v(0) + gt$ as required

b Now when $t = 0$, $s(0) = 0$

$$\therefore a \times 0^2 + b \times 0 + c = 0$$

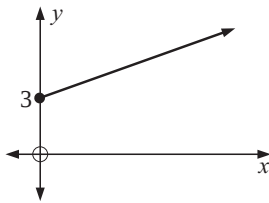
$$\therefore c = 0$$

$$\text{and so } s(t) = \left(\frac{1}{2}g\right)t^2 + v(0)t$$

$$\text{i.e., } s(t) = v(0) \times t + \frac{1}{2}gt^2 \text{ as required}$$

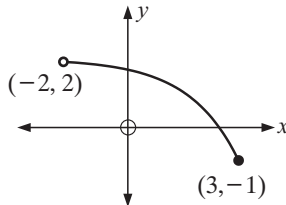
EXERCISE 22E.1

1 a



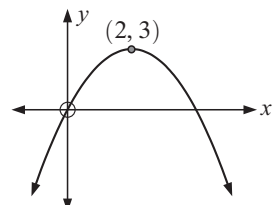
- i** $x \geq 0$
ii never

b



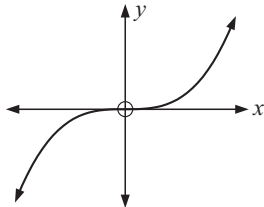
- i** never
ii $-2 < x \leq 3$

c



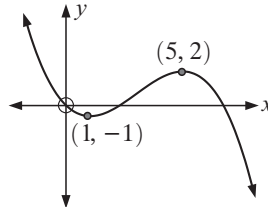
- i** $x \leq 2$
ii $x \geq 2$

d



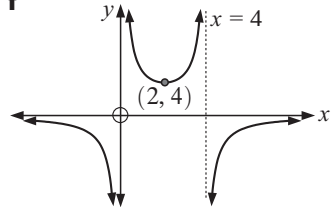
- i** all real x
ii never

e



- i** $1 \leq x \leq 5$
ii $x \leq 1, x \geq 5$

f



- i** $2 \leq x < 4, x > 4$
ii $x < 0, 0 < x \leq 2$

EXERCISE 22E.2

1 a

$$f(x) = x^2, \quad f'(x) = 2x$$

Sign $f'(x)$: $\xleftarrow{-} \quad \quad \quad \xrightarrow{+}$

increasing when $x \geq 0$,
 decreasing when $x \leq 0$

b

$$f(x) = -x^3, \quad f'(x) = -3x^2$$

Sign $f'(x)$: $\xleftarrow{-} \quad \quad \quad \xleftarrow{-}$

decreasing for all x

c

$$f(x) = 2x^2 + 3x - 4, \quad f'(x) = 4x + 3$$

Sign $f'(x)$: $\xleftarrow{-} \quad \quad \quad \xrightarrow{+}$

increasing when $x \geq -\frac{3}{4}$,
 decreasing when $x \leq -\frac{3}{4}$

d

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}, \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Sign $f'(x)$: $\xleftarrow{-} \quad \quad \quad \xrightarrow{+}$

only defined when $x \geq 0$,
 increasing when $x \geq 0$, never decreasing

e

$$f(x) = \frac{2}{\sqrt{x}}, \quad f'(x) = -x^{-\frac{3}{2}} = \frac{-1}{x\sqrt{x}} = 2x^{-\frac{1}{2}}$$

Sign $f'(x)$: $\xleftarrow{-} \quad \quad \quad \xrightarrow{-}$

$f(x)$ is only defined for $x > 0$
 decreasing when $x > 0$, never increasing

f

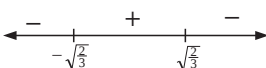
$$f(x) = x^3 - 6x^2, \quad f'(x) = 3x^2 - 12x = 3x(x - 4)$$

Sign $f'(x)$: $\xrightarrow{+} \quad \quad \quad \xleftarrow{-} \quad \quad \quad \xrightarrow{+}$

increasing when $x \leq 0$ or $x \geq 4$,
 decreasing when $0 \leq x \leq 4$

g $f(x) = -2x^3 + 4x$

$$\begin{aligned} f'(x) &= -6x^2 + 4 \\ &= -2(3x^2 - 2) \end{aligned}$$

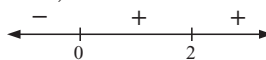
Sign $f'(x)$: 

increasing for $-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$,

decreasing for $x \leq -\sqrt{\frac{2}{3}}$ or $x \geq \sqrt{\frac{2}{3}}$

i $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$,

$$\begin{aligned} f'(x) &= 12x^3 + 48x^2 + 48x \\ &= 12x(x^2 - 4x + 4) \\ &= 12x(x - 2)^2 \end{aligned}$$

Sign $f'(x)$: 

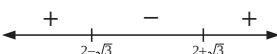
increasing when $x \geq 0$,

decreasing when $x \leq 0$

k $f(x) = x^3 - 6x^2 + 3x - 1$,

$$\begin{aligned} f(x) &= 3x^2 - 12x + 3 \\ &= 3(x^2 - 4x + 1) \end{aligned}$$

$$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Sign $f'(x)$: 

increasing when $x \leq 2 - \sqrt{3}$

or $x \geq 2 + \sqrt{3}$,

decreasing when $2 - \sqrt{3} \leq x \leq 2 + \sqrt{3}$

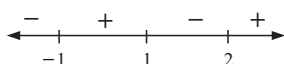
m $y = 3x^4 - 8x^3 - 6x^2 + 24x + 11$

$$\begin{aligned} \frac{dy}{dx} &= 12x^3 - 24x^2 - 12x + 24 \\ &= 12(x^3 - 2x^2 - x + 2) \end{aligned}$$

Using technology, a root is -1 ,

$$\begin{aligned} \therefore \frac{dy}{dx} &= 12(x+1)(x^2 - 3x + 2) \\ &= 12(x+1)(x-1)(x-2) \end{aligned}$$

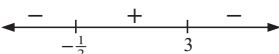
Sign diagram of $\frac{dy}{dx}$:



So $f(x)$ is increasing for $-1 \leq x \leq 1$ and $x \geq 2$, and decreasing for $x \leq -1$ and $1 \leq x \leq 2$.

h $f(x) = -4x^3 + 15x^2 + 18x + 3$

$$\begin{aligned} f'(x) &= -12x^2 + 30x + 18 \\ &= -6(2x^2 - 5x - 3) \\ &= -6(2x+1)(x-3) \end{aligned}$$

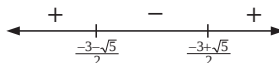
Sign $f'(x)$: 

increasing when $-\frac{1}{2} \leq x \leq 3$,

decreasing when $x \leq -\frac{1}{2}$ and $x \geq 3$

j $f(x) = 2x^3 + 9x^2 + 6x - 7$,

$$\begin{aligned} f'(x) &= 6x^2 + 18x + 6 \\ &= 6(x^2 + 3x + 1) \end{aligned}$$

Sign $f'(x)$: 

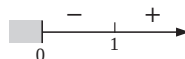
$$x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

increasing for $x \leq \frac{-3-\sqrt{5}}{2}$ or $x \geq \frac{-3+\sqrt{5}}{2}$,

decreasing for $\frac{-3-\sqrt{5}}{2} \leq x \leq \frac{-3+\sqrt{5}}{2}$

l $f(x) = x - 2\sqrt{x}$, $f'(x) = 1 - x^{-\frac{1}{2}}$

$$\begin{aligned} &= x - 2x^{\frac{1}{2}} &= 1 - \frac{1}{\sqrt{x}} \\ & &= \frac{\sqrt{x} - 1}{\sqrt{x}} \end{aligned}$$

Sign $f'(x)$: 

increasing when $x \geq 1$,

decreasing when $0 \leq x \leq 1$

n $y = x^4 - 4x^3 + 2x^2 + 4x + 1$,

$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 12x^2 + 4x + 4 \\ &= 4(x^3 - 3x^2 + x + 1) \end{aligned}$$

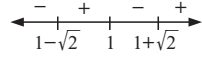
Using technology, a root is 1 ,

$$\begin{aligned} \therefore \frac{dy}{dx} &= 4(x-1)(x^2 - 2x - 1) \\ &= 0 \end{aligned}$$

when $x = 1$ or $x = \frac{2 \pm \sqrt{8}}{2}$

i.e., $x = 1$ or $x = 1 \pm \sqrt{2}$

Sign diagram

of $\frac{dy}{dx}$: 

$\therefore$ increasing for $1 - \sqrt{2} \leq x \leq 1$

and $x \geq 1 + \sqrt{2}$, decreasing for

$x \leq 1 - \sqrt{2}$ and $1 \leq x \leq 1 + \sqrt{2}$

2 a i $f(x) = \frac{4x}{x^2 + 1},$

let $u = 4x, \quad u' = 4,$
 $v = x^2 + 1, \quad v' = 2x$

$$\begin{aligned} \therefore f'(x) &= \frac{4(x^2 + 1) - 4x \times 2x}{(x^2 + 1)^2} \\ &= \frac{4x^2 + 4 - 8x^2}{(x^2 + 1)^2} \\ &= \frac{4 - 4x^2}{(x^2 + 1)^2} \\ &= \frac{-4(x^2 - 1)}{(x^2 + 1)^2} \\ &= \frac{-4(x + 1)(x - 1)}{(x^2 + 1)^2} \end{aligned}$$

Sign diagram of $\frac{dy}{dx}$:

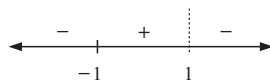
ii $\therefore$ increasing for $-1 \leq x \leq 1,$
 decreasing for $x \leq -1$ and $x \geq 1$

b i $f(x) = \frac{4x}{(x-1)^2},$

let $u = 4x, \quad u' = 4,$
 $v = (x-1)^2, \quad v' = 2(x-1)^1$

$$\begin{aligned} \therefore f'(x) &= \frac{4(x-1)^2 - 8x(x-1)}{(x-1)^4} \\ &= \frac{4(x-1)((x-1) - 2x)}{(x-1)^4} \\ &= \frac{4(-1-x)}{(x-1)^3} \\ &= \frac{-4(x+1)}{(x-1)^3} \end{aligned}$$

Sign diagram of $\frac{dy}{dx}$:

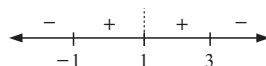


ii $\therefore$ increasing for $-1 \leq x < 1,$
 decreasing for $x \leq -1$ and $x > 1$

c i $f(x) = \frac{-x^2 + 4x - 7}{x - 1},$ let $u = -x^2 + 4x - 7, \quad u' = -2x + 4, \quad v = x - 1, \quad v' = 1$

$$\begin{aligned} \therefore f'(x) &= \frac{(-2x + 4)(x - 1) - (-x^2 + 4x - 7)(1)}{(x - 1)^2} \\ &= \frac{-2x^2 + 6x - 4 + x^2 - 4x + 7}{(x - 1)^2} \\ &= \frac{-x^2 + 2x + 3}{(x - 1)^2} \\ &= \frac{-(x^2 - 2x - 3)}{(x - 1)^2} \\ &= \frac{-(x - 3)(x + 1)}{(x - 1)^2} \end{aligned}$$

Sign diagram of $\frac{dy}{dx}$:

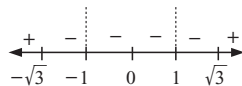


ii $\therefore f(x)$ is increasing for $-1 \leq x < 1$ and $1 < x \leq 3,$ and
 decreasing for $x \leq -1$ and $x \geq 3$

3 a $f(x) = \frac{x^3}{x^2 - 1},$ let $u = x^3, \quad u' = 3x^2, \quad v = x^2 - 1, \quad v' = 2x$

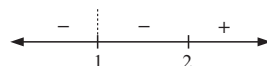
$$\begin{aligned} \therefore f'(x) &= \frac{3x^2(x^2 - 1) - x^3 \times 2x}{(x^2 - 1)^2} \\ &= \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2} \\ &= \frac{x^2(x^2 - 3)}{(x^2 - 1)^2} \\ &= \frac{x^2(x + \sqrt{3})(x - \sqrt{3})}{(x^2 - 1)^2} \end{aligned}$$

Sign diagram of $\frac{dy}{dx}$:



$\therefore f(x)$ is increasing for $x \geq \sqrt{3}$ and $x \leq -\sqrt{3},$ and
 decreasing for $-\sqrt{3} \leq x < -1, \quad -1 < x \leq 0, \quad 0 \leq x < 1$ and $1 < x \leq \sqrt{3}.$

$$\begin{aligned}
 \text{b } f(x) &= x^2 + \frac{4}{x-1}, & f'(x) &= 2x - 4(x-1)^{-2} \times 1 \\
 &= x^2 + 4(x-1)^{-1} & &= 2x - \frac{4}{(x-1)^2} \\
 & & &= \frac{2x(x-1)^2 - 4}{(x-1)^2} \\
 & & &= \frac{2x(x^2 - 2x + 1) - 4}{(x-1)^2} \\
 & & &= \frac{2x^3 - 4x^2 + 2x - 4}{(x-1)^2} \\
 & & &= \frac{(x-2)(2x^2 + 2)}{(x-1)^2}
 \end{aligned}$$

 Sign of $\frac{dy}{dx}$:


$\therefore f(x)$ is increasing for $x \geq 2$, and decreasing for $1 < x \leq 2$ and $x < 1$.

EXERCISE 22E.3

1 a A is a *local minimum*, B is a *local maximum*, C is a *horizontal inflection*.

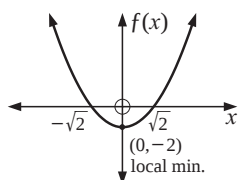
b $f'(x)$ has sign diagram:

c i $f(x)$ is increasing for $x \leq -2$ and $x \geq 3$ ii $f(x)$ is decreasing for $-2 \leq x \leq 3$.

2 a $f(x) = x^2 - 2 \therefore f'(x) = 2x$

with sign diagram:

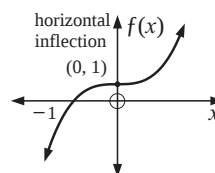
local minimum at $(0, -2)$



b $f(x) = x^3 + 1 \therefore f'(x) = 3x^2$

with sign diagram:

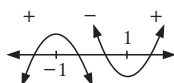
horizontal inflection at $(0, 1)$



c $f(x) = x^3 - 3x + 2$

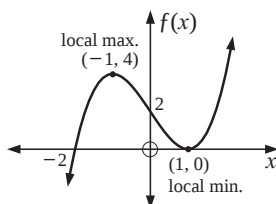
$$\begin{aligned}
 \therefore f'(x) &= 3x^2 - 3 \\
 &= 3(x^2 - 1) \\
 &= 3(x+1)(x-1)
 \end{aligned}$$

with sign diagram:



local maximum at $(-1, 4)$,

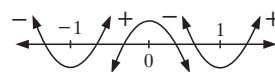
local minimum at $(1, 0)$



d $f(x) = x^4 - 2x^2$

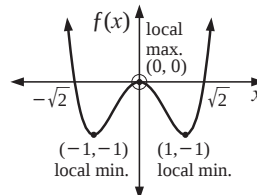
$$\begin{aligned}
 \therefore f'(x) &= 4x^3 - 4x \\
 &= 4x(x^2 - 1) \\
 &= 4x(x+1)(x-1)
 \end{aligned}$$

with sign diagram:



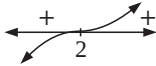
local minima at $(-1, -1)$ and $(1, -1)$,

local maximum at $(0, 0)$

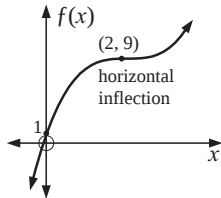


$$\begin{aligned} \text{e } f(x) &= x^3 - 6x^2 + 12x + 1 \\ \therefore f'(x) &= 3x^2 - 12x + 12 \\ &= 3(x^2 - 4x + 4) \\ &= 3(x - 2)^2 \end{aligned}$$

with sign diagram:

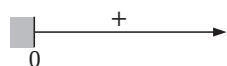


$\therefore$ horizontal inflection at (2, 9)

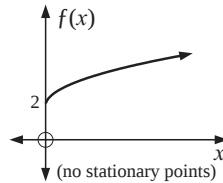


$$\begin{aligned} \text{f } f(x) &= \sqrt{x} + 2 \\ \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \neq 0 \end{aligned}$$

with sign diagram:

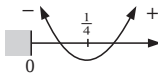


$\therefore$ no stationary points.



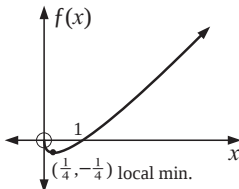
$$\begin{aligned} \text{g } f(x) &= x - \sqrt{x} \\ \therefore f'(x) &= 1 - \frac{1}{2}x^{-\frac{1}{2}} \\ &= 1 - \frac{1}{2\sqrt{x}} \\ &= \frac{2\sqrt{x} - 1}{2\sqrt{x}} \end{aligned}$$

with sign diagram:



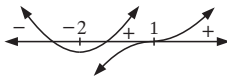
$f(x)$ is defined for all $x \geq 0$

local minimum at $(\frac{1}{4}, -\frac{1}{4})$



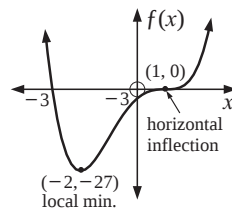
$$\begin{aligned} \text{h } f(x) &= x^4 - 6x^2 + 8x - 3 \\ \therefore f'(x) &= 4x^3 - 12x + 8 \\ &= 4(x^3 - 3x + 2) \\ &= 4(x - 1)(x^2 + x - 2) \\ &= 4(x - 1)(x + 2)(x - 1) \end{aligned}$$

with sign diagram:



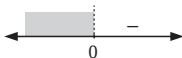
local minimum at $(-2, -27)$,

horizontal inflection at $(1, 0)$



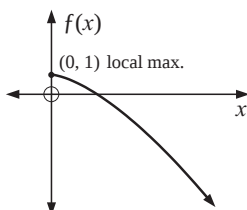
$$\begin{aligned} \text{i } f(x) &= 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}} \\ \therefore f'(x) &= -\frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{-3\sqrt{x}}{2} \end{aligned}$$

with sign diagram:



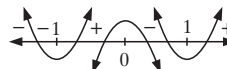
$f(x)$ is only defined when $x \geq 0$

$\therefore$ local maximum at (0, 1)



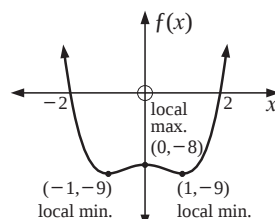
$$\begin{aligned} \text{j } f(x) &= x^4 - 2x^2 - 8 \\ \therefore f'(x) &= 4x^3 - 4x \\ &= 4x(x^2 - 1) \\ &= 4x(x + 1)(x - 1) \end{aligned}$$

with sign diagram:



local minima at $(-1, -9)$ and $(1, -9)$,

local maximum at $(0, -8)$



3 $f(x) = ax^2 + bx + c, \quad a \neq 0$

$f'(x) = 2ax + b$ and $f(x)$ has a stationary point when $f'(x) = 0$ i.e., $x = -\frac{b}{2a}$

There is a local maximum when $a < 0$  and there is a local minimum when $a > 0$ .

4 $f(x) = 2x^3 + ax^2 - 24x + 1, \quad \therefore f'(x) = 6x^2 + 2ax - 24$

But $f'(-4) = 0, \quad \therefore 96 - 8a - 24 = 0$

$\therefore 72 = 8a$ and so $a = 9$.

5 a $f(x) = x^3 + ax + b, \quad \therefore f'(x) = 3x^2 + a$

Now $3x^2 + a = 0$ when $x = -2$

$\therefore 12 + a = 0$

i.e., $a = -12$

Now $(-2, 3)$ is a point on the curve.

$\therefore f(x) = x^3 - 12x + b$

$\therefore 3 = -8 + 24 + b$

$\therefore b = -13$

b $\therefore f(x) = x^3 - 12x - 13$ and $f'(x) = 3x^2 - 12$

$f'(x) = 3(x+2)(x-2)$

where $f'(x)$ has sign diagram:



$\therefore$ there is a local maximum at $(-2, 3)$ and a local minimum at $(2, -29)$

6 Let the cubic polynomial be $P(x) = ax^3 + bx^2 + cx + d$ (1)

$\therefore P'(x) = 3ax^2 + 2bx + c$ (2)

$(0, 2)$ lies on $P(x) \quad \therefore P(0) = 2$ and so $a(0) + b(0) + c(0) + d = 2$
i.e., $d = 2$

Now the tangent is $y = 9x + 2, \quad \therefore$ slope at $(0, 2)$ is 9 $\therefore P'(0) = 9$

$\therefore 3a(0) + 2b(0) + c = 9$

$\therefore c = 9$

and a stationary point at $(-1, -7)$ means that

$P'(-1) = 0$

$\therefore 3a(-1)^2 + 2b(-1) + c = 0$ {using (2)}

$\therefore 3a - 2b + c = 0$

but since $c = 9$

$3a - 2b = -9$ (3)

Now $(-1, -7)$ also lies on $P(x) \quad \therefore a(-1)^3 + b(-1)^2 + c(-1) + d = -7$

$\therefore -a + b - 9 + 2 = -7$

$\therefore a - b = 0$ (4)

$\therefore a = b$

$\therefore 3a - 2a = -9$ {in (3)}

$\therefore a = -9$

$\therefore a = b = -9$

$\therefore P(x) = -9x^3 - 9x^2 + 9x + 2$

7 a $f(x) = x^3 - 12x - 2, \quad \text{for } -3 \leq x \leq 5$

$\therefore f'(x) = 3x^2 - 12$

$= 3(x+2)(x-2)$

which is 0 when $x = -2$ or 2

$\therefore$ maximum value is 63 when $x = 5$, and minimum value is -18 , when $x = 2$.

x	-3	-2	2	5
$f(x)$	7	14	-18	63

$$\begin{aligned} \mathbf{b} \quad f(x) &= 4 - 3x^2 + x^3, \quad \text{for } -2 \leq x \leq 3 \\ \therefore f'(x) &= -6x + 3x^2 \\ &= 3x(x - 2) \\ &\text{which is 0 when } x = 0 \text{ or } 2 \end{aligned}$$

x	-2	0	2	3
$f(x)$	-16	4	0	4

$\therefore$ maximum value is 4 when $x = 0$ or $x = 3$, minimum value is -16, when $x = -2$.

$$\begin{aligned} \mathbf{8} \quad C(x) &= 0.0007x^3 - 0.1796x^2 + 14.663x + 160 \\ C'(x) &= 0.0021x^2 - 0.3592x + 14.663 \\ C'(x) &= 0 \quad \text{when } 0.0021x^2 - 0.3592x + 14.663 = 0 \\ \text{Using technology, } x &\doteq 103.74 \text{ or } x \doteq 67.30 \end{aligned}$$

x	50	67.30	103.74	150
$C(x)$	531.65	546.73	529.80	680.95

$\therefore$ the maximum hourly cost is \$680.95 when 150 hinges are made. The minimum hourly cost is \$529.80 when 104 hinges are made.

EXERCISE 22F.1

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad f(x) &= \frac{3x-2}{x+1} \\ &= \frac{3(x+1)-5}{x+1} \\ &= 3 - \frac{5}{x+1} \end{aligned}$$

$\therefore$ VA is $x+1=0$, i.e., $x=-1$
and the HA is $y=3$
{as $|x| \rightarrow \infty$, $f(x) \rightarrow 3$ }

$$\begin{aligned} \mathbf{c} \quad f(x) &= \frac{4-2x}{x-1} \\ &= \frac{-2(x-1)+2}{x-1} \\ &= -2 + \frac{2}{x-1} \end{aligned}$$

$\therefore$ VA is $x-1=0$, i.e., $x=1$ and the HA is $y=-2$ {as $|x| \rightarrow \infty$, $f(x) \rightarrow -2$ }

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \mathbf{i} \quad f(x) &= -3 + \frac{1}{4-x} \\ \therefore \text{VA is } 4-x &= 0 \quad \text{i.e., } x=4 \\ \text{and HA is } y &= -3 \\ \{\text{as } |x| \rightarrow \infty, y &\rightarrow -3\} \end{aligned}$$

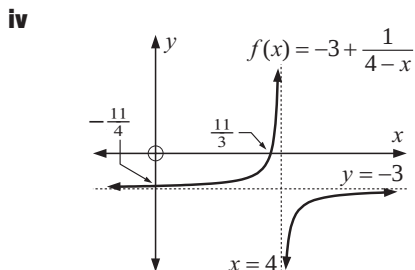
$$\begin{aligned} \mathbf{iii} \quad \text{When } x=0, y &= -3 + \frac{1}{4} = -2\frac{3}{4} \\ \therefore y \text{ intercept is } &-2\frac{3}{4} \\ \text{When } y=0, &-3 + \frac{1}{4-x} = 0 \\ \therefore \frac{1}{4-x} &= 3 \\ \therefore 4-x &= \frac{1}{3} \\ \therefore x &= 3\frac{2}{3} \\ \therefore x\text{-intercept is } &3\frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= \frac{x-4}{2x-1} \\ &= \frac{\frac{1}{2}(2x-1) - 3\frac{1}{2}}{2x-1} \\ &= \frac{1}{2} + \frac{-3\frac{1}{2}}{2x-1} \\ &= \frac{1}{2} - \frac{7}{2(2x-1)} \end{aligned}$$

$\therefore$ VA is $2x-1=0$, i.e., $x=\frac{1}{2}$
and the HA is $y=\frac{1}{2}$
{as $|x| \rightarrow \infty$, $f(x) \rightarrow \frac{1}{2}$ }

$$\begin{aligned} \mathbf{ii} \quad f(x) &= -3 + (4-x)^{-1} \\ \therefore f'(x) &= -(4-x)^{-2} \times (-1) \\ &= \frac{1}{(4-x)^2} \end{aligned}$$

Sign diagram
of $f'(x)$ is



$$\begin{aligned} \text{b i } f(x) &= \frac{x}{x+2} \\ &= \frac{(x+2)-2}{x+2} \\ &= 1 - \frac{2}{x+2} \end{aligned}$$

$\therefore$ VA is $x = -2$ $\{x+2=0\}$

and HA is $y = 1$.

$\{\text{as } |x| \rightarrow \infty, y \rightarrow 1\}$

iii when $x = 0$, $f(0) = 0$

$\therefore$ y -intercept is 0

when $y = 0$, $\frac{x}{x+2} = 0$

$\therefore x = 0$

$\therefore$ x -intercept is 0

$$\begin{aligned} \text{c i } f(x) &= \frac{4x+3}{x-2} \\ &= \frac{4(x-2)+11}{x-2} \\ &= 4 + \frac{11}{x-2} \end{aligned}$$

$\therefore$ VA is $x - 2 = 0$ i.e., $x = 2$

and HA is $y = 4$

$\{\text{as } |x| \rightarrow \infty, y \rightarrow 4\}$

iii when $x = 0$, $f(0) = \frac{3}{-2}$

$\therefore$ y -intercept is $-1\frac{1}{2}$

when $y = 0$, $4x + 3 = 0$

$\therefore x = -\frac{3}{4}$

$\therefore$ x -intercept is $-\frac{3}{4}$

$$\begin{aligned} \text{d i } f(x) &= \frac{1-x}{x+2} \\ &= \frac{-(x+2)+3}{x+2} \\ &= -1 + \frac{3}{x+2} \end{aligned}$$

$\therefore$ VA is $x + 2 = 0$ i.e., $x = -2$

and HA is $y = -1$

$\{\text{as } |x| \rightarrow \infty, y \rightarrow -1\}$

iii when $x = 0$, $f(0) = \frac{1}{2}$

$\therefore$ y -intercept is $\frac{1}{2}$

when $y = 0$, $\frac{1-x}{x+2} = 0$

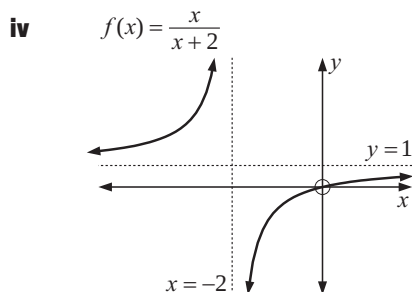
$\therefore 1 - x = 0$

$\therefore x = 1$

$\therefore$ x -intercept is 1

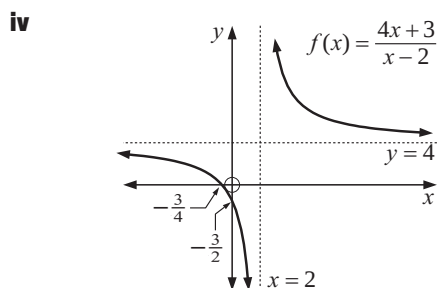
$$\begin{aligned} \text{ii } f'(x) &= \frac{1(x+2) - x(1)}{(x+2)^2} \\ &= \frac{2}{(x+2)^2} \end{aligned}$$

and has sign diagram $\begin{array}{ccc} & + & + \\ & | & \\ -2 & & \end{array}$



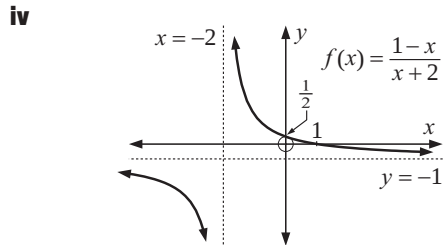
$$\begin{aligned} \text{ii } f'(x) &= \frac{4(x-2) - (4x+3)1}{(x-2)^2} \\ &= \frac{-11}{(x-2)^2} \end{aligned}$$

and has sign diagram $\begin{array}{ccc} & - & - \\ & | & \\ 2 & & \end{array}$



$$\begin{aligned} \text{ii } f'(x) &= \frac{-1(x+2) - (1-x)1}{(x+2)^2} \\ &= \frac{-3}{(x+2)^2} \end{aligned}$$

and has sign diagram $\begin{array}{ccc} & - & - \\ & | & \\ -2 & & \end{array}$



EXERCISE 22F.2

1 a $y = \frac{2x}{x^2 - 4} = \frac{2x}{(x+2)(x-2)}$
 VAs are $x+2=0$ and $x-2=0$
 i.e., $x=-2$ and $x=2$
 HA is $y=0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

b $y = \frac{1-x}{(x+2)^2}$
 VA is $x+2=0$ i.e., $x=-2$
 HA is $y=0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

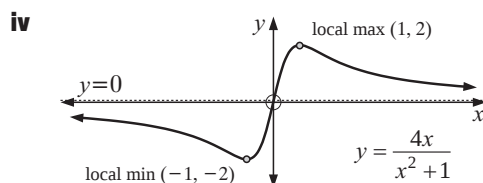
c $y = \frac{3x+2}{x^2+1}$ has no VAs {as $x^2+1=0$ has no real solutions}
 and a HA of $y=0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

2 a i $f(x) = \frac{4x}{x^2+1}$
 has no VAs {as $x^2+1=0$
 has no real solutions}
 and a HA of $y=0$
 {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

iii when $x=0$, $f(0)=0$
 $\therefore$ y -intercept is 0

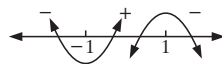
when $y=0$, $\frac{4x}{x^2+1}=0$

$\therefore x=0$ and so the x -intercept is 0



ii $f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2}$
 $= \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2}$
 $= \frac{4 - 4x^2}{(x^2+1)^2}$
 $= \frac{4(1+x)(1-x)}{(x^2+1)^2}$

and has a sign diagram of



$\therefore$ local min. is $\left(-1, \frac{4(-1)}{1+1}\right)$
 i.e., $(-1, -2)$

and local max is $\left(1, \frac{4(1)}{1+1}\right)$
 i.e., $(1, 2)$

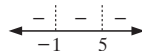
b $f(x) = \frac{4x}{x^2 - 4x - 5} = \frac{4x}{(x-5)(x+1)}$

i Vertical asymptotes occur when
 $(x-5)(x+1)=0$,
 i.e., at $x=-1$ and $x=5$.

Horizontal asymptote is $y=0$
 {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

ii $f'(x) = \frac{4(x^2 - 4x - 5) - 4x(2x - 4)}{(x^2 - 4x - 5)^2}$
 $= \frac{4x^2 - 16x - 20 - 8x^2 + 16x}{(x^2 - 4x - 5)^2}$
 $= \frac{-4x^2 - 20}{(x-5)^2(x+1)^2}$

Since $-4x^2 - 20$ is always negative we have no turning points, and sign diagram is:

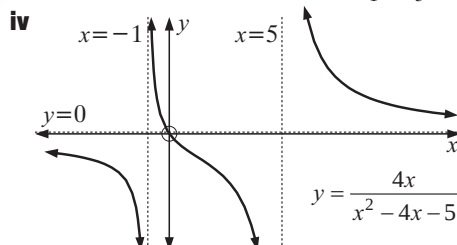


iii When $x=0$, $y = \frac{0}{-5} = 0$
 $\therefore$ y -intercept is 0

When $y=0$, $\frac{4x}{x^2 - 4x - 5} = 0$

$\therefore x=0$

$\therefore$ x -intercept is 0



c i $f(x) = \frac{4x}{(x-1)^2}$

has VA $x - 1 = 0$ i.e., $x = 1$
and a HA of $y = 0$
{as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

ii $f'(x) = \frac{4(x-1)^2 - 4x \times 2(x-1)1}{(x-1)^4}$
 $= \frac{4(x-1)[x-1-2x]}{(x-1)^4}$
 $= \frac{4(-x-1)}{(x-1)^3}$
 $= \frac{-4(x+1)}{(x-1)^3}$

which has sign
diagram



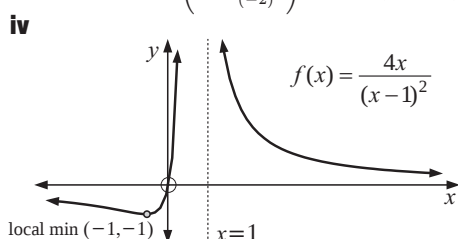
local min. at $\left(-1, \frac{4(-1)}{(-2)^2}\right)$ i.e., at $(-1, -1)$

iii when $x = 0$, $y = \frac{0}{1} = 0$

$\therefore$ y -intercept is 0

when $y = 0$, $\frac{4x}{(x-1)^2} = 0$

$\therefore$ x -intercept is 0



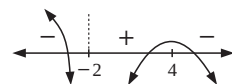
d $f(x) = \frac{3x-3}{(x+2)^2}$

i Vertical asymptotes occur when
 $(x+2)^2 = 0$, i.e., at $x = -2$.

HA is $y = 0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

ii $f'(x) = \frac{3(x+2)^2 - 2(x+2)(3x-3)}{(x+2)^4}$
 $= \frac{3(x+2) - 2(3x-3)}{(x+2)^3}$
 $= \frac{3x+6-6x+6}{(x+2)^3}$
 $= \frac{12-3x}{(x+2)^3}$
 $= \frac{-3(x-4)}{(x+2)^3}$

Sign diagram
of $f'(x)$ is:



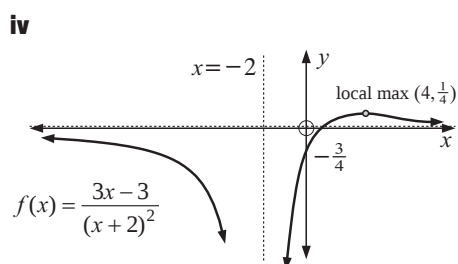
$\therefore$ there is a local maximum at $(4, \frac{1}{4})$.

iii when $x = 0$, $y = \frac{-3}{2^2} = -\frac{3}{4}$

$\therefore$ y -intercept is $-\frac{3}{4}$

when $y = 0$, $3x - 3 = 0$
 $\therefore x = 1$

$\therefore$ x -intercept is 1



EXERCISE 22F.3

1 a $y = \frac{2x^2 - x + 2}{x^2 - 1} = \frac{2x^2 - x + 2}{(x+1)(x-1)}$

has VA when $x+1=0$, $x-1=0$

i.e., $x=-1$, $x=1$

and as $y = \frac{2 - \frac{1}{x} + \frac{2}{x^2}}{1 - \frac{1}{x^2}}$

HA is $y=2$

{as $|x| \rightarrow \infty$, $y \rightarrow \frac{2}{1}$ }

c $y = \frac{3x^2 - x + 2}{(x+2)^2} = \frac{3x^2 - x + 2}{x^2 + 4x + 4} = \frac{3 - \frac{1}{x} + \frac{2}{x^2}}{1 + \frac{4}{x} + \frac{4}{x^2}}$

has VA $x+2=0$ i.e., $x=-2$ had HA $y=3$ {as $|x| \rightarrow \infty$, $y \rightarrow \frac{3}{1}$ }

b $y = \frac{-x^2 + 2x - 1}{x^2 + x + 1}$

has a VA when $x^2 + x + 1 = 0$

But $\Delta = 1^2 - 4(1)(1) < 0 \therefore$ no real solutions $\therefore$ no VA's exist

and as $y = \frac{-1 + \frac{2}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}}$

HA is $y=-1$ {as $|x| \rightarrow \infty$, $y = \frac{-1}{1}$ }

2 a i $y = \frac{x^2 - x}{x^2 - x - 6} = \frac{x(x-1)}{(x-3)(x+2)}$

has VA when $x-3=0$, $x+2=0$

i.e., $x=3$, $x=-2$

and since $y = \frac{1 - \frac{1}{x}}{1 - \frac{1}{x} - \frac{6}{x^2}}$

the HA is $y=1$

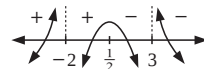
{as $|x| \rightarrow \infty$, $y \rightarrow \frac{1}{1}$ }

ii $f'(x) = \frac{(2x-1)(x^2-x-6) - (x^2-x)(2x-1)}{(x^2-x-6)^2}$
 $= \frac{(2x-1)(x^2-x-6-x^2+x)}{(x^2-x-6)^2}$
 $= \frac{-6(2x-1)}{(x^2-x-6)^2}$

Turning points are when

$f'(x)=0$, i.e., when $x=\frac{1}{2}$.

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $(\frac{1}{2}, \frac{1}{25})$.

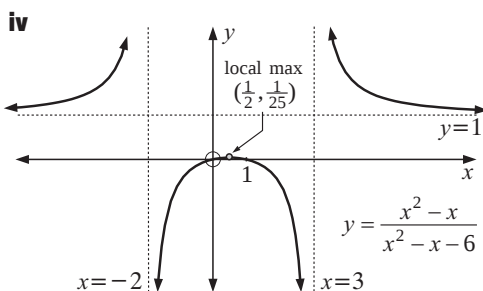
iii When $x=0$, $y = \frac{0}{-6} = 0$

$\therefore$ y -intercept is 0

When $y=0$, $x(x-1)=0$

$\therefore$ $x=0$ or 1

$\therefore$ x -intercepts are 0 and 1



b i $y = \frac{x^2 - 1}{x^2 + 1} = \frac{(x+1)(x-1)}{x^2 + 1}$

has no VA as $x^2 + 1 = 0$ has no real solutions and since

$$y = \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} \quad \text{the HA is } y = 1$$

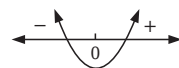
{as $|x| \rightarrow \infty, y \rightarrow \frac{1}{1}$ }

ii $f'(x) = \frac{2x(x^2 + 1) - (x^2 - 1)2x}{(x^2 + 1)^2}$

$$= \frac{2x[x^2 + 1 - x^2 + 1]}{(x^2 + 1)^2}$$

$$= \frac{4x}{(x^2 + 1)^2}$$

and has sign diagram



$\therefore$ a local minimum of $(0, -1)$.

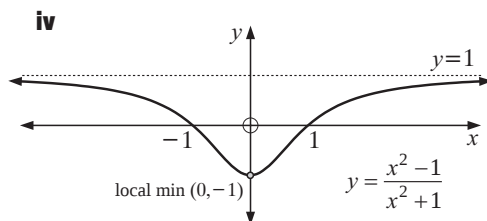
iii When $x = 0, y = \frac{-1}{1} = -1$

$\therefore$ y -intercept is -1

When $y = 0, (x+1)(x-1) = 0$

$\therefore x = \pm 1$

$\therefore$ x -intercepts are ± 1



c i $y = \frac{x^2 - 5x + 4}{x^2 + 5x + 4} = \frac{(x-1)(x-4)}{(x+1)(x+4)}$

has VAs of $x + 1 = 0$ and $x + 4 = 0$
i.e., $x = -1, x = -4$

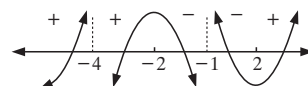
and as $y = \frac{1 - \frac{5}{x} + \frac{4}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}}$ the HA is $y = 1$ {as $|x| \rightarrow \infty, y \rightarrow \frac{1}{1}$ }

ii $\frac{dy}{dx} = \frac{(2x-5)(x^2+5x+4) - (x^2-5x+4)(2x+5)}{(x+1)^2(x+4)^2}$

$$= \frac{[2x^3 + 10x^2 + 8x - 5x^2 - 25x - 20] - [2x^3 - 10x^2 + 8x + 5x^2 - 25x + 20]}{(x+1)^2(x+4)^2}$$

$$= \frac{10x^2 - 40}{(x+1)^2(x+4)^2}$$

$$= \frac{10(x+2)(x-2)}{(x+1)^2(x+4)^2} \quad \text{which has sign diagram}$$



$\therefore$ local max at $\left(-2, \frac{4+10+4}{4-10+4}\right)$ i.e., at $(-2, -9)$

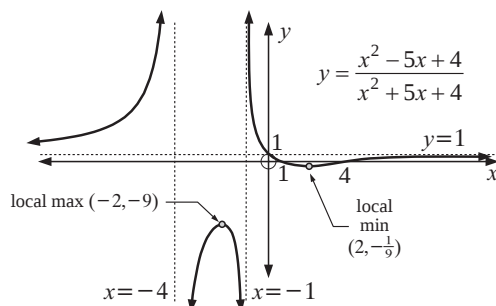
local min at $\left(2, \frac{4-10+4}{4+10+4}\right)$ i.e., at $(2, -\frac{1}{9})$

iii When $x = 0, y = \frac{4}{4} = 1$ $\therefore$ the y -intercept is 1

When $y = 0, (x-1)(x-4) = 0$ $\therefore x = 1$ or 4

$\therefore$ x -intercepts are 1, 4

iv



$$\mathbf{d} \quad \mathbf{i} \quad y = \frac{x^2 - 6x + 5}{(x+1)^2} = \frac{(x-1)(x-5)}{(x+1)^2}$$

has VA $x+1=0$ i.e., $x=-1$ and HA $y=1$

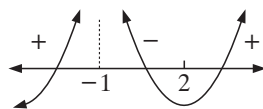
$$\left\{ \text{as } y = \frac{1 - \frac{6}{x} + \frac{5}{x^2}}{1 + \frac{2}{x} + \frac{1}{x^2}} \rightarrow \frac{1}{1} \text{ as } |x| \rightarrow \infty \right\}$$

$$\mathbf{ii} \quad \frac{dy}{dx} = \frac{(2x-6)(x+1)^2 - (x^2-6x+5)2(x+1)^1}{(x+1)^4}$$

$$= \frac{(x+1)[2x^2 - 4x - 6 - 2x^2 + 12x - 10]}{(x+1)^4}$$

$$= \frac{8x-16}{(x+1)^3}$$

$$= \frac{8(x-2)}{(x+1)^3} \text{ and has sign diagram}$$



$\therefore$ local minimum at $(2, \frac{4-12+5}{3^2})$ i.e., $(2, -\frac{1}{3})$

$$\mathbf{iii} \quad \text{When } x=0, y = \frac{5}{1} = 5$$

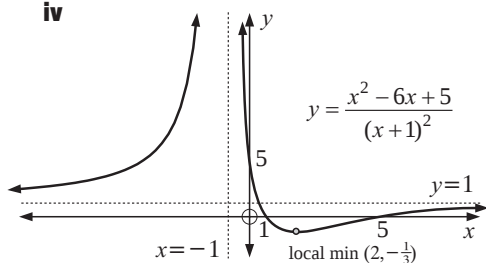
$\therefore$ y -intercept is 5

$$\text{When } y=0, (x-1)(x-5)=0$$

$$\therefore x=1 \text{ or } 5$$

$\therefore$ x -intercepts are 1, 5

iv



EXERCISE 22G

$$\mathbf{1} \quad \mathbf{a} \quad f(x) = x^2 + 3$$

$$\therefore f'(x) = 2x \text{ and } f''(x) = 2$$

Since $f''(x) \neq 0$,

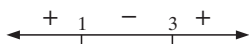
no points of inflection exist.

$$\mathbf{c} \quad f(x) = x^3 - 6x^2 + 9x + 1$$

$$\therefore f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x-3)(x-1)$$



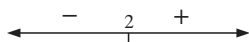
$$\text{and } f''(x) = 6x - 12$$

$$= 6(x-2)$$

$$\text{Now } f''(x) = 0 \text{ when } x = 2$$

$$\text{and } f'(2) \neq 0$$

$\therefore$ there is a non-horizontal inflection at (2, 3)



$$\mathbf{b} \quad f(x) = 2 - x^3$$

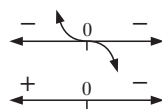
$$\therefore f'(x) = -3x^2$$

$$\text{and } f''(x) = -6x$$

$$\text{Now } f''(x) = 0 \text{ when } x = 0$$

$$\text{and } f'(0) = 0$$

$\therefore$ there is a horizontal inflection at (0, 2)

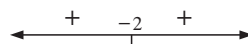


$$\mathbf{d} \quad f(x) = x^3 + 6x^2 + 12x + 5$$

$$\therefore f'(x) = 3x^2 + 12x + 12$$

$$= 3(x^2 + 4x + 4)$$

$$= 3(x+2)^2$$



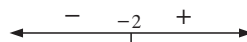
$$\text{and } f''(x) = 6x + 12$$

$$= 6(x+2)$$

$$\text{Now } f''(x) = 0 \text{ when } x = -2$$

$$\text{and } f'(-2) = 0$$

$\therefore$ there is a horizontal inflection at (-2, -3)



e $f(x) = -3x^4 - 8x^3 + 2$

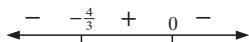
$$\therefore f'(x) = -12x^3 - 24x^2$$

$$= -12x^2(x + 2)$$



and $f''(x) = -36x^2 - 48x$

$$= -12x(3x + 4)$$



$\therefore$ horizontal inflection at $(0, 2)$,

non-horizontal inflection at $(-\frac{4}{3}, \frac{310}{27})$

f $f(x) = 3 - \frac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{3}{2}}$$

and $f''(x) = -\frac{3}{4}x^{-\frac{5}{2}}$

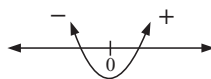
$$= \frac{-3}{4x^2\sqrt{x}}$$

$$f''(x) \neq 0 \text{ for all } x$$

$\therefore$ no points of inflection.

2 a $f(x) = x^2$

$$\therefore f'(x) = 2x \text{ which has sign diagram:}$$



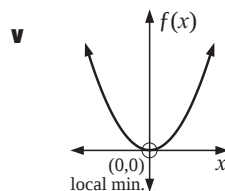
and $f''(x) = 2$

i There is a local minimum at $(0, 0)$.

ii There are no points of inflection as $f''(x) \neq 0$.

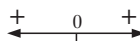
iii $f(x)$ is increasing when $x \geq 0$, and decreasing when $x \leq 0$.

iv $f(x)$ is concave up for all x as $f''(x) > 0$ for all x .

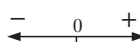


b $f(x) = x^3$

$$\therefore f'(x) = 3x^2 \text{ which has sign diagram:}$$



and $f''(x) = 6x$ which has sign diagram:

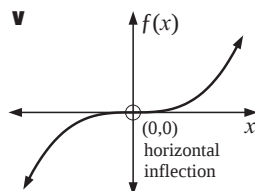


i A horizontal inflection at $(0, 0)$. $\{f'(x) = 0\}$

ii see **i**

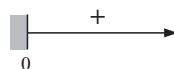
iii $f(x)$ is increasing for all x .

iv $f(x)$ is concave up $x \geq 0$, and concave down $x \leq 0$

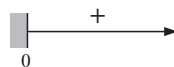


c $f(x) = \sqrt{x}$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \text{ which has sign diagram:}$$



and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \frac{-1}{4x\sqrt{x}}$ which has sign diagram:

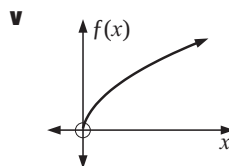


i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x \geq 0$.

iv $f(x)$ is concave down for all $x \geq 0$ as $f''(x) < 0$ for all $x > 0$.



d $f(x) = x^3 - 3x^2 - 24x + 1$

$$\therefore f'(x) = 3x^2 - 6x - 24$$

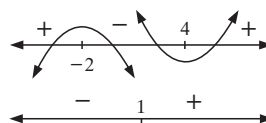
$$= 3(x^2 - 2x - 8)$$

$$= 3(x - 4)(x + 2) \text{ which has sign diagram:}$$

and $f''(x) = 6x - 6$

$$= 6(x - 1)$$

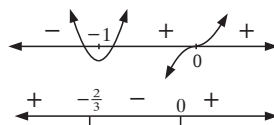
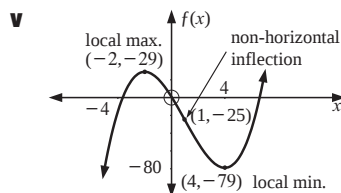
which has sign diagram:



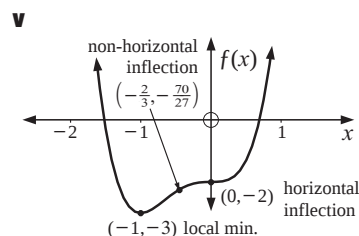
i There is a local maximum at $(-2, 29)$, and a local minimum at $(4, -79)$.

- ii There is a non-horizontal inflection at $(1, -25)$.
- iii $f(x)$ is increasing for $x \leq -2$ or $x \geq 4$, and decreasing for $-2 \leq x \leq 4$.
- iv $f(x)$ is concave down when $x \leq 1$, and concave up when $x \geq 1$.

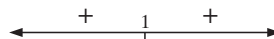
e $f(x) = 3x^4 + 4x^3 - 2$
 $\therefore f'(x) = 12x^3 + 12x^2$
 $= 12x^2(x + 1)$ which has sign diagram:
 and $f''(x) = 36x^2 + 24x$
 $= 12x(3x + 2)$ which has sign diagram:



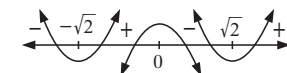
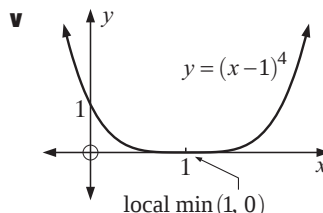
- i There is a local minimum at $(-1, -3)$, and a horizontal inflection at $(0, -2)$
- ii There is a non-horizontal inflection at $(-\frac{2}{3}, -\frac{70}{27})$ and a horizontal inflection at $(0, -2)$
- iii $f(x)$ is increasing for $x \geq -1$, and decreasing for $x \leq -1$.
- iv $f(x)$ is concave down for $-\frac{2}{3} \leq x \leq 0$, and concave up for $x \leq -\frac{2}{3}$ or $x \geq 0$.



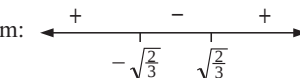
f $f(x) = (x - 1)^4$
 $\therefore f'(x) = 4(x - 1)^3 \times 1$
 $= 4(x - 1)^3$ which has sign diagram:
 and $f''(x) = 12(x - 1)^2 \times 1$
 $= 12(x - 1)^2$ which has sign diagram:



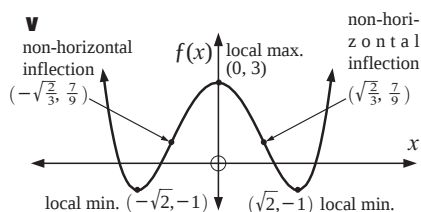
- i There is a local minimum at $(1, 0)$.
- ii There are no points of inflection.
- iii $f(x)$ is increasing for $x \geq 1$, and decreasing for $x \leq 1$.
- iv $f(x)$ is concave up for all x .



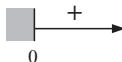
g $f(x) = x^4 - 4x^2 + 3$
 $f'(x) = 4x^3 - 8x$
 $= 4x(x^2 - 2)$
 $= 4x(x + \sqrt{2})(x - \sqrt{2})$ which has sign diagram:
 $f''(x) = 12x^2 - 8$
 $= 4(3x^2 - 2)$
 $= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$ which has sign diagram:



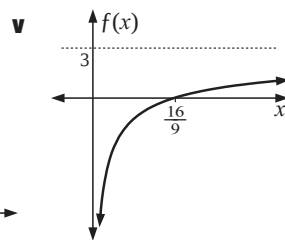
- i There is a local maximum at $(0, 3)$, and local min. at $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$.
- ii There are non-horizontal inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and $(-\sqrt{\frac{2}{3}}, \frac{7}{9})$. $\{f''(x) = 0\}$
- iii $f(x)$ is increasing for $-\sqrt{2} \leq x \leq 0$ and $x \geq \sqrt{2}$, and decreasing for $0 \leq x \leq \sqrt{2}$ and $x \leq -\sqrt{2}$.
- iv $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$, and concave up for $x \leq -\sqrt{\frac{2}{3}}$ and $x \geq \sqrt{\frac{2}{3}}$.



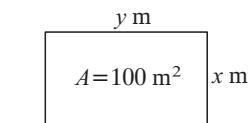
h $f(x) = 3 - \frac{4}{\sqrt{x}}, \quad x > 0$
 $= 3 - 4x^{-\frac{1}{2}}$

$\therefore f'(x) = 2x^{-\frac{3}{2}} = \frac{2}{x\sqrt{x}}$ with sign diag: 

and $f''(x) = -3x^{-\frac{5}{2}} = -\frac{3}{x^2\sqrt{x}}$ with sign diag: 



- i** There are no stationary points as $f'(x) \neq 0$.
- ii** There are no points of inflection as $f''(x) \neq 0$.
- iii** $f(x)$ is increasing for all $x > 0$ as $f'(x) > 0$ for all x .
- iv** $f(x)$ is concave down for all $x > 0$ as $f''(x) < 0$ for all x .

EXERCISE 22H**1 a**

$L = 2x + y$
 but $xy = 100$

$\therefore y = \frac{100}{x}$

$\therefore L = 2x + \frac{100}{x}$

c $\frac{dL}{dx} = 2 - 100x^{-2} = 2 - \frac{100}{x^2}$

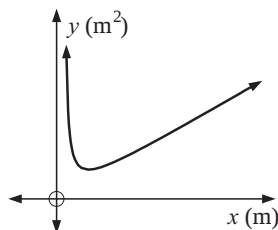
which is 0 when $\frac{100}{x^2} = 2$

$\therefore x^2 = 50$

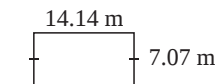
$\therefore x = \sqrt{50} \quad \{x > 0\}$

$\frac{d^2L}{dx^2} = 200x^{-3} = \frac{200}{x^3} > 0$ for $x > 0$

$\therefore \min L = 28.28 \text{ m}$ when $x = \sqrt{50} \text{ m}$

b

$\therefore L_{\min} = 2\sqrt{50} + \frac{100}{\sqrt{50}}$
 $= 2\sqrt{50} + 2\sqrt{50}$
 $= 4\sqrt{50}$
 $= 20\sqrt{2} \text{ m}$ when $x = 5\sqrt{2} \text{ m}$

d**2 a** Inner length of box = $2x$ cm**b**

Volume = 200 cm^3

$\therefore x \times 2x \times h = 200$

$2x^2h = 200$

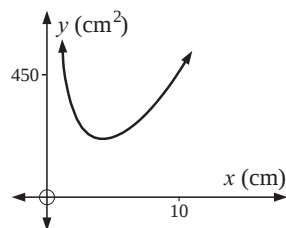
$\therefore x^2h = 100 \quad \dots\dots (1)$

c From (1) $h = \frac{100}{x^2}$.

Now area of inner surface is

$A(x) = 2(2x \times x) + 2(2x \times h) + 2(x \times h)$
 $= 4x^2 + 4xh + 2xh$
 $= 4x^2 + 6xh$

i.e., $A(x) = 4x^2 + \frac{600}{x} \text{ cm}^2$

d

$$\begin{aligned} \mathbf{e} \quad A(x) &= 4x^2 + 600x^{-1} \\ \therefore A'(x) &= 8x - 600x^{-2} \\ &= 8x - \frac{600}{x^2} \end{aligned}$$

$$\therefore A'(x) = 0 \quad \text{when}$$

$$8x = \frac{600}{x^2}$$

$$8x^3 = 600$$

$$x^3 = 75$$

$$x = \sqrt[3]{75}$$

$$x \doteq 4.127 \text{ cm}$$

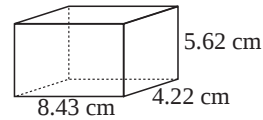
$$\begin{aligned} A''(x) &= 8 + 1200x^{-3} \\ &= 8 + \frac{1200}{x^3} \end{aligned}$$

$$\therefore A''(x) > 0 \quad \{\text{as } x > 0\}$$

$$\therefore \text{minimum when } x \doteq 4.22 \text{ cm}$$

$$\begin{aligned} \therefore A_{\min} &= 4(4.217)^2 + \frac{600}{(4.217)} \\ &\doteq 213.41 \text{ cm}^2 \end{aligned}$$

f

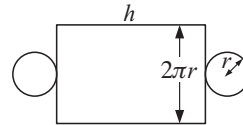


3 a Volume of can = $\pi r^2 h$

$$\therefore 1000 = \pi r^2 h \quad (\text{in cm})$$

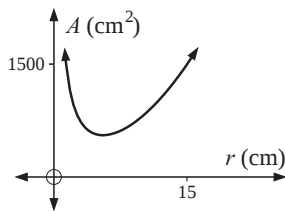
$$\therefore h = \frac{1000}{\pi r^2} \text{ cm}$$

b Opening the can up we get



$$\begin{aligned} \therefore A(r) &= \pi r^2 + \pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2000}{r} \text{ cm}^2 \end{aligned}$$

c



d

$$A(r) = 2\pi r^2 + 2000r^{-1}$$

$$A'(r) = 4\pi r - 2000r^{-2} = 4\pi r - \frac{2000}{r^2}$$

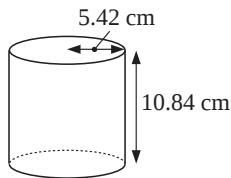
$$\text{So, } A'(r) = 0 \quad \text{when} \quad 4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

$$\therefore r \doteq 5.419 \text{ cm}$$

e



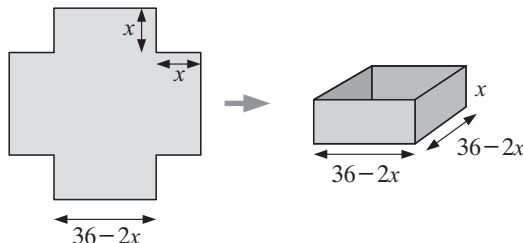
$$A''(r) = 4\pi + 4000r^{-3} = 4\pi + \frac{4000}{r^3}$$

$$\text{and as } r > 0, \quad A''(r) > 0$$

$$\therefore \text{area is a minimum when } r \doteq 5.42 \text{ cm}$$

$$\text{and } h = \frac{1000}{\pi r^2} \doteq 10.84 \text{ cm}$$

4



$$\begin{aligned}
 \text{Now volume of container is } V &= lbd \\
 &= x(36 - 2x)(36 - 2x) \\
 \therefore V &= x(36 - 2x)^2 \text{ cm}^3
 \end{aligned}$$

$$\text{let } u = x, \quad u' = 1, \quad v = (36 - 2x)^2, \quad v' = 2(36 - 2x)(-2)$$

$$\begin{aligned}
 \therefore V'(x) &= (36 - 2x)^2 - 4x(36 - 2x) \\
 &= (36 - 2x)[(36 - 2x) - 4x] \\
 &= (36 - 2x)(36 - 6x)
 \end{aligned}$$

$$\therefore V'(x) = 0 \quad \text{when } x = 6 \quad \text{or } x = 18$$

Sign diagram of $V'(x)$ is:



$\therefore$ volume is maximised when $x = 6$ cm

$\therefore$ cut out $6 \text{ cm} \times 6 \text{ cm}$ squares.

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad P &= 2\pi r + 2l \\
 \therefore 400 &= 2\pi(x) + 2l \\
 \therefore 200 &= \pi x + l \\
 \therefore l &= 200 - \pi x
 \end{aligned}$$

$$\text{Now } 0 \leq l \leq 200$$

and $l > 0$ means that

$$\pi x < 200$$

$$\therefore x < \frac{200}{\pi}$$

$$\text{and } x > 0$$

$$\therefore 0 \leq x \leq \frac{200}{\pi}$$

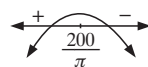
$$\begin{aligned}
 \mathbf{b} \quad \text{Area, } A &= \pi r^2 + (2x) \times l \\
 &= \pi x^2 + 2xl \\
 &= \pi x^2 + 2x(200 - \pi x) \\
 &= \pi x^2 + 400x - 2\pi x^2 \\
 \therefore A &= 400x - \pi x^2
 \end{aligned}$$

$$\mathbf{c} \quad \text{Now } \frac{dA}{dx} = 400 - 2\pi x$$

$$\text{which is 0 when } 2\pi x = 400$$

$$x = \frac{200}{\pi} \quad \text{and } l = 0$$

Sign diagram of $A'(x)$:



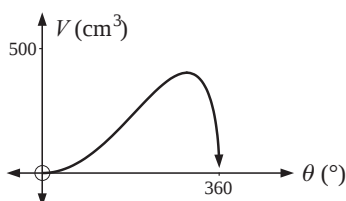
$\therefore$ area will be a maximum when the track is a circle.

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad \text{Arc AC} &= \frac{\theta}{360} \times (2\pi r) \\
 &= \frac{\theta}{360} (2 \times \pi \times 10) \\
 \text{Arc AC} &= \frac{\pi\theta}{18}
 \end{aligned}$$

$$\mathbf{c} \quad \text{Height of cone} = \sqrt{10^2 - r^2} \quad \{\text{Pythagoras}\}$$

$$\therefore h = \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$$

$\mathbf{e}$



$\mathbf{b}$ Now arc AC forms the base of the cone.

$$\therefore 2\pi r = \frac{\theta}{360} \times 2\pi \times 10$$

$$\therefore r = \frac{\theta}{36}$$

$$\begin{aligned}
 \mathbf{d} \quad V &= \frac{1}{3}\pi r^2 h \\
 V &= \frac{1}{3}\pi \left(\frac{\theta}{36}\right)^2 \sqrt{100 - \left(\frac{\theta}{36}\right)^2} \\
 &= \frac{\pi\theta^2}{3 \times 36^2} \sqrt{\frac{129\,600 - \theta^2}{36^2}} \\
 &= \frac{\pi\theta^2}{139\,968} \sqrt{129\,600 - \theta^2}
 \end{aligned}$$

- 9 Let the circle have a radius of x cm.

$$\therefore \text{area of circle} = \pi x^2$$

$$\text{and circumference of circle} = 2\pi x$$

$\therefore (24 - 2\pi x)$ cm is used to form the square

$$\therefore \text{side of the square} = \frac{24 - 2\pi x}{4} = \left(6 - \frac{\pi x}{2}\right) \text{ cm}$$

$$\therefore \text{area of the square} = \left(6 - \frac{\pi x}{2}\right)^2$$

$\therefore$ total area, A = area of square + area of circle

$$= \left(6 - \frac{\pi x}{2}\right)^2 + \pi x^2$$

$$= 36 - 6\pi x + \frac{\pi^2 x^2}{4} + \pi x^2$$

$$\therefore \frac{dA}{dx} = -6\pi + \frac{2\pi^2 x}{4} + 2\pi x$$

$$\text{which is 0 when } \frac{2\pi^2 x}{4} + 2\pi x = 6\pi \quad \therefore \quad \frac{\pi x}{2} + 2x = 6$$

$$\therefore \pi x + 4x = 12$$

$$\therefore x(\pi + 4) = 12$$

$$\therefore x = \frac{12}{\pi + 4} \text{ cm}$$

$$\text{and } \frac{d^2 A}{dx^2} = \frac{2\pi^2}{4} + 2\pi \text{ which is } > 0$$

$$\therefore \text{minimum occurs when } x = \frac{12}{\pi + 4} \text{ cm} \doteq 1.68 \text{ cm}$$

$$\text{i.e., when the circumference of the circle} = 2\pi \left[\frac{12}{\pi + 4}\right] \text{ cm} = \frac{24\pi}{\pi + 4} \text{ cm} \doteq 10.56 \text{ cm}$$

$\therefore$ make the cut 10.56 cm from RH end or 13.44 cm from LH end.

- 10 a X must lie between A and C (or at A or C).

If $x = 0$, then he rows straight to the shore and runs to C.

If $x = 6$, then he rows straight to C.

$$\therefore 0 \leq x \leq 6$$

b Now $XC = 6 - x$

$$\therefore \frac{dT}{dx} = 0$$

$\therefore$ time to row from B to X

$$= \frac{BX}{8} = \frac{\sqrt{25 + x^2}}{8}$$

$$\text{when } \frac{x}{8\sqrt{25 + x^2}} = \frac{1}{17}$$

$$\therefore 17x = 8\sqrt{25 + x^2}$$

$$\therefore 289x^2 = 64(25 + x^2)$$

$$\therefore 289x^2 = 1600 + 64x^2$$

$$\therefore 225x^2 = 1600$$

$$\therefore x^2 = \frac{1600}{225}$$

$$\therefore x = \frac{40}{15} = \frac{8}{3} \text{ km}$$

and time to run from X to C

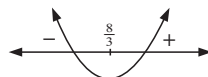
$$= \frac{XC}{17} = \frac{6 - x}{17}$$

$$\therefore \text{total time } T(x) = \frac{\sqrt{25 + x^2}}{8} + \frac{6 - x}{17} \text{ hours}$$

$$= \frac{1}{8}(25 + x^2)^{\frac{1}{2}} + \frac{6}{17} - \frac{x}{17}$$

$$\text{c Now } \frac{dT}{dx} = \frac{1}{16}(25 + x^2)^{-\frac{1}{2}}(2x) - \frac{1}{17}$$

$$\therefore \frac{dT}{dx} = \frac{x}{8\sqrt{25 + x^2}} - \frac{1}{17} \quad \text{Sign diagram of } \frac{dT}{dx}:$$



Thus, the time taken is a minimum if Peter aims for X such that $x = \frac{8}{3}$ km.

11Let $MX = x$ km, then $XN = 5 - x$ km

$$\therefore AX = \sqrt{4 + x^2} \text{ km} \quad \text{and} \quad XB = \sqrt{1 + (5 - x)^2} \text{ km} \quad \{\text{Pythagoras}\}$$

Now $P = AX + XB$

$$P = (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$

$$\begin{aligned} \therefore \frac{dP}{dx} &= \frac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(2x - 10) \\ &= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}} \end{aligned}$$

$$\text{Thus, } \frac{dP}{dx} = 0 \quad \text{when}$$

$$\frac{x}{\sqrt{4 + x^2}} = \frac{5 - x}{\sqrt{x^2 - 10x + 26}}$$

$$\therefore \frac{x^2}{4 + x^2} = \frac{(5 - x)^2}{x^2 - 10x + 26} \quad \{\text{squaring both sides}\}$$

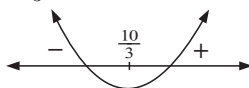
$$\therefore x^2(x^2 - 10x + 26) = (4 + x^2)(25 - 10x + x^2)$$

$$\therefore x^4 - 10x^3 + 26x^2 = 100 - 40x + 4x^2 + 25x^2 - 10x^3 + x^4$$

$$\therefore 3x^2 - 40x + 100 = 0$$

$$\therefore (3x - 10)(x - 10) = 0$$

$$\therefore x = \frac{10}{3} \quad \{\text{as } x \text{ cannot be } 10\}$$

Sign diagram of $\frac{dP}{dx}$ is:
 $\therefore$ minimum length pipeline occurs when $x = \frac{10}{3}$ km
12

$$V = \pi r^2 h$$

$$\therefore 0.1 = \pi r^2 h \quad \{\text{as } 100 \text{ L} = 0.1 \text{ m}^3\}$$

$$\therefore h = \frac{0.1}{\pi r^2}$$

$$\text{Now } A = \pi r^2 + (2\pi r)h = \pi r^2 + 2\pi r \left(\frac{0.1}{\pi r^2} \right)$$

$$\text{i.e., } A(r) = \pi r^2 + 0.2r^{-1}$$

$$\therefore A'(r) = 2\pi r - 0.2r^{-2} = 2\pi r - \frac{0.2}{r^2}$$

$$\therefore A'(r) = 0 \quad \text{when} \quad 2\pi r = \frac{0.2}{r^2}$$

$$r^3 = \frac{0.2}{2\pi}$$

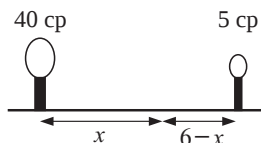
$$r = \sqrt[3]{\frac{0.2}{2\pi}} \doteq 0.3169 \text{ m}$$

$$r \doteq 31.7 \text{ cm}$$

$$\text{Now } A''(r) = 2\pi + 0.4r^{-3} = 2\pi + \frac{0.4}{r^3} \quad \text{which is } > 0 \quad \text{as } r > 0$$

$$\therefore \text{minimum area occurs when } r \doteq 31.7 \text{ cm} \quad \text{and} \quad h \doteq \frac{1}{10\pi(31.69)} \doteq 31.7 \text{ cm}$$

$$\therefore r = h \doteq 31.7 \text{ cm}$$

13


$$I \propto \frac{s}{d^2} \quad \text{where } s \text{ is the power of the source and } d \text{ is the distance from it}$$

$$\therefore I = \frac{kS}{d^2} \quad \{k \text{ is a constant}\}$$

$$\therefore \text{intensity due to 40 cp} = \frac{40k}{x^2}$$

$$\text{and intensity due to 5 cp} = \frac{5k}{(6-x)^2}$$

$$\therefore \text{total intensity, } I = \frac{40k}{x^2} + \frac{5k}{(6-x)^2}$$

$$= k[40x^{-2} + 5(6-x)^{-2}]$$

$$\text{Now } \frac{dI}{dx} = k[-80x^{-3} - 10(6-x)^{-3}(-1)]$$

$$= k \left[\frac{-80}{x^3} + \frac{10}{(6-x)^3} \right]$$

$$\therefore \frac{dI}{dx} = 0 \quad \text{when } \frac{80}{x^3} = \frac{10}{(6-x)^3}$$

$$\text{i.e., } 8(6-x)^3 = x^3$$

$$\therefore 2(6-x) = x \quad \{\text{finding cube roots}\}$$

$$\therefore 12 - 2x = x$$

$$\therefore 12 = 3x$$

$$\therefore x = 4$$

 Sign diagram of $\frac{dI}{dx}$ is:


$\therefore$ the minimum intensity, i.e., the darkest point occurs when $x = 4$ m i.e., at 4 m from the 40 cp lamp.

14 a

$$AB = x \text{ m}$$

$$\therefore BC = (24 - x) \text{ m}$$

$$\therefore AC^2 = AB^2 + BC^2 \quad \{\text{Pythagoras}\}$$

$$= x^2 + (24 - x)^2$$

$$= x^2 + 576 - 48x + x^2$$

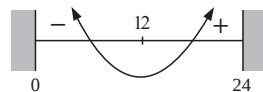
$$= 2x^2 - 48x + 576$$

$$\text{i.e., } [D(x)]^2 = 2x^2 - 48x + 576 \quad \text{and so } D(x) = \sqrt{2x^2 - 48x + 576}$$

b

$$\therefore \frac{d[D(x)]^2}{dx} = 4x - 48$$

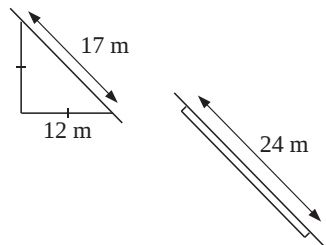
$$\therefore \frac{d[D(x)]^2}{dx} = 0 \quad \text{when } x = 12 \quad \text{and the sign diagram is:}$$



c i.e., when $AB = BC = 12$ m, $D(x)$ is a minimum, and minimum $D(x) = 12\sqrt{2}$ m $\doteq$ 16.97 m.

$D(x)$ is a maximum when either $x = 0$ or $x = 24$,

i.e., when the pen ceases to exist and $D(x) = 24$ m.



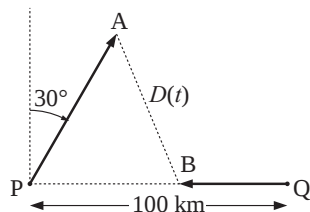
- 15 a** Consider each boat's position t hours after 1.00 pm.

$$AP = 12t \quad BQ = 8t$$

$$\therefore PB = 100 - 8t$$

Using the Cosine rule in $\triangle PAB$

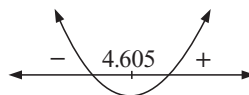
$$\begin{aligned} D(t)^2 &= AP^2 + BP^2 - 2AP \times BP \cos 60^\circ \\ &= (12t)^2 + (100 - 8t)^2 - 2(12t)(100 - 8t)\frac{1}{2} \\ &= 144t^2 + (100 - 8t)^2 - 12t(100 - 8t) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2 \\ &= 304t^2 - 2800t + 10\,000 \quad \text{and so} \quad D(t) = \sqrt{304t^2 - 2800t + 10\,000} \end{aligned}$$



b Now $\frac{d[D(t)]^2}{dt} = 608t - 2800$

$$\therefore \frac{d[D(t)]^2}{dt} = 0 \quad \text{when} \quad t = \frac{2800}{608} \div 4.605\,26$$

and has sign diagram:



$\therefore D(t)$ is a minimum when $t \div 4.60526$ hours after 1.00 pm

$$\text{and} \quad [D(t)]_{\min}^2 \div 304(4.6053)^2 - 2800(4.6053) + 10\,000$$

$$\therefore [D(t)]_{\min}^2 = 3552.63 \text{ km}^2$$

- c** The ships are closest when $t = 4.60526$ hours
i.e., when the time is 4 hours 36 minutes
 $\therefore$ time is approximately 5.36 pm

- 16 a** $\triangle$ s PAB and PRQ are similar.

$$\therefore \frac{PA}{PR} = \frac{PB}{PQ} = \frac{AB}{RQ}$$

$$\therefore \frac{x}{x+2} = \frac{1}{QR} \quad \text{and} \quad \therefore QR = \frac{x+2}{x}$$

- b** Now $[L(x)]^2 = RP^2 + QR^2$ {Pythagoras}

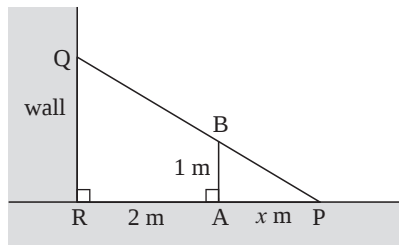
$$\begin{aligned} &= (x+2)^2 + \left(\frac{x+2}{x}\right)^2 \\ &= (x+2)^2 \times 1 + (x+2)^2 \times \frac{1}{x^2} \end{aligned}$$

$$\therefore [L(x)]^2 = (x+2)^2 \left[1 + \frac{1}{x^2}\right] \quad \text{as required}$$

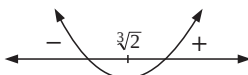
$$\therefore [L(x)]^2 = (x+2)^2 [1 + x^{-2}]$$

- c**
$$\begin{aligned} \frac{d[L(x)]^2}{dx} &= 2(x+2)[1 + x^{-2}] + (x+2)^2[-2x^{-3}] \quad \{\text{product rule}\} \\ &= 2(x+2)[1 + x^{-2} - (x+2)x^{-3}] \\ &= 2(x+2)[1 + x^{-2} - x^{-2} - 2x^{-3}] \\ &= 2(x+2)\left(1 - \frac{2}{x^3}\right) \\ &= 2(x+2)\left(\frac{x^3 - 2}{x^3}\right) \end{aligned}$$

$$\therefore \frac{d[L(x)]^2}{dx} = 0 \quad \text{when} \quad x = \sqrt[3]{2} \div 1.2599 \quad \{\text{as } x > 0 \text{ and } L(x) > 0\}$$



d Sign diagram of $L'(x)$ is:



$\therefore$ the ladder is shortest when $x = \sqrt[3]{2} \text{ m}$ and minimum $L \doteq \sqrt{(x+2)^2(1 + \frac{1}{x^2})} \doteq 4.16 \text{ m}$

17 Suppose $PN = x \text{ m}$

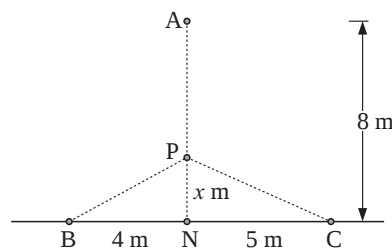
$\therefore$ length of cable = $PA + PB + PC$

$$\therefore L = 8 - x + \sqrt{x^2 + 16} + \sqrt{x^2 + 25}$$

Using technology we graph this and find the minimum,

i.e., the minimum length occurs when $x = 2.57798$

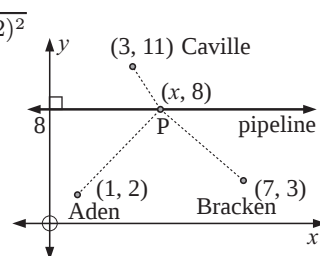
i.e., $x \doteq 2.578 \text{ m}$ from N



18 Suppose P has coordinates $(x, 8)$

$$\begin{aligned} \therefore PC &= \sqrt{(x-3)^2 + (8-11)^2} & AP &= \sqrt{(x-1)^2 + (8-2)^2} \\ &= \sqrt{x^2 - 6x + 9 + 9} & &= \sqrt{x^2 - 2x + 1 + 36} \\ &= \sqrt{x^2 - 6x + 18} & &= \sqrt{x^2 - 2x + 37} \end{aligned}$$

$$\begin{aligned} PB &= \sqrt{(x-7)^2 + (8-3)^2} \\ &= \sqrt{x^2 - 14x + 49 + 25} \\ &= \sqrt{x^2 - 14x + 74} \end{aligned}$$



$$\therefore \text{length of pipeline } L = \sqrt{x^2 - 6x + 18} + \sqrt{x^2 - 2x + 37} + \sqrt{x^2 - 14x + 74}$$

Use technology to graph L and find a minimum. This occurs when $x \doteq 3.54366$,

i.e., P is at $(3.544, 8)$.

19 $y = mx + c$ is the equation of any straight line.

If this line is to pass through $(2a, a)$ then

$$a = m(2a) + c$$

$$a = 2am + c$$

$$\therefore c = a - 2am$$

$$\therefore y = mx + (a - 2am) \quad \dots (1)$$

This line cuts the x -axis when $y = 0$,

$$\therefore 0 = mx + a - 2am$$

$$\therefore mx = 2am - a$$

$$\therefore x = \frac{2am - a}{m}$$

This line cuts the y -axis when $x = 0$,

$$\therefore y = m \times 0 + (a - 2am)$$

$$\text{i.e., } y = a - 2am$$

$$\therefore \text{area of } \triangle AOB, A(m) = \frac{1}{2} \left(\frac{2am - a}{m} \right) (a - 2am)$$

$$= \frac{a^2}{2m} (2m - 1)(1 - 2m)$$

$$= \frac{a^2}{2} \left(\frac{4m - 1 - 4m^2}{m} \right)$$

$$\therefore A(m) = \frac{a^2}{2} (4 - m^{-1} - 4m)$$

$$\text{Now } A'(m) = \frac{a^2}{2} (m^{-2} - 4) \text{ which is 0 when } m^{-2} = 4 \text{ i.e., } m = \pm \frac{1}{2}$$

Now since the graphs are in Q_1 the slopes are negative and so $m = -\frac{1}{2}$

$$\text{Now } A''(m) = \frac{a^2}{2} \times -2m^{-3} = \frac{-a^2}{m^3}$$

$$\text{and } A''(-\frac{1}{2}) = \frac{-a^2}{-\frac{1}{8}} = 8a^2 \text{ which is } > 0$$

$\therefore$ minimum area occurs when $m = -\frac{1}{2}$

$$\begin{aligned} \text{Using (1) } y &= -\frac{1}{2}x + \left(a - 2a(-\frac{1}{2})\right) \\ y &= -\frac{1}{2}x + 2a \end{aligned}$$

Thus the graph meets the x -axis when $y = 0$, $\therefore 0 = -\frac{1}{2}x + 2a$
 $\therefore \frac{1}{2}x = 2a$
 $\therefore x = 4a$ i.e., at $(4a, 0)$

20

$$r^2 + h^2 = s^2$$

$$\therefore h^2 = s^2 - r^2$$

$$h = \sqrt{s^2 - r^2}$$

$$\text{But } V = \frac{1}{3}\pi r^2 h$$

$$\therefore V = \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2}$$

$$\therefore V^2 = \frac{\pi^2}{9} r^4 (s^2 - r^2)$$

$$= \frac{\pi^2}{9} (r^4 s^2 - r^6)$$

$$\therefore \frac{dV^2}{dr} = \frac{\pi^2}{9} (4r^3 s^2 - 6r^5)$$

$$= \frac{\pi^2}{9} 2r^3 (2s^2 - 3r^2)$$

$\frac{dV^2}{dr}$ is 0 when

$$2s^2 - 3r^2 = 0$$

$$\therefore 2s^2 = 3r^2 \quad \{\text{as } r > 0\}$$

$$\therefore \frac{s^2}{r^2} = \frac{3}{2}$$

$$\frac{s}{r} = \sqrt{\frac{3}{2}}$$

$$\text{i.e., } s : r = \sqrt{\frac{3}{2}} : 1$$

Sign diagram of $\frac{dV^2}{dr}$ is:

$\therefore V$ is a maximum when $s : r = \sqrt{\frac{3}{2}} : 1 = \sqrt{3} : \sqrt{2}$

21 a

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore x^2 b^2 + y^2 a^2 = a^2 b^2 \quad \{\times \text{ by } a^2 b^2\}$$

$$\therefore a^2 y^2 = a^2 b^2 - x^2 b^2$$

$$\therefore y^2 = \frac{a^2 b^2 - x^2 b^2}{a^2}$$

$$\therefore y = \pm \sqrt{b^2 - \frac{b^2}{a^2} x^2}$$

Since A lies in Q_1 , $y > 0$

$$\therefore y = \sqrt{b^2 - \frac{b^2}{a^2} x^2}$$

$$\therefore y = \frac{b}{a} \sqrt{a^2 - x^2}$$

b Seating area is

$$A = 2x \times 2y$$

$$= 4xy$$

$$= 4x \left[\frac{b}{a} \sqrt{a^2 - x^2} \right]$$

$$\therefore A(x) = \frac{4bx}{a} \sqrt{a^2 - x^2} \quad \text{as required}$$

$$\therefore A^2 = \frac{16b^2 x^2}{a^2} (a^2 - x^2)$$

$$= \frac{16b^2}{a^2} (a^2 x^2 - x^4)$$

$$c \quad \frac{dA^2}{dx} = \frac{16b^2}{a^2} [2a^2x - 4x^3]$$

which is 0 when

$$2a^2x - 4x^3 = 0$$

$$\therefore 2x(a^2 - 2x^2) = 0$$

$$\therefore 2x^2 = a^2 \quad \{\text{as } x > 0\}$$

$$\therefore x = \pm \frac{a}{\sqrt{2}}$$

$$\therefore x = \frac{a}{\sqrt{2}} \quad \{\text{as } x \text{ is in } Q_1\}$$

[Note that A is a maximum when A^2 is a maximum.]

$$e \quad \% \text{ occupied} = \frac{2ab}{\pi ab} \times 100\% = 63.66\%$$

$$d \quad \text{Sign diagram of } \frac{dA^2}{dx} \text{ is: } \begin{array}{c} \xleftarrow{-} \quad \xrightarrow{+} \quad \xleftarrow{-} \\ \downarrow \quad \downarrow \quad \downarrow \\ 0 \quad \frac{a}{\sqrt{2}} \quad a \\ r=0.5a \quad \quad r=0.8a \end{array}$$

$$\therefore \text{maximum area occurs when } x = \frac{a}{\sqrt{2}}$$

$$\begin{aligned} \text{Max. area} &= \frac{4b}{a} \frac{a}{\sqrt{2}} \sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2} \\ &= \frac{4b}{\sqrt{2}} \times \sqrt{\frac{a^2}{2}} \\ &= \frac{4b}{\sqrt{2}} \times \frac{a}{\sqrt{2}} \\ &= 2ab \end{aligned}$$

EXERCISE 22I

$$1 \quad C(x) = 38\,000 + 250x + x^2$$

$$\begin{aligned} a \quad \text{Cost} &= C(800) \\ &= 38\,000 + 250(800) + (800)^2 \\ &= \$878\,000 \end{aligned}$$

$$\begin{aligned} \text{Average cost} &= \frac{\$878\,000}{800} \\ &= \$1097.50 \end{aligned}$$

$$\begin{aligned} \text{Marginal cost} &= C'(x) \\ &= 250 + 2x \end{aligned}$$

$$\begin{aligned} \therefore C'(800) &= 250 + 2(800) \\ &= \$1850 \end{aligned}$$

$$b \quad A(x) \text{ is minimised when } A(x) = C'(x)$$

$$\text{i.e., } \frac{C(x)}{x} = C'(x)$$

$$C(x) = xC'(x)$$

$$38\,000 + 250x + x^2 = x(250 + 2x)$$

$$38\,000 + 250x + x^2 = 250x + 2x^2$$

$$\therefore 38\,000 = x^2$$

$$\therefore x = \sqrt{38\,000} \quad \{\text{as } x > 0\}$$

$$\therefore x \doteq 195$$

i.e., A is minimised when approximately 195 items are produced.

$$\begin{aligned} \text{Average cost} &= \frac{C(x)}{x} \\ &= \frac{C(195)}{195} \\ &= \frac{\$124\,775}{195} \\ &\doteq \$639.87 \end{aligned}$$

$$2 \quad a \quad C(x) = 295 + 24x - 0.08x^2 + 0.0008x^3$$

$$\text{Average cost, } A(x) = \frac{C(x)}{x} = \frac{295}{x} + 24 - 0.08x + 0.0008x^2$$

$$\text{Marginal cost} = C'(x) = 24 - 0.16x + 0.0024x^2$$

$$b \quad \text{Average cost is minimised when } C'(x) = A(x)$$

$$\therefore 24 - 0.16x + 0.0024x^2 = \frac{295}{x} + 24 - 0.08x + 0.0008x^2$$

$$\therefore 24x - 0.16x^2 + 0.0024x^3 = 295 + 24x - 0.08x^2 + 0.0008x^3$$

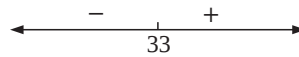
$$\therefore 0.0016x^3 - 0.08x^2 - 295 = 0$$

Using technology, $x \doteq 79.311$ and so, $x > 79$ items

$$\begin{aligned}
 \therefore \text{minimum average cost} &= A(79) \\
 &= \frac{295}{79} + 24 - 0.08(79) + 0.0008(79) \\
 &= \$26.41
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad C'(x) &= 24 - 0.16x + 0.0024x^2 \\
 C''(x) &= -0.16 + 0.0048x \\
 \text{which is 0 when } 0.0048x &= 0.16 \\
 \therefore x &= \frac{0.16}{0.0048} \doteq 33.33 \\
 \text{i.e., } x &= 33 \text{ items}
 \end{aligned}$$

Sign diagram for $C''(x)$ is:



$$\begin{aligned}
 \therefore C'(x) \text{ is a minimum when } x &= 33 \text{ items} \\
 \therefore \text{minimum marginal cost} &= C'(33) = \$21.33
 \end{aligned}$$

3 Suppose x fittings are produced daily.

$$\therefore C(x) = 1000 + 2x + \frac{5000}{x} = 1000 + 2x + 5000x^{-1} \text{ dollars}$$

$$\therefore C'(x) = 2 - \frac{5000}{x^2}$$

$$\begin{aligned}
 C'(x) = 0 \quad \text{when} \quad x^2 &= 2500 \\
 \text{i.e., } x &= 50 \quad \{\text{as } x > 0\}
 \end{aligned}$$

$$\begin{aligned}
 C''(x) &= 10000x^{-3} \\
 &= \frac{10000}{x^3}
 \end{aligned}$$

which is > 0 when $x > 0$

$\therefore$ there is minimum cost when 50 fittings are produced.

$$\mathbf{4} \quad C(x) = 720 + 4x + 0.02x^2 \text{ dollars} \quad \text{and} \quad p(x) = 15 - 0.002x \text{ dollars}$$

$$\therefore R(x) = xp(x) = 15x - 0.002x^2 \quad \text{and} \quad R'(x) = 15 - 0.004x$$

If $C'(x) = R'(x)$ then

$$4 + 0.04x = 15 - 0.004x$$

$$\therefore 0.044x = 11$$

$$\therefore x = 250$$

And as $R''(x) = -0.004$ and $C''(x) = 0.04$ i.e., $R''(250) < C''(250)$

then maximum profit is made when 250 items are produced.

$$\mathbf{5} \quad C(x) = \frac{1}{4}x^2 + 8x + 200 \quad \text{and} \quad P(x) = 23 - \frac{1}{2}x$$

$$\therefore R(x) = xP(x) = x(23 - \frac{1}{2}x) = 23x - \frac{1}{2}x^2 \quad \therefore R'(x) = 23 - x$$

If $C'(x) = R'(x)$ then

$$\frac{1}{2}x + 8 = 23 - x$$

$$\therefore \frac{3}{2}x = 15$$

$$\therefore x = 10$$

And as $R''(x) = -1$ and $C''(x) = \frac{1}{2}$ i.e., $R''(10) < C''(10)$

then maximum profit is made when 10 blankets/day are produced.

- 6 a** Let the demand function be $p(x) = A + Bx$

$$\text{Now } p(800) = 150$$

$$\therefore A + 800B = 150 \dots\dots (1)$$

$$\text{and } p(840) = 145$$

$$\therefore A + 840B = 145 \dots\dots (2)$$

$$(2) - (1) \text{ gives } 40B = -5 \text{ and so } B = -\frac{1}{8}$$

$$\text{and } A = 150 - 800B = 150 - 800\left(-\frac{1}{8}\right) = 250$$

$$\text{i.e., the demand function is } p(x) = 250 - \frac{1}{8}x$$

b Revenue $R(x) = xp(x) = x\left(250 - \frac{x}{8}\right) = 250x - \frac{x^2}{8}$

$$\text{Now } R'(x) = 250 - \frac{2x}{8} \text{ which is 0 when } \frac{x}{4} = 250 \text{ i.e., } x = 1000$$

$$\text{and as } R''(x) = -\frac{1}{4} < 0, \text{ maximum revenue occurs when } x = 1000$$

$$\therefore \text{ need to sell at } p(1000) = 250 - \frac{1000}{8} = \$125$$

$$\text{and must have a rebate of } \$150 - \$125 = \$25$$

c $C(x) = 20\,000 + 30x$ dollars

$$\text{If } C'(x) = R'(x) \text{ then } 30 = 250 - \frac{x}{4}$$

$$\therefore \frac{x}{4} = 220$$

$$\therefore x = 880$$

$$\text{And as } R''(x) = -\frac{1}{4} \text{ and } C''(x) = 0 \text{ i.e., } R''(880) < C''(880)$$

then maximum profit is made when 880 DVD players/week are produced.

$$\therefore \text{ need to sell 880 DVDs and the cost per item } = p(880)$$

$$= \$\left(250 - \frac{880}{8}\right)$$

$$= \$250 - \$110$$

$$= \$140.00$$

$$\text{i.e., offer a rebate of } \$150 - \$140 = \$10.$$

7 Cost/hour = running costs + other costs = $\frac{v^2}{10} + 62.5$

$$\text{Now } C(v) = \frac{\frac{v^2}{10} + 62.5}{v} \quad \left\{ \text{cost/km} = \frac{\text{cost/hour}}{\text{km/hour}} \right\}$$

$$\therefore C(v) = \frac{v}{10} + \frac{62.5}{v} = 0.1v + 62.5v^{-1}$$

$$\therefore C'(v) = 0.1 - 62.5v^{-2}$$

$$\text{which is 0 when } 0.1 = \frac{62.5}{v^2}$$

$$\therefore v^2 = 625$$

$$\therefore v = 25 \quad \{\text{as } v > 0\}$$

$$\text{and } C''(x) = 62.5 \times 2v^{-3} = \frac{125}{v^3} \text{ which is } > 0 \text{ when } v > 0$$

$$\therefore \text{ minimum cost/km occurs when } v = 25 \text{ km/h}$$

REVIEW SET 22A

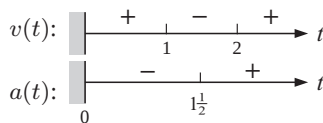
1 a $s(t) = 2t^3 - 9t^2 + 12t - 5 \text{ cm}, \quad t \geq 0$

$$v(t) = 6t^2 - 18t + 12$$

$$= 6(t^2 - 3t + 2)$$

$$= 6(t-2)(t-1) \text{ cm s}^{-1}$$

b When $t = 0$, $s(0) = -5 \text{ cm}$
 $v(0) = 12 \text{ cm s}^{-1}$
 $a(0) = -18 \text{ cm s}^{-2}$



and $a(t) = 12t - 18$
 $= 6(2t - 3) \text{ cm s}^{-2}$

c When $t = 2$, $s(2) = -1 \text{ cm}$
 $v(2) = 0 \text{ cm s}^{-1}$
 $a(2) = 6 \text{ cm s}^{-2}$

i.e., when $t = 2$, the particle is 1 cm to the left of O, instantaneously at rest and increasing in speed towards O.

d Particle changes direction when $t = 1$ and $t = 2$ $s(1) = 0 \text{ cm}$ $s(2) = -1 \text{ cm}$

e **f** Speed is increasing when $1 \leq t \leq \frac{3}{2}$ and $t \geq 2$
 $\{v(t) \text{ and } a(t) \text{ have the same sign}\}$

2 $C(v) = 10v + \frac{90}{v}$ dollars/hour

a i $t = 2$ hours at $v = 15 \text{ kmph}$
 $\therefore \text{cost} = \$ \left(10 \times 15 + \frac{90}{15} \right) \times 2$
 $= \$312.00$

ii $t = 5$ hours at $v = 24 \text{ kmph}$
 $\therefore \text{cost} = \$ \left(10 \times 24 + \frac{90}{24} \right) \times 5$
 $= \$1218.75$

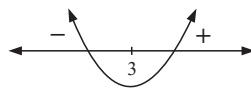
b $C'(v) = 10 - 90v^{-2} = 10 - \frac{90}{v^2}$

i If $v = 10 \text{ kmph}$
 $\therefore C'(10) = 10 - \frac{90}{10^2}$
 $= 10 - 0.9$
 $= \$9.10 \text{ per kmph}$

ii If $v = 6 \text{ kmph}$
 $\therefore C'(6) = 10 - \frac{90}{6^2}$
 $= \$7.50 \text{ per kmph}$

c Now $C'(v) = 10 - \frac{90}{v^2} = \frac{10v^2 - 90}{v^2}$
 $\therefore C'(v) = 0$ when $v^2 = 9$
i.e., $v = 3$ {as $v > 0$ }

$\therefore$ minimum cost occurs when $v = 3 \text{ kmph}$



3 a $f(x) = 2x^3 - 3x^2 - 36x + 7$
 $\therefore f'(x) = 6x^2 - 6x - 36$
 $= 6(x^2 - x - 6)$
 $= 6(x-3)(x+2)$

and $f'(x)$ has sign diagram:

and $f(-2) = 51$, $f(3) = -74$

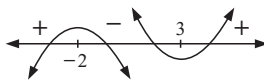
So there is a local maximum at $(-2, 51)$, and a local minimum at $(3, -74)$

$$f''(x) = 12x - 6$$

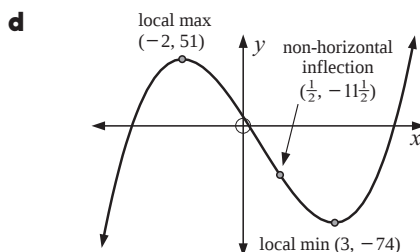
$$= 6(2x - 1)$$

and $f''(x)$ has sign diagram:

and $f(\frac{1}{2}) = -\frac{23}{2}$ so there is a non-horizontal inflection at $(\frac{1}{2}, -\frac{23}{2})$



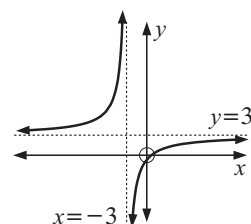
- b** $f(x)$ is increasing when $x \leq -2$ or $x \geq 3$,
and decreasing when $-2 \leq x \leq 3$.
- c** $f(x)$ is concave up when $x \geq \frac{1}{2}$,
and concave down when $x \leq \frac{1}{2}$.



4 $f(x) = \frac{3x-2}{x+3}$

- a** The vertical asymptote is $x = -3$ {when $x+3=0$ }
- b** Cuts x -axis at $(\frac{2}{3}, 0)$, y -axis at $(0, -\frac{2}{3})$
- c** $f'(x) = \frac{3(x+3) - (3x-2)(1)}{(x+3)^2} = \frac{11}{(x+3)^2}$

which has sign diagram: $\leftarrow + \quad -3 \quad + \rightarrow x$



- d** No stationary points exist since $f'(x)$ is never 0.

- 5** **a** Now if $OD = x$, the coordinates of C are $(x, k - x^2)$

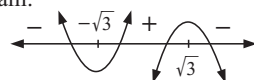
$$\therefore \text{area of ABCD} = 2x \times (k - x^2)$$

$$\text{i.e., } A = 2kx - 2x^3$$

- b** $\frac{dA}{dx} = 2k - 6x^2$ and $\frac{dA}{dx} = 0$ when $AD = 2\sqrt{3}$ i.e., $x = \sqrt{3}$
- $$\therefore 2k - 6(\sqrt{3})^2 = 0$$
- $$\therefore 2k - 18 = 0$$
- $$\therefore 2k = 18$$
- $$\therefore k = 9$$

$$\begin{aligned} \text{Checking } \frac{dA}{dx} &= 18 - 6x^2 \\ &= 6(3 - x^2) \\ &= 6(\sqrt{3} + x)(\sqrt{3} - x) \end{aligned}$$

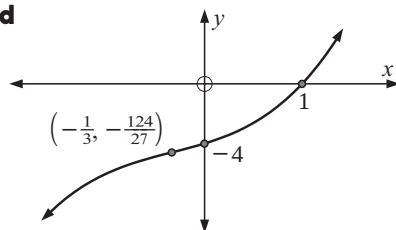
which has sign diagram:



i.e., maximum occurs at $x = \sqrt{3}$ i.e., when $AD = 2\sqrt{3}$

6 $f(x) = x^3 + x^2 + 2x - 4$

- a** y -intercept is -4
- b** x -intercepts occur when $x^3 + x^2 + 2x - 4 = 0$
 $\therefore (x-1)(x^2 + 2x + 4) = 0$ given $x = 1$ is one root
 Now $x^2 + 2x + 4 = 0$ has $\Delta < 0$ $\therefore$ the only x -intercept is 1.
- c** $f'(x) = 3x^2 + 2x + 2$ has $\Delta = 4 - 24 = -20$
 which is < 0 , $\therefore f'(x)$ is never 0,
 so there are no stationary points
 $f''(x) = 6x + 2$
 $\therefore f''(x) = 0$ when $x = -\frac{1}{3}$
 $\therefore$ a non-horizontal inflection occurs at $(-\frac{1}{3}, -\frac{124}{27})$
- d**



$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad A &= 500 \text{ m}^2 \\ \therefore xy &= 500 \\ \therefore y &= \frac{500}{x}, \quad x > 0 \end{aligned}$$

c As the width of the rectangle increases, the length must decrease to maintain a constant area.

$$\begin{aligned} \mathbf{b} \quad y &= 500x^{-1} \\ \therefore \frac{dy}{dx} &= -500x^{-2} = -\frac{500}{x^2} \\ \text{and since } x^2 &\geq 0 \text{ for all } x, \\ \frac{dy}{dx} &< 0 \text{ for all } x \end{aligned}$$

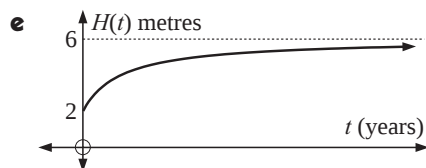
$$\mathbf{8} \quad H(t) = 6 \left(1 - \frac{2}{t+3} \right) = 6 - 12(t+3)^{-1} \text{ m}, \quad t > 0$$

$$\mathbf{a} \quad \text{When } t = 0, \quad H(0) = 6 \left(1 - \frac{2}{3} \right) = 6 \left(\frac{1}{3} \right) = 2 \text{ m}$$

$$\begin{aligned} \mathbf{b} \quad H(3) &= 6 \left(1 - \frac{2}{6} \right) & H(6) &= 6 \left(1 - \frac{2}{9} \right) & H(9) &= 6 \left(1 - \frac{2}{12} \right) \\ &= 6 \left(\frac{2}{3} \right) & &= 6 \left(\frac{7}{9} \right) & &= 6 \left(\frac{5}{6} \right) \\ &= 4 \text{ m} & &= \frac{14}{3} \text{ m} & &= 5 \text{ m} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad H'(t) &= \frac{12}{(t+3)^2} & \therefore H'(0) &= \frac{12}{9} = \frac{4}{3} \text{ m yr}^{-1} & H'(6) &= \frac{12}{81} = \frac{4}{27} \text{ m yr}^{-1} \\ & & H'(3) &= \frac{12}{36} = \frac{1}{3} \text{ m yr}^{-1} & H'(9) &= \frac{12}{144} = \frac{1}{12} \text{ m yr}^{-1} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \text{Since } H'(t) &= \frac{12}{(t+3)^2} \\ H'(t) &> 0 \text{ for all } t \geq 0 \\ \text{i.e., } H(t) &\text{ continually increases.} \\ \text{i.e., the tree} &\text{ is continually growing.} \end{aligned}$$



$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad \text{Volume} &= lbd \\ \therefore x^2y &= 1 \\ \therefore y &= \frac{1}{x^2}, \quad x > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{area} &= x^2 + 4xy \\ \therefore \text{cost} &= (x^2 + 4xy) \times 2 \\ &= 2x^2 + 8xy \\ \text{But } y &= \frac{1}{x^2} \\ \therefore C &= 2x^2 + \frac{8}{x} \text{ dollars} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{Thus } \frac{dC}{dx} &= 4x - 8x^{-2} \\ &= 4x - \frac{8}{x^2} \\ &= \frac{4(x^3 - 2)}{x^2} \end{aligned}$$

$$\text{which is 0 when } x = \sqrt[3]{2} \text{ m}$$

$$\text{and } \frac{dC}{dx} \text{ has sign diagram: } \begin{array}{c} \leftarrow - \quad \quad \quad \sqrt[3]{2} \quad \quad \quad + \rightarrow \\ \quad \quad \quad \cup \end{array}$$

$$\begin{aligned} \therefore \text{minimum cost when } x &= \sqrt[3]{2} \text{ m i.e., } x = 1.26 \\ \therefore y &= \frac{1}{x^2} \div 0.630 \end{aligned}$$

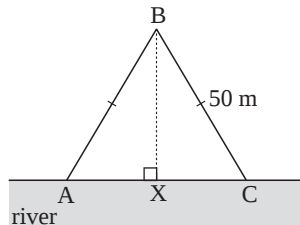
$$\mathbf{10} \quad \mathbf{a} \quad AC = 2x \text{ m}$$

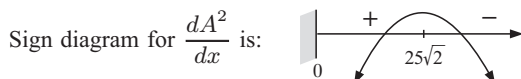
Now, ABC is an isosceles triangle.

$$\begin{aligned} \therefore XC &= x \\ \text{But, } BC^2 &= BX^2 + XC^2 \quad \{\text{Pythagoras}\} \\ \therefore 2500 &= BX^2 + x^2 \\ \therefore BX &= \sqrt{2500 - x^2} \\ \therefore A(x) &= \frac{1}{2}(2x)\sqrt{2500 - x^2} = x\sqrt{2500 - x^2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{Now } [A(x)]^2 &= x^2(2500 - x^2) \\ \text{i.e., } A^2 &= 2500x^2 - x^4 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dA^2}{dx} &= 5000x - 4x^3 \\ &= 4x(1250 - x^2) \\ &= 2x(\sqrt{1250} + x)(\sqrt{1250} - x) \end{aligned}$$





$\therefore$ maximum area occurs when $x = 25\sqrt{2} \text{ m} \doteq 35.4 \text{ m}$

The corresponding maximum area $\doteq 1250 \text{ m}^2$.

11 $s(t) = 15t - \frac{60}{(t-1)^2} \text{ cm}, \quad t \geq 0$

a $\therefore s(t) = 15t - 60(t-1)^{-2} \text{ cm}$

$\therefore v(t) = 15 + 120(t-1)^{-3} \text{ cm s}^{-1}$

$\therefore a(t) = -360(t-1)^{-4} \text{ cm s}^{-2}$

c $v(t) = 15 + \frac{120}{(t-1)^3} \text{ cm s}^{-1}$

$\therefore v(t) = 0$

when $15 + \frac{120}{(t-1)^3} = 0$

$\therefore 15(t-1)^3 + 120 = 0$

$\therefore (t-1)^3 = -8$

$\therefore t = -1$

$a(t) = -360(t-1)^{-4} = \frac{-360}{(t-1)^4} \text{ cm s}^{-2}$

where $(t-1)^4$ is always positive.

$\therefore a(t) < 0$ for all $t > 0$

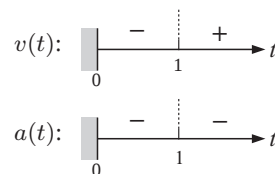
So, speed increases for $0 \leq t < 1$ as $v(t)$, $a(t)$ have the same sign.

b When $t = 3$, $s(t) = 30 \text{ cm}$

$v(t) = 30 \text{ cm s}^{-1}$

$a(t) = -22.5 \text{ cm s}^{-2}$

The particle is 30 cm right of O, travelling right at 30 cm s^{-1} and is slowing down at 22.5 cm s^{-2}



- 12** Suppose the sheet is bent x cm from each end.
To maximise the water carried we need to maximise the area of cross-section.

$A = x(24 - 2x)$

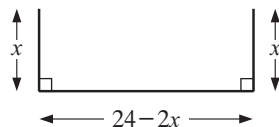
$\therefore A = 24x - 2x^2$

and $\frac{dA}{dx} = 24 - 4x$

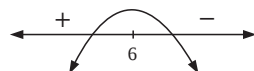
which is 0 when $x = 6$

$\therefore$ maximum water when $x = 6 \text{ cm}$

i.e., bends must be made 6 cm from each end.



Sign diagram of $\frac{dA}{dx}$ is:



REVIEW SET 22B

1 $W(t) = 5000 - \frac{4900}{t^2 + 1} \text{ g}, \quad t \geq 0$

a When $t = 0$, $W(0) = 5000 - \frac{4900}{1} = 100 \text{ g}$

b i When $t = 1$,

$W(1) = 5000 - \frac{4900}{2}$
 $= 2550 \text{ g}$

ii When $t = 4$,

$W(4) = 5000 - \frac{4900}{17}$
 $\doteq 4711.8 \text{ g}$

iii When $t = 26$,

$W(26) = 5000 - \frac{4900}{677}$
 $\doteq 4992.8 \text{ g}$

$$\text{c } W'(t) = 4900(t^2 + 1)^{-2} \times 2t = \frac{9800t}{(t^2 + 1)^2}$$

$$\text{i } t = 0 \text{ days} = 0 \text{ weeks} \quad \text{ii } t = 10 \text{ days} = \frac{10}{7} \text{ weeks} \quad \text{iii } t = 20 \text{ days} = \frac{20}{7} \text{ weeks}$$

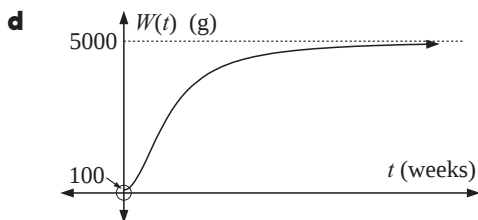
$$\therefore W'(0) = 0$$

$$\therefore W'(\frac{10}{7}) = \frac{9800(\frac{10}{7})}{((\frac{10}{7})^2 + 1)^2}$$

$$\div 1514 \text{ g/week}$$

$$\therefore W'(\frac{20}{7}) = \frac{9800(\frac{20}{7})}{((\frac{20}{7})^2 + 1)^2}$$

$$\div 333.5 \text{ g/week}$$



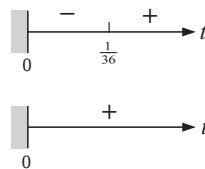
$$\text{2 } x(t) = 3t - \sqrt{t} \text{ cm, } t \geq 0$$

$$\text{a } x(t) = 3t - t^{\frac{1}{2}}$$

$$\therefore v(t) = 3 - \frac{1}{2}t^{-\frac{1}{2}} = 3 - \frac{1}{2\sqrt{t}} = \frac{6\sqrt{t} - 1}{2\sqrt{t}}$$

$$\text{which is 0 when } \sqrt{t} = \frac{1}{6}, \text{ i.e., } t = \frac{1}{36}$$

$$\text{and } a(t) = \frac{1}{4}t^{-\frac{3}{2}} = \frac{1}{4t\sqrt{t}} \text{ which is always positive}$$



$$\text{b } \text{When } t = 0, \quad x(0) = 0 \text{ cm}$$

$v(0)$ is infinitely large and negative, so is not defined

$a(0)$ is infinitely large and positive, so is not defined

i.e., the particle is at O, moving left and slowing down.

$$\text{c } \text{When } t = 9, \quad x(9) = 24 \text{ cm} \quad v(9) = \frac{17}{6} \text{ cm s}^{-1} \quad a(9) = \frac{1}{108} \text{ cm s}^{-2}$$

the particle is 24 cm right of O, moving right at $\frac{17}{6} \text{ cm s}^{-1}$ and increasing its speed.

$$\text{d } \text{The particle reverses direction when } t = \frac{1}{36} \text{ seconds.}$$

$$x\left(\frac{1}{36}\right) = \frac{3}{36} - \frac{1}{6} = -\frac{3}{36} = -\frac{1}{12}, \text{ i.e., it is } \frac{1}{12} \div 0.083 \text{ cm to the left of O.}$$

$$\text{e } \text{The particle's speed decreases when } v(t) \text{ and } a(t) \text{ have different signs,}$$

$$\text{i.e., for } 0 \leq t \leq \frac{1}{36}.$$

$$\text{3 a } G = \frac{1}{3}(t-2)^2 + 5$$

$$\therefore \frac{dG}{dt} = \frac{2}{3}(t-2)^1$$

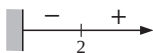
$$\text{which is 0 when } t = 2$$

$$\therefore G \text{ is increasing for } t \geq 2$$

$$\text{b } \frac{dG}{dt} > 10 \quad \therefore \frac{2}{3}(t-2) > 10$$

$$\therefore t-2 > 15$$

$$\therefore t > 17$$



$$\text{4 } f(x) = x^3 - 4x^2 + 4x$$

$$= x(x^2 - 4x + 4)$$

$$= x(x-2)^2$$

$$\text{a } \text{Cuts the } y\text{-axis when } x = 0 \text{ i.e., at } (0, 0)$$

$$\text{Cuts the } x\text{-axis when } y = 0 \text{ i.e., } x(x-2)^2 = 0,$$

$$\text{i.e., when } x = 0 \text{ or } 2$$

$$\text{i.e., at } (0, 0) \text{ and } (2, 0)$$

$$\begin{aligned} \mathbf{b} \quad f'(x) &= 3x^2 - 8x + 4 \\ &= (3x - 2)(x - 2) \\ \text{which is 0 when } x &= \frac{2}{3} \text{ or } 2 \end{aligned}$$

Sign diagram of $f'(x)$ is:

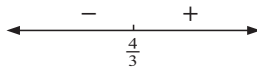


$\therefore$ there is a local maximum at $(\frac{2}{3}, \frac{32}{27})$,

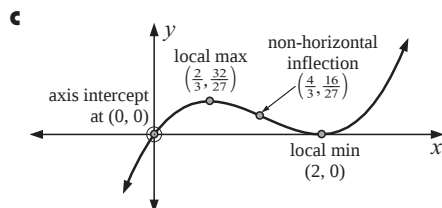
and a local minimum at $(2, 0)$

$$f''(x) = 6x - 8 = 2(3x - 4)$$

Sign diagram of $f''(x)$ is:



$\therefore$ there is a non-horizontal inflection at $(\frac{4}{3}, \frac{16}{27})$



$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad P &= 200 \text{ m} \\ \therefore P &= 2x + 2y + \pi x \\ \therefore 200 &= 2x + 2y + \pi x \\ \therefore 2y &= 200 - 2x - \pi x \\ \therefore y &= 100 - x - \frac{\pi}{2}x \end{aligned}$$

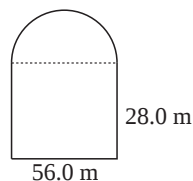
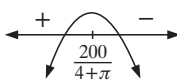
$$\begin{aligned} \mathbf{b} \quad \text{Area of lawn} &= 2x \times y + \frac{1}{2}\pi x^2 \\ &= 2x \left[100 - x - \frac{\pi}{2}x \right] + \frac{1}{2}\pi x^2 \\ &= 200x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2 \\ &= 200x - 2x^2 - \frac{\pi}{2}x^2 \\ \therefore A &= 200x - \left(2 + \frac{\pi}{2} \right) x^2 \text{ m}^2 \end{aligned}$$

$$\mathbf{c} \quad \frac{dA}{dx} = 200 - 2 \left(2 + \frac{\pi}{2} \right) x$$

$$= 200 - (4 + \pi)x \quad \text{which is 0 when } (4 + \pi)x = 200$$

$$x = \frac{200}{4 + \pi}$$

and the sign diagram for $\frac{dA}{dx}$ is:



$\therefore$ maximum area occurs when $x = \frac{200}{4 + \pi} \div 28.00 \text{ m}$

$$\mathbf{6} \quad f(x) = x\sqrt{x} - x$$

$\mathbf{a} \quad f(x)$ has meaning provided $x \geq 0$

$\mathbf{b} \quad$ Cuts y -axis when $x = 0$, i.e., at $(0, 0)$

Cuts x -axis when $x\sqrt{x} - x = 0$

$$\therefore x(\sqrt{x} - 1) = 0$$

i.e., when $x = 0$ or $\sqrt{x} - 1 = 0$

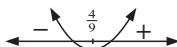
$$\therefore x = 0 \text{ or } \sqrt{x} = 1$$

$$\therefore x = 0 \text{ or } x = 1$$

i.e., at $(0, 0)$ and $(1, 0)$

$$\begin{aligned} \mathbf{c} \quad f(x) &= x^{\frac{3}{2}} - x \\ \therefore f'(x) &= \frac{3}{2}x^{\frac{1}{2}} - 1 \\ \text{which is 0 when } \frac{3}{2}\sqrt{x} &= 1 \\ \therefore \sqrt{x} &= \frac{2}{3} \\ x &= \frac{4}{9} \end{aligned}$$

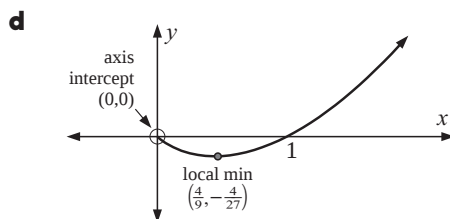
Sign diagram of $f'(x)$ is:



$\therefore$ there is a local minimum at $(\frac{4}{9}, -\frac{4}{27})$

$$\text{and } f''(x) = \frac{3}{2}x^{-\frac{1}{2}} = \frac{3}{4\sqrt{x}}$$

Since $f''(x) \neq 0$ no inflections.



7 $f(x) = \frac{x^2 - 1}{x^2 + 1}$

a Cuts x -axis when $y = 0$

i.e., $x^2 - 1 = 0$, $\therefore x = \pm 1$

i.e., at $(1, 0)$ and $(-1, 0)$

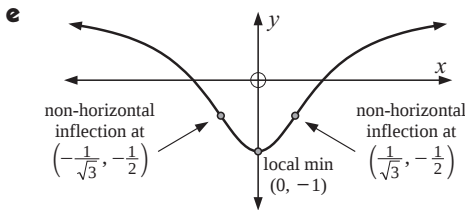
Cuts y -axis when $x = 0$, i.e., $(0, -1)$

c
$$f'(x) = \frac{2x(x^2 + 1) - (x^2 - 1)2x}{(x^2 + 1)^2}$$

$$= \frac{4x}{(x^2 + 1)^2}$$

Sign diagram of $f'(x)$ is: 

$\therefore$ there is a local minimum at $(0, -1)$



b As $x^2 \geq 0$, $x^2 + 1$ can never be 0.

$\therefore$ there are no vertical asymptotes.

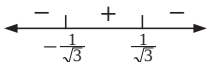
d
$$f''(x) = \frac{4(x^2 + 1)^2 - 4x \times 2(x^2 + 1)(2x)}{(x^2 + 1)^4}$$

$$= \frac{4(x^2 + 1) - 16x^2}{(x^2 + 1)^3}$$

$$= \frac{4 - 12x^2}{(x^2 + 1)^3}$$

$$= \frac{4(1 + \sqrt{3}x)(1 - \sqrt{3}x)}{(x^2 + 1)^3}$$

$\therefore f''(x) = 0$ when $x = \pm \frac{1}{\sqrt{3}}$

and sign diagram is: 

$\therefore f(x)$ has nonstationary inflections at $x = \pm \sqrt{\frac{1}{3}}$.

8 $f(x) = \frac{x - 2}{x^2 + x - 2} = \frac{x - 2}{(x + 2)(x - 1)}$

a Vertical asymptotes are: $x = -2$ and $x = 1$

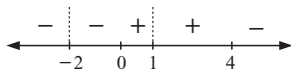
b
$$f'(x) = \frac{1(x^2 + x - 2) - (x - 2)(2x + 1)}{(x^2 + x - 2)^2}$$

$$= \frac{x^2 + x - 2 - 2x^2 + 3x + 2}{(x + 2)^2(x - 1)^2}$$

$$= \frac{4x - x^2}{(x + 2)^2(x - 1)^2}$$

$$= \frac{-x(x - 4)}{(x^2 + x - 2)^2}$$

which has sign diagram:



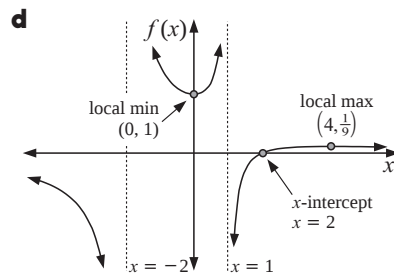
$\therefore$ there is a local minimum at $(0, 1)$,

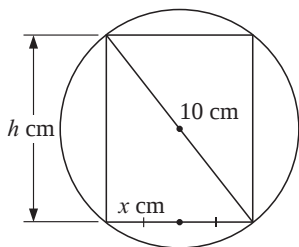
and a local maximum at $(4, \frac{1}{9})$

e For $\frac{x - 2}{x^2 + x - 2} = p$ to have 2 real distinct roots the horizontal line $y = p$ has to cut the curve in exactly two different places, i.e., $p < 0$ or $p > 1$ or $0 < p < \frac{1}{9}$

{when the denominator is 0}

c Cuts y -axis when $x = 0$ i.e., at $(0, 1)$
Cuts x -axis when $y = 0$ i.e., at $(2, 0)$



9 aLet the height of the cylinder be h cm.

$$\therefore (2x)^2 + h^2 = 10^2 \quad \{\text{Pythagoras}\}$$

$$\therefore h = \sqrt{100 - 4x^2}$$

$$\therefore V(x) = \text{area of base} \times \text{height} \\ = \pi x^2 \times \sqrt{100 - 4x^2}$$

$$\text{So, } V(x) = \pi x^2 \sqrt{100 - 4x^2} \text{ cm}^2$$

$$\begin{aligned} \text{b Now } V^2 &= \pi^2 x^4 (100 - 4x^2) \\ &= \pi^2 (100x^4 - 4x^6) \end{aligned}$$

$$\begin{aligned} \therefore \frac{dV^2}{dx} &= \pi^2 (400x^3 - 24x^5) \\ &= 8\pi^2 x^3 (50 - 3x^2) \\ &= 8\pi^2 x^3 (\sqrt{50} + \sqrt{3}x)(\sqrt{50} - \sqrt{3}x) \end{aligned}$$

$$\text{which is 0 when } x = \sqrt{\frac{50}{3}} \quad \{\text{as } x > 0\}$$

$$\text{and } \frac{dV^2}{dx} \text{ has sign diagram: } \begin{array}{c} \text{+} \\ \text{0} \quad \sqrt{\frac{50}{3}} \quad \text{---} \end{array}$$

$$\therefore \text{maximum } V \text{ occurs when } x = \sqrt{\frac{50}{3}} \div 4.08$$

$$\therefore \text{radius} = 4.08 \text{ cm, height} = \sqrt{100 - 4\left(\frac{50}{3}\right)} \div 5.77 \text{ cm}$$

10 a $\triangle s$ LQX and XPM are similar.

$$\therefore \frac{LQ}{XP} = \frac{LX}{XM} = \frac{QX}{PM}$$

$$\therefore \frac{LQ}{1} = \frac{8}{PM}$$

$$\therefore LQ = \frac{8}{PM}$$

$$\therefore LQ = \frac{8}{x} \text{ km}$$

b

$$L(x) = LX + XM$$

$$= \sqrt{\left(\frac{8}{x}\right)^2 + 8^2} + \sqrt{1^2 + x^2}$$

$$= 8\sqrt{\frac{1}{x^2} + 1} + \sqrt{x^2 + 1}$$

$$= \frac{8}{x}\sqrt{1 + x^2} + 1\sqrt{1 + x^2}$$

$$= \sqrt{x^2 + 1} \left(\frac{8}{x} + 1\right)$$

$$\therefore L(x) = \sqrt{x^2 + 1} \left(1 + \frac{8}{x}\right)$$

$$\text{and } L^2 = (x^2 + 1) \left(1 + \frac{8}{x}\right)^2$$

$$\text{c } \frac{d[L(x)]^2}{dx} = 2x \left(1 + \frac{8}{x}\right)^2 + (x^2 + 1)2 \left(1 + \frac{8}{x}\right) \left(-\frac{8}{x^2}\right) \quad \{\text{product rule}\}$$

$$= 2 \left(1 + \frac{8}{x}\right) \left[x \left(1 + \frac{8}{x}\right) - (x^2 + 1) \left(\frac{8}{x^2}\right) \right]$$

$$= 2 \left(1 + \frac{8}{x}\right) \left[x + 8 - 8 - \frac{8}{x^2} \right]$$

$$= 2 \left(\frac{x+8}{x}\right) \left(\frac{x^3-8}{x^2}\right)$$

$$\frac{d[L(x)]^2}{dx} = 0 \quad \text{when } x = -8 \quad \text{or } x^3 = 8$$

$$\text{but } x > 0, \therefore \text{when } x = 2$$

Sign diagram for $\frac{d[L(x)]^2}{dx}$ is:

$\therefore$ minimum $L(x)$ occurs when $x = 2$ and the shortest length is $\sqrt{2^2 + 1} \left(1 + \frac{8}{2}\right)$
 $= 5\sqrt{5}$
 $\doteq 11.2 \text{ km}$

11 When the box is manufactured its base is $(k - 2x)$ by $(k - 2x)$ and height is x cm.

$$\therefore V = x(k - 2x)(k - 2x)$$

$$\text{i.e., } V = x(k - 2x)^2$$

$$\begin{aligned} \therefore \frac{dV}{dx} &= 1(k - 2x)^2 + x \cdot 2(k - 2x)^1(-2) \\ &= (k - 2x)^2 - 4x(k - 2x) \\ &= (k - 2x)[k - 2x - 4x] \\ &= (k - 2x)(k - 6x) \end{aligned}$$

which is 0 when $x = \frac{k}{6}$ or $\frac{k}{2}$, but $k - 2x > 0$ and so $x < \frac{k}{2}$.

and the sign diagram of $\frac{dV}{dx}$ is:

$\therefore$ maximum capacity occurs when $x = \frac{k}{6}$.

12 a $ON = x$, $OC = r$

$$\therefore OB = r \text{ (radii)}$$

$$AN = \sqrt{r^2 - x^2}$$

$$\therefore A(x) = \frac{1}{2}AB \times CN = AN \times CN$$

$$\therefore A(x) = \sqrt{r^2 - x^2}(r + x)$$

b Now $A^2 = (r^2 - x^2)(r + x)^2 = (r - x)(r + x)^3$

$$\begin{aligned} \therefore \frac{dA^2}{dx} &= (-1)(r + x)^3 + (r - x)3(r + x)^2(1) \\ &= (r + x)^2 [-(r + x) + 3(r - x)] \\ &= (r + x)^2 [-r - x + 3r - 3x] = (r + x)^2 [2r - 4x] \end{aligned}$$

which is 0 only when $x = \frac{r}{2}$ {as $x > 0$ }

Sign diagram for $\frac{dA^2}{dx}$ is: $\therefore$ maximum area occurs when $x = \frac{r}{2}$

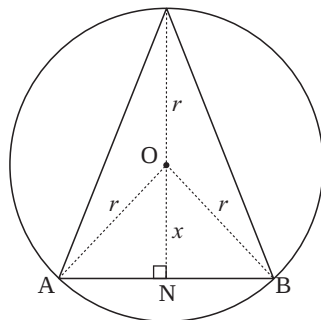
and in this case $CN = r + \frac{r}{2} = \frac{3}{2}r$

$$\text{and } AN = \sqrt{r^2 - \frac{r^2}{4}} = \sqrt{\frac{3r^2}{4}} = \frac{\sqrt{3}}{2}r$$

$$\therefore \tan \angle CAN = \frac{CN}{AN} = \frac{\frac{3}{2}r}{\frac{\sqrt{3}}{2}r} = \sqrt{3}$$

$$\therefore \angle CAN = 60^\circ$$

$\therefore \Delta$ is equilateral {as it is isosceles with base angles 60° }



Chapter 23

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXERCISE 23A

1 a $f(x) = e^{4x}$
 $\therefore f'(x) = 4e^{4x}$

b $f(x) = e^x + 3$
 $\therefore f'(x) = e^x + 0$
 $= e^x$

c $f(x) = \exp(-2x)$
 $\therefore f(x) = e^{-2x}$
 $\therefore f'(x) = -2e^{-2x}$

d $f(x) = e^{\frac{x}{2}}$
 $\therefore f'(x) = \frac{1}{2}e^{\frac{x}{2}}$

e $f(x) = 2e^{-\frac{x}{2}}$
 $\therefore f'(x) = 2e^{-\frac{x}{2}} \left(-\frac{1}{2}\right)$
 $= -e^{-\frac{x}{2}}$

f $f(x) = 1 - 2e^{-x}$
 $\therefore f'(x) = 0 - 2e^{-x}(-1)$
 $= 2e^{-x}$

g $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$
 $\therefore f'(x) = 4e^{\frac{x}{2}} \left(\frac{1}{2}\right)$
 $- 3e^{-x}(-1)$
 $= 2e^{\frac{x}{2}} + 3e^{-x}$

h $f(x) = \frac{e^x + e^{-x}}{2}$
 $\therefore f'(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$
 $\therefore f'(x) = \frac{e^x}{2} + \frac{e^{-x}(-1)}{2}$
 $= \frac{e^x - e^{-x}}{2}$

i $f(x) = e^{-x^2}$
 $\therefore f'(x) = e^{-x^2}(-2x)$
 $= -2xe^{-x^2}$

j $f(x) = e^{\frac{1}{x}}$
 $\therefore f'(x) = e^{\frac{1}{x}} \left(-\frac{1}{x^2}\right)$
 $= -\frac{e^{\frac{1}{x}}}{x^2}$

k $f(x) = 10(1 + e^{2x})$
 $= 10 + 10e^{2x}$
 $\therefore f'(x) = 0 + 10e^{2x}(2)$
 $= 20e^{2x}$

l $f(x) = 20(1 - e^{-2x})$
 $= 20 - 20e^{-2x}$
 $\therefore f'(x) = 0 - 20e^{-2x}(-2)$
 $= 40e^{-2x}$

m $f(x) = e^{2x+1}$
 $\therefore f'(x) = e^{2x+1} \times 2$
 $= 2e^{2x+1}$

n $y = e^{\frac{x}{4}}$
 $\therefore \frac{dy}{dx} = e^{\frac{x}{4}} \times \frac{1}{4}$
 $= \frac{1}{4}e^{\frac{x}{4}}$

o $y = e^{1-2x^2}$
 $\therefore \frac{dy}{dx} = e^{1-2x^2}(-4x)$
 $= -4xe^{1-2x^2}$

p $y = e^{-0.02x} \quad \therefore \frac{dy}{dx} = e^{-0.02x} \times (-0.02) = -0.02e^{-0.02x}$

2 a $f(x) = xe^x$
 $\therefore f'(x) = 1e^x + e^xx$
 $= e^x + xe^x$

$u = x, \quad v = e^x, \quad u' = 1, \quad v' = e^x$
 {product rule}

b $f(x) = x^3e^{-x}$
 $\therefore f'(x) = 3x^2e^{-x} + x^3(-e^{-x})$
 $= 3x^2e^{-x} - x^3e^{-x}$

$u = x^3, \quad v = e^{-x}, \quad u' = 3x^2, \quad v' = -e^{-x}$
 {product rule}

c $f(x) = \frac{e^x}{x}$
 $\therefore f'(x) = \frac{e^x x - e^x(1)}{x^2}$
 $= \frac{xe^x - e^x}{x^2}$

$u = e^x, \quad v = x, \quad u' = e^x, \quad v' = 1$
 {quotient rule}

$$\mathbf{d} \quad f(x) = \frac{x}{e^x} \quad u = x, \quad v = e^x, \quad u' = 1, \quad v' = e^x$$

$$\begin{aligned} \therefore f'(x) &= \frac{1e^x - xe^x}{(e^x)^2} \quad \{\text{quotient rule}\} \\ &= \frac{e^x(1-x)}{(e^x)^2} \\ &= \frac{1-x}{e^x} \end{aligned}$$

$$\mathbf{e} \quad f(x) = x^2 e^{3x} \quad u = x^2, \quad v = e^{3x}, \quad u' = 2x, \quad v' = 3e^{3x}$$

$$\therefore f'(x) = 2xe^{3x} + 3x^2 e^{3x}$$

$$\mathbf{f} \quad f(x) = \frac{e^x}{\sqrt{x}} \quad u = e^x, \quad v = x^{\frac{1}{2}}, \quad u' = e^x, \quad v' = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\begin{aligned} \therefore f'(x) &= \frac{e^x \sqrt{x} - \frac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2} \quad \{\text{quotient rule}\} \\ &= \frac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}} \end{aligned}$$

$$\mathbf{g} \quad f(x) = \sqrt{x}e^{-x} \quad u = x^{\frac{1}{2}}, \quad v = e^{-x}, \quad u' = \frac{1}{2}x^{-\frac{1}{2}}, \quad v' = -e^{-x}$$

$$\therefore f'(x) = \frac{1}{2\sqrt{x}}e^{-x} - \sqrt{x}e^{-x} \quad \{\text{product rule}\}$$

$$\mathbf{h} \quad f(x) = \frac{e^x + 2}{e^{-x} + 1} \quad u = e^x + 2, \quad v = e^{-x} + 1, \quad u' = e^x, \quad v' = -e^{-x}$$

$$\begin{aligned} \therefore f'(x) &= \frac{e^x(e^{-x} + 1) - (e^x + 2)(-e^{-x})}{(e^{-x} + 1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x} + 1)^2} \\ &= \frac{2 + e^x + 2e^{-x}}{(e^{-x} + 1)^2} \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad f(x) = (e^x + 2)^4 = u^4 \quad \text{where} \quad u = e^x + 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 4u^3(e^x)$$

$$\therefore f'(x) = 4(e^x + 2)^3(e^x) = 4e^x(e^x + 2)^3$$

$$\mathbf{b} \quad f(x) = \frac{1}{1 - e^{-x}} = u^{-1} \quad \text{where} \quad u = 1 - e^{-x}$$

$$\therefore f'(x) = \frac{dy}{du} \frac{du}{dx} = -u^{-2}(e^{-x})$$

$$\therefore f'(x) = -\frac{e^{-x}}{(1 - e^{-x})^2}$$

$$\mathbf{c} \quad f(x) = \sqrt{e^{2x} + 10} = u^{\frac{1}{2}} \quad \text{where} \quad u = e^{2x} + 10$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(2e^{2x})$$

$$\therefore f'(x) = \frac{e^{2x}}{\sqrt{e^{2x} + 10}}$$

$$\mathbf{d} \quad f(x) = \frac{1}{(1 - e^{3x})^2} = u^{-2} \quad \text{where} \quad u = 1 - e^{3x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -2u^{-3}(-3e^{3x}) = \frac{6e^{3x}}{u^3}$$

$$\therefore f'(x) = \frac{6e^{3x}}{(1 - e^{3x})^3}$$

$$\mathbf{e} \quad f(x) = \frac{1}{\sqrt{1 - e^{-x}}} = u^{-\frac{1}{2}} \quad \text{where} \quad u = 1 - e^{-x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -\frac{1}{2}u^{-\frac{3}{2}}(e^{-x}) = \frac{-e^{-x}}{2u^{\frac{3}{2}}}$$

$$\therefore f'(x) = \frac{-e^{-x}}{2(1 - e^{-x})^{\frac{3}{2}}}$$

$$\mathbf{f} \quad f(x) = x\sqrt{1 - 2e^{-x}} = x u^{\frac{1}{2}} \quad \text{where} \quad u = 1 - 2e^{-x}$$

$$\therefore f'(x) = 1 u^{\frac{1}{2}} + x \frac{1}{2} u^{-\frac{1}{2}} \frac{du}{dx} \quad \{\text{product and chain rules}\}$$

$$= 1 \sqrt{u} + x \frac{1}{2} u^{-\frac{1}{2}} 2e^{-x}$$

$$= \frac{\sqrt{1 - 2e^{-x}}}{1} + \frac{x e^{-x}}{\sqrt{1 - 2e^{-x}}}$$

$$= \frac{(1 - 2e^{-x}) + x e^{-x}}{\sqrt{1 - 2e^{-x}}}$$

$$\therefore f'(x) = \frac{1 - 2e^{-x} + x e^{-x}}{\sqrt{1 - 2e^{-x}}}$$

$$\mathbf{4} \quad \mathbf{a} \quad y = Ae^{kx} \quad \mathbf{i} \quad \frac{dy}{dx} = Ae^{kx}(k) \quad \mathbf{ii} \quad \frac{d^2y}{dx^2} = k \frac{dy}{dx}$$

$$= k(Ae^{kx})$$

$$= ky$$

$$= k(ky)$$

$$= k^2y$$

$$\mathbf{b} \quad \text{Prediction: } \frac{d^n y}{dx^n} = k^n y$$

$$\mathbf{5} \quad y = 2e^{3x} + 5e^{4x}$$

$$\therefore \frac{dy}{dx} = 6e^{3x} + 20e^{4x}$$

$$\therefore \frac{d^2y}{dx^2} = 18e^{3x} + 80e^{4x}$$

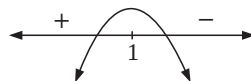
$$\begin{aligned} \text{Now } \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 12y &= (18e^{3x} + 80e^{4x}) - 7(6e^{3x} + 20e^{4x}) + 12(2e^{3x} + 5e^{4x}) \\ &= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x} \\ &= e^{3x}[18 - 42 + 24] + e^{4x}[80 - 140 + 60] \\ &= e^{3x}(0) + e^{4x}(0) \\ &= 0 \end{aligned}$$

$$\text{i.e., } \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 12y = 0$$

6 a $y = xe^{-x}$ $u = x, v = e^{-x}, u' = 1, v' = -e^{-x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 1e^{-x} - xe^{-x} \quad \{\text{product rule}\} \\ &= e^{-x}(1 - x) \\ &= \frac{1 - x}{e^x}\end{aligned}$$

which has sign diagram:



$\therefore$ at $x = 1, y = 1e^{-1} = \frac{1}{e}$ we have a maximum turning point.

$\therefore$ has a local maximum at $(1, \frac{1}{e})$

b $y = x^2e^x$ $u = x^2, v = e^x, u' = 2x, v' = e^x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 2xe^x + x^2e^x \quad \{\text{product rule}\} \\ &= xe^x(2 + x)\end{aligned}$$

which has sign diagram:



$\therefore$ at $x = -2, y = 4e^{-2}$, we have a maximum turning point

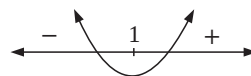
and at $x = 0, y = 0$, we have a minimum turning point.

$\therefore$ we have a local maximum at $(-2, \frac{4}{e^2})$, and a local minimum at $(0, 0)$.

c $y = \frac{e^x}{x}$ $u = e^x, v = x, u' = e^x, v' = 1$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{e^x x - e^x(1)}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{e^x(x - 1)}{x^2}\end{aligned}$$

which has sign diagram:



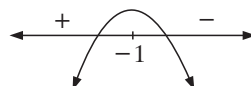
$\therefore$ at $x = 1, y = \frac{e}{1} = e$ we have a minimum turning point.

$\therefore$ we have a local minimum at $(1, e)$

d $y = e^{-x}(x + 2)$ $u = e^{-x}, v = x + 2, u' = -e^{-x}, v' = 1$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -e^{-x}(x + 2) + e^{-x} \quad \{\text{product rule}\} \\ &= e^{-x}(-x - 2 + 1) \\ &= e^{-x}(-x - 1)\end{aligned}$$

which has sign diagram:



$\therefore$ at $x = -1, y = e(-1 + 2) = e$ we have a maximum turning point.

$\therefore$ we have a local maximum at $(-1, e)$

EXERCISE 23B

1 a $N = 50e^{2t}$

$$\begin{aligned}\therefore \ln N &= \ln(50e^{2t}) \quad \{\ln ab = \ln a + \ln b\} \\ &= \ln 50 + \ln e^{2t} \quad \{\text{as } \ln e^n = n\}\end{aligned}$$

$$\therefore \ln N = \ln 50 + 2t$$

b $P = 8.69e^{-0.0541t}$

$$\begin{aligned}\therefore \ln P &= \ln(8.69e^{-0.0541t}) \\ &= \ln 8.69 + \ln e^{-0.0541t}\end{aligned}$$

$$\therefore \ln P = \ln 8.69 - 0.0541t$$

c $S = a^2e^{-kt}$

$$\begin{aligned}\therefore \ln S &= \ln(a^2e^{-kt}) \\ &= \ln a^2 + \ln e^{-kt}\end{aligned}$$

$$\therefore \ln S = 2 \ln a - kt \quad \{\ln a^n = n \ln a\}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \ln D &\doteq 2.1 + 0.69t \\ \therefore D &\doteq e^{2.1+0.69t} \\ &\doteq e^{2.1} \times e^{0.69t} \\ \therefore D &\doteq 8.166e^{0.69t} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \ln P &= \ln g - 2t \\ &= \ln g - \ln(e^{2t}) \\ &= \ln\left(\frac{g}{e^{2t}}\right) \\ \therefore P &= \frac{g}{e^{2t}} \\ \therefore P &= ge^{-2t} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \ln e^2 &= 2 \ln e \\ &= 2(1) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \ln\left(\frac{1}{\sqrt{e}}\right) &= \ln e^{-\frac{1}{2}} \\ &= -\frac{1}{2} \ln e \\ &= -\frac{1}{2}(1) \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \ln \sqrt{e} &= \ln e^{\frac{1}{2}} \\ &= \frac{1}{2} \ln e \\ &= \frac{1}{2} \end{aligned}$$

$$\mathbf{e} \quad e^{\ln 3} = 3$$

$$\begin{aligned} \mathbf{g} \quad e^{-\ln 5} &= e^{\ln 5^{-1}} \\ &= e^{\ln \frac{1}{5}} \\ &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \ln G &\doteq -31.64 + 0.0173t \\ \therefore G &\doteq e^{-31.64+0.0173t} \\ &\doteq e^{-31.64} \times e^{0.0173t} \\ \therefore G &\doteq 1.815 \times 10^{-14} \times e^{0.0173t} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \ln F &= 2 \ln x - 0.03t \\ &= \ln x^2 - \ln e^{0.03t} \\ &= \ln(x^2 e^{-0.03t}) \\ \therefore F &= x^2 e^{-0.03t} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \ln\left(\frac{1}{e}\right) &= \ln e^{-1} \\ &= -1 \ln e \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad e^{2 \ln 3} &= e^{\ln 3^2} \\ &= e^{\ln 9} \\ &= 9 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad e^{-2 \ln 2} &= e^{\ln 2^{-2}} \\ &= e^{\ln \frac{1}{4}} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \ln 5 + \ln 6 \\ &= \ln(5 \times 6) \\ &= \ln 30 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 3 \ln 4 - 2 \ln 2 \\ &= \ln 4^3 - \ln 2^2 \\ &= \ln \frac{4^3}{2^2} \\ &= \ln 16 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2 \ln 5 \\ &= \ln 5^2 \\ &= \ln 25 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 2 + 3 \ln 2 \\ &= \ln e^2 + \ln 2^3 \\ &= \ln(8e^2) \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad \text{Let } 2 &= e^x \\ \therefore \ln 2 &= x \\ \therefore 2 &= e^{\ln 2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{Let } 10 &= e^x \\ \therefore \ln 10 &= x \\ \therefore 10 &= e^{\ln 10} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{Let } a &= e^x \\ \therefore \ln a &= x \\ \therefore a &= e^{\ln a} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \text{Let } a^x &= e^x \\ \therefore \ln a^x &= x \\ \therefore x \ln a &= x \\ \therefore a^x &= e^{x \ln a} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad e^x &= 2 \\ \therefore \ln e^x &= \ln 2 \\ x &= \ln 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad e^x &= 0 \quad \text{has no solutions} \\ \text{as } e^x &> 0 \quad \text{for all } x \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad e^x &= e^{-x} \\ \therefore x &= -x \\ \therefore 2x &= 0 \\ \therefore x &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad e^x &= -2 \quad \text{has no solutions} \\ \text{as } e^x &> 0 \quad \text{for all } x \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad e^{2x} &= 2e^x \\ \therefore e^x(e^x - 2) &= 0 \\ \therefore e^x &= 2 \quad \{\text{as } e^x > 0\} \\ \therefore x &= \ln 2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad e^{2x} - 5e^x + 6 &= 0 \\ \therefore (e^x - 3)(e^x - 2) &= 0 \\ \therefore e^x &= 3 \text{ or } 2 \\ \therefore x &= \ln 3 \text{ or } \ln 2 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & e^x + 2 = 3e^{-x} \\
 & \therefore e^{2x} + 2e^x = 3 \quad \{\times e^x\} \\
 & \therefore e^{2x} + 2e^x - 3 = 0 \\
 & \therefore (e^x + 3)(e^x - 1) = 0 \\
 & \therefore e^x = -3 \text{ or } 1 \\
 & \therefore e^x = 1 \quad \{\text{as } e^x > 0\} \\
 & \therefore x = \ln 1 \\
 & \therefore x = 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & e^x + e^{-x} = 3 \\
 & \therefore e^{2x} + 1 = 3e^x \quad \{\times e^x\} \\
 & \therefore e^{2x} - 3e^x + 1 = 0 \\
 & \therefore e^x = \frac{3 \pm \sqrt{9-4}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 1 + 12e^{-x} = e^x \\
 & \therefore e^x + 12 = e^{2x} \quad \{\times e^x\} \\
 & \therefore e^{2x} - e^x - 12 = 0 \\
 & \therefore (e^x - 4)(e^x + 3) = 0 \\
 & \therefore e^x = 4 \text{ or } -3 \\
 & \therefore e^x = 4 \quad \{\text{as } e^x > 0\} \\
 & \therefore x = \ln 4
 \end{aligned}$$

$$\begin{aligned}
 & \therefore e^x = \frac{3 \pm \sqrt{5}}{2} \\
 & \therefore x = \ln \left(\frac{3 + \sqrt{5}}{2} \right) \text{ or } \ln \left(\frac{3 - \sqrt{5}}{2} \right) \\
 & \quad \div 0.962 \quad \text{or} \quad -0.962
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad & \ln x + \ln(x+2) = \ln 8 \\
 & \therefore \ln[x(x+2)] = \ln 8 \\
 & \therefore x(x+2) = 8 \\
 & \therefore x^2 + 2x = 8 \\
 & \therefore x^2 + 2x - 8 = 0 \\
 & \therefore (x+4)(x-2) = 0 \\
 & \therefore x = 2 \text{ or } -4 \\
 & \therefore x = 2 \\
 & \{\text{since } \ln(-4) \text{ is undefined}\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \ln(x-2) - \ln(x+3) = \ln 2 \\
 & \therefore \ln \left(\frac{x-2}{x+3} \right) = \ln 2 \\
 & \therefore \frac{x-2}{x+3} = 2 \\
 & \therefore x-2 = 2(x+3) \\
 & \therefore x-2 = 2x+6 \\
 & \therefore -8 = x \\
 & \text{As } \ln(-8) \text{ is undefined, no solutions exist.}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & y_1 = e^x, \quad y_2 = e^{2x} - 6 \quad \text{meet when } y_1 = y_2 \\
 & \therefore e^x = e^{2x} - 6 \\
 & \therefore e^{2x} - e^x - 6 = 0 \\
 & \therefore (e^x - 3)(e^x + 2) = 0 \\
 & \therefore e^x = 3 \text{ or } -2 \\
 & \therefore e^x = 3 \quad \{\text{as } e^x > 0\} \\
 & \therefore x = \ln 3 \quad \text{and } y = e^x = 3 \\
 & \therefore \text{meet at } (\ln 3, 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & y_1 = 2e^x + 1, \quad y_2 = 7 - e^x \quad \text{meet when } y_1 = y_2 \\
 & \therefore 2e^x + 1 = 7 - e^x \\
 & \therefore 3e^x = 6 \\
 & \therefore e^x = 2 \\
 & \therefore x = \ln 2 \quad \text{and } y = 7 - e^x = 7 - 2 = 5 \\
 & \therefore \text{meet at } (\ln 2, 5)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & y_1 = 3 - e^x, \quad y_2 = 5e^{-x} - 3 \quad \text{meet when } y_1 = y_2 \\
 & \therefore 3 - e^x = 5e^{-x} - 3 \\
 & \therefore 3e^x - e^{2x} = 5 - 3e^x \quad \{\times \text{ by } e^x\} \\
 & \therefore e^{2x} - 6e^x + 5 = 0 \\
 & \therefore (e^x - 5)(e^x - 1) = 0 \\
 & \therefore e^x = 1 \text{ or } 5 \\
 & \therefore x = 0 \text{ or } \ln 5 \\
 & \text{and } y = 3 - e^0 \quad \text{or} \quad 3 - e^{\ln 5} \\
 & \quad = 3 - 1 \quad \quad = 3 - 5 \\
 & \quad = 2 \quad \quad = -2 \\
 & \text{i.e., meet at } (0, 2) \text{ and } (\ln 5, -2)
 \end{aligned}$$

9 a $f(x) = 3 - e^x$

cuts x -axis when $y = 0$ i.e., $3 - e^x = 0$

$$\therefore e^x = 3$$

$$\therefore x = \ln 3 \quad \text{i.e., } A \text{ is } (\ln 3, 0)$$

cuts y -axis when $x = 0$ $\therefore y = 3 - e^0$

$$= 3 - 1$$

$$= 2 \quad \text{i.e., } B \text{ is } (0, 2)$$

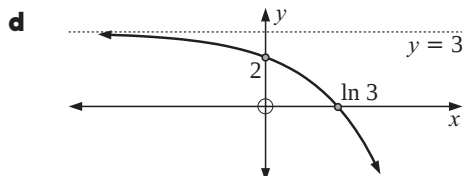
b $f'(x) = -e^x$ where

Now $e^x > 0$ for all x , $\therefore f'(x) < 0$ for all x

$\therefore f(x)$ is decreasing for all x

c $f''(x) = -e^x$ where $e^x > 0$ for all x , $\therefore f'(x) < 0$ for all x

$\therefore f(x)$ is concave down for all x



e as $x \rightarrow +\infty$, $f(x) \rightarrow -\infty$,
as $x \rightarrow -\infty$, $e^x \rightarrow 0$ $\therefore 3 - e^x \rightarrow 3$,
i.e., $f(x) \rightarrow 3$ (below)
 $\therefore y = 3$ is a horizontal asymptote.

10 a $y = e^x - 3e^{-x}$

cuts the x axis at P when $y = 0$ $\therefore e^x - 3e^{-x} = 0$

$$\therefore e^{2x} - 3 = 0 \quad \{\times \text{ each term by } e^x\}$$

$$\therefore e^{2x} = 3$$

$$\therefore 2x = \ln 3$$

$$\therefore x = \frac{1}{2} \ln 3 \quad \therefore P \text{ is } (\frac{1}{2} \ln 3, 0)$$

cuts the y -axis at Q when $x = 0$ $\therefore y = e^0 - 3e^0$

$$= 1 - 3$$

$$= -2 \quad \therefore Q \text{ is } (0, -2)$$

b $\frac{dy}{dx} = e^x + 3e^{-x}$
 $= e^x + \frac{3}{e^x}$

Since $e^x > 0$ for all x

$$\frac{dy}{dx} > 0 \text{ for all } x$$

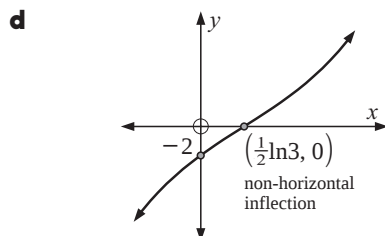
$\therefore$ the function is increasing for all x

c $\frac{dy}{dx} = e^x + 3e^{-x}$
 $\therefore \frac{d^2y}{dx^2} = e^x - 3e^{-x}$
 $= y$

Above x -axis $y > 0$ $\therefore \frac{d^2y}{dx^2} > 0$
 $\therefore$ the function is concave up

Below x -axis $y < 0$ $\therefore \frac{d^2y}{dx^2} < 0$
 $\therefore$ the function is concave down

i.e., a non-horizontal inflection occurs
when $y = 0$



11 a For $f(x) = e^x - 3$ and $g(x) = 3 - \frac{5}{e^x}$

the x -intercept occurs when $y = 0$ $\therefore e^x - 3 = 0$ and $3 - \frac{5}{e^x} = 0$
 $\therefore e^x = 3$
 $\therefore x = \ln 3$ $\therefore \frac{3e^x - 5}{e^x} = 0$
 i.e., at $(\ln 3, 0)$ $\therefore 3e^x - 5 = 0$
 $\therefore e^x = \frac{5}{3}$
 $\therefore x = \ln \frac{5}{3}$
 i.e., at $(\ln \frac{5}{3}, 0)$

the y -intercept occurs when $x = 0$ $\therefore y = e^0 - 3$ and $y = 3 - \frac{5}{e^0}$
 $y = -2$ $= 3 - 5$
 i.e., at $(0, -2)$ $= -2$
 i.e., $(0, -2)$

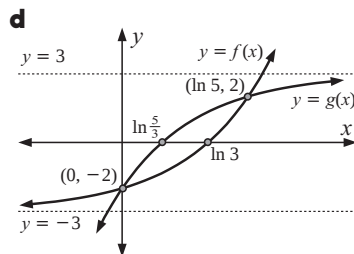
b as $x \rightarrow +\infty$ $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ $g(x) \rightarrow 3$ (below)
 $x \rightarrow -\infty$ $f(x) \rightarrow -3$ (above) $x \rightarrow -\infty$ $g(x) \rightarrow -\infty$

c $f(x)$ and $g(x)$ meet when $e^x - 3 = 3 - 5e^{-x}$
 $\therefore e^{2x} - 3e^x = 3e^x - 5 \quad \{\times e^x\}$
 $\therefore e^{2x} - 6e^x + 5 = 0$
 $\therefore (e^x - 5)(e^x - 1) = 0$
 i.e., when $e^x = 5$ or 1
 $\therefore x = \ln 5$ or 0

when $x = \ln 5$, $f(x) = e^{\ln 5} - 3 = 5 - 3 = 2$

when $x = 0$, $f(x) = e^0 - 3 = 1 - 3 = -2$

$\therefore f(x)$ and $g(x)$ meet at $(0, -2)$ and $(\ln 5, 2)$



EXERCISE 23C

1 a $y = \ln(7x)$ or $y = \ln(7x)$
 $\therefore y = \ln 7 + \ln x$
 $\therefore \frac{dy}{dx} = 0 + \frac{1}{x}$
 $= \frac{1}{x}$
 $\therefore \frac{dy}{dx} = \frac{7}{7x} \leftarrow f'(x)$
 $= \frac{1}{x}$
 $= \frac{1}{x} \leftarrow f(x)$

b $y = \ln(2x + 1)$
 $\therefore \frac{dy}{dx} = \frac{2}{2x + 1} \leftarrow f'(x)$
 $= \frac{1}{x + 0.5} \leftarrow f(x)$

d $y = 3 - 2 \ln x$
 $\therefore \frac{dy}{dx} = 0 - 2 \left(\frac{1}{x} \right)$
 $= -\frac{2}{x}$

c $y = \ln(x - x^2)$
 $\therefore \frac{dy}{dx} = \frac{1 - 2x}{x - x^2} \leftarrow f'(x)$
 $= \frac{1 - 2x}{x(1 - x)} \leftarrow f(x)$

e $y = x^2 \ln x$
 $\therefore \frac{dy}{dx} = 2x \ln x + x^2 \left(\frac{1}{x} \right)$
 $= 2x \ln x + x$

$$\begin{aligned}\mathbf{f} \quad y &= \frac{\ln x}{2x} \\ \therefore \frac{dy}{dx} &= \frac{\left(\frac{1}{x}\right)2x - \ln x(2)}{(2x)^2} \\ &= \frac{2 - 2\ln x}{4x^2} \\ &= \frac{1 - \ln x}{2x^2}\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad y &= (\ln x)^2 \\ \therefore \frac{dy}{dx} &= 2(\ln x)^1 \left(\frac{1}{x}\right) \\ &= \frac{2\ln x}{x}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad y &= e^{-x} \ln x \\ \therefore \frac{dy}{dx} &= -e^{-x} \ln x + e^{-x} \left(\frac{1}{x}\right) \\ &= \frac{e^{-x}}{x} - e^{-x} \ln x\end{aligned}$$

$$\begin{aligned}\mathbf{l} \quad y &= \frac{2\sqrt{x}}{\ln x} \\ \therefore \frac{dy}{dx} &= \frac{\frac{1}{\sqrt{x}} \ln x - 2\sqrt{x} \left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{\frac{1}{\sqrt{x}} \ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2} \\ &= \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}\end{aligned}$$

$$\begin{aligned}\mathbf{n} \quad y &= \sqrt{x} \ln 4x \\ \therefore \frac{dy}{dx} &= \frac{1}{2\sqrt{x}} \ln 4x + \sqrt{x} \left(\frac{1}{x}\right) \\ &= \frac{\ln 4x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\end{aligned}$$

$$\begin{aligned}\mathbf{2} \quad \mathbf{a} \quad y &= x \ln 5 \\ \therefore \frac{dy}{dx} &= \ln 5\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad y &= \ln(x^4 + x) \\ \therefore \frac{dy}{dx} &= \frac{4x^3 + 1}{x^4 + x}\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad y &= e^x \ln x \\ \therefore \frac{dy}{dx} &= e^x \ln x + \frac{e^x}{x}\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad y &= \sqrt{\ln x} = (\ln x)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \frac{1}{2}(\ln x)^{-\frac{1}{2}} \left(\frac{1}{x}\right) \\ &= \frac{1}{2x\sqrt{\ln x}}\end{aligned}$$

$$\begin{aligned}\mathbf{k} \quad y &= \sqrt{x} \ln 2x \\ \therefore \frac{dy}{dx} &= \frac{1}{2\sqrt{x}} \ln 2x + \sqrt{x} \left(\frac{1}{x}\right) \\ &= \frac{\ln 2x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\end{aligned}$$

$$\begin{aligned}\mathbf{m} \quad y &= 3 - 4 \ln(1 - x) \\ \therefore \frac{dy}{dx} &= 0 - 4 \times \frac{-1}{1 - x} \\ &= \frac{4}{1 - x}\end{aligned}$$

$$\begin{aligned}\mathbf{o} \quad y &= x \ln(x^2 + 1) \\ \therefore \frac{dy}{dx} &= \ln(x^2 + 1) + x \left(\frac{2x}{x^2 + 1}\right) \\ &= \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad y &= \ln(x^3) = 3 \ln x \\ \therefore \frac{dy}{dx} &= 3 \left(\frac{1}{x}\right) = \frac{3}{x}\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad y &= \ln(10 - 5x) \\ \therefore \frac{dy}{dx} &= \frac{-5}{10 - 5x} = \frac{1}{x - 2}\end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= [\ln(2x+1)]^3 \\
 \therefore \frac{dy}{dx} &= 3[\ln(2x+1)]^2 \times \frac{2}{2x+1} \\
 &= \frac{6[\ln(2x+1)]^2}{2x+1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= \frac{\ln(4x)}{x} \\
 \therefore \frac{dy}{dx} &= \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2} \\
 &= \frac{1 - \ln(4x)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad y &= \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) \\
 \therefore y &= -\ln x \\
 \therefore \frac{dy}{dx} &= -\frac{1}{x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad y &= \ln(\ln x) \\
 \therefore \frac{dy}{dx} &= \frac{\frac{1}{x}}{\ln x} \\
 &= \frac{1}{x \ln x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad y &= \frac{1}{\ln x} = [\ln x]^{-1} \quad \therefore \frac{dy}{dx} = -1[\ln x]^{-2} \times \frac{1}{x} \\
 &= \frac{-1}{x[\ln x]^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad y &= \ln \sqrt{1-2x} \\
 &= \ln(1-2x)^{\frac{1}{2}} \\
 &= \frac{1}{2} \ln(1-2x) \\
 \therefore \frac{dy}{dx} &= \frac{1}{2} \times \frac{-2}{1-2x} \\
 &= \frac{1}{2x-1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \ln \frac{1}{(2x+3)} \\
 &= -\ln(2x+3) \\
 \therefore \frac{dy}{dx} &= -\frac{2}{2x+3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \ln(e^x \sqrt{x}) \\
 &= \ln e^x + \ln x^{\frac{1}{2}} \\
 &= \ln e^x + \frac{1}{2} \ln x \\
 &= x + \frac{1}{2} \ln x \\
 \therefore \frac{dy}{dx} &= 1 + \frac{1}{2} \frac{1}{x} = 1 + \frac{1}{2x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= \ln(x\sqrt{2-x}) \\
 &= \ln x + \ln(2-x)^{\frac{1}{2}} \\
 &= \ln x + \frac{1}{2} \ln(2-x) \\
 \therefore \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2} \left(\frac{-1}{2-x} \right) \\
 &= \frac{1}{x} - \frac{1}{2(2-x)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= \ln\left(\frac{x+3}{x-1}\right) \\
 &= \ln(x+3) - \ln(x-1) \\
 \therefore \frac{dy}{dx} &= \frac{1}{x+3} - \frac{1}{x-1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= \ln\left(\frac{x^2}{3-x}\right) \\
 &= \ln x^2 - \ln(3-x) \\
 &= 2 \ln x - \ln(3-x) \\
 \therefore \frac{dy}{dx} &= \frac{2}{x} - \frac{-1}{3-x} \\
 &= \frac{2}{x} + \frac{1}{3-x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad f(x) &= \ln((3x-4)^3) \\
 &= 3 \ln(3x-4) \\
 \therefore f'(x) &= 3 \times \frac{3}{3x-4} \\
 &= \frac{9}{3x-4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad f(x) &= \ln(x(x^2+1)) \\
 &= \ln x + \ln(x^2+1) \\
 \therefore f'(x) &= \frac{1}{x} + \frac{2x}{x^2+1}
 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad f(x) &= \ln\left(\frac{x^2+2x}{x-5}\right) \\ &= \ln(x^2+2x) - \ln(x-5) \\ f'(x) &= \frac{2x+2}{x^2+2x} - \frac{1}{x-5} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad f(x) &= \ln\left(\frac{x^3}{(2-3x)^2}\right) \\ &= 3\ln x - 2\ln(2-3x) \\ \therefore f'(x) &= \frac{3}{x} - 2\left(\frac{-3}{2-3x}\right) \\ &= \frac{3}{x} + \frac{6}{2-3x} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad f(x) &= \ln\left(\frac{\sqrt{x+1}}{x(x+2)}\right) \\ &= \frac{1}{2}\ln(x+1) - \ln x - \ln(x+2) \\ \therefore f'(x) &= \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x} - \frac{1}{x+2} \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad f(x) &= \ln\left(\frac{(x+1)(2x-1)}{3x^2}\right) \\ &= \ln(x+1) + \ln(2x-1) - \ln(3x^2) \\ \therefore f'(x) &= \frac{1}{x+1} + \frac{2}{2x-1} - \frac{6x}{3x^2} \\ &= \frac{1}{x+1} + \frac{2}{2x-1} - \frac{2}{x} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad y &= 2^x \quad \therefore \frac{dy}{dx} = e^{x \ln 2} \times \ln 2 \\ &= (e^{\ln 2})^x = 2^x \ln 2 \\ &= e^{x \ln 2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= a^x \quad \therefore \frac{dy}{dx} = e^{x \ln a} \times \ln a \\ &= (e^{\ln a})^x = a^x \ln a \\ &= e^{x \ln a} \end{aligned}$$

$$\mathbf{5} \quad f(x) = \ln(2x-1) - 3$$

$$\begin{aligned} \mathbf{a} \quad \text{when } y &= 0, \quad \ln(2x-1) = 3 \\ \therefore 2x-1 &= e^3 \\ \therefore 2x &= e^3 + 1 \\ \therefore x &= \frac{e^3+1}{2} \div 10.54 \quad \text{i.e., the } x \text{ intercept is } \frac{e^3+1}{2}. \end{aligned}$$

$$\mathbf{b} \quad f(0) = \ln(-1) \text{ cannot be found as } \ln(-1) \text{ is not defined. } \therefore \text{ there is no } y\text{-intercept.}$$

$$\begin{aligned} \mathbf{c} \quad f'(x) &= \frac{2}{2x-1}, \quad \therefore f'(1) = \frac{2}{2-1} = 2 \\ \therefore \text{slope of tangent} &= 2 \end{aligned}$$

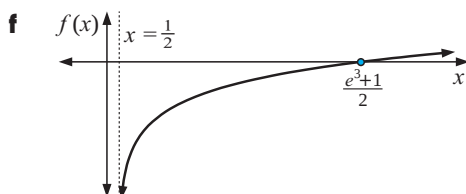
$$\begin{aligned} \mathbf{d} \quad \ln(2x-1) \text{ has meaning provided } 2x-1 &> 0 \\ \text{i.e., } 2x &> 1 \\ x &> \frac{1}{2} \end{aligned}$$

$$\therefore f(x) \text{ has meaning provided } x > \frac{1}{2}$$

$$\begin{aligned} \mathbf{e} \quad f'(x) &= 2(2x-1)^{-1} \\ \therefore f''(x) &= -2(2x-1)^{-2} \cdot 2 = \frac{-4}{(2x-1)^2} \end{aligned}$$

$$\text{provided } f(x) \text{ has meaning i.e., provided } x > \frac{1}{2}, \quad f''(x) < 0$$

$$\therefore f(x) \text{ is always concave down for } x > \frac{1}{2}$$



6 a $f(x) = x \ln x$

$f(x)$ is defined provided $\ln x$ is defined i.e., $x > 0$

b $f(x) = x \ln x$ Let $u = x$, $v = \ln x$, $u' = 1$, $v = \frac{1}{x}$

$$\therefore f'(x) = (1) \ln x + x \left(\frac{1}{x} \right) \quad \{\text{product rule}\}$$

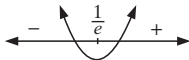
$$= \ln x + 1$$

which is 0 when $\ln x = -1$

$$\text{i.e., } x = e^{-1}$$

$$\text{i.e., } x = \frac{1}{e}, \text{ and } f\left(\frac{1}{e}\right) = \frac{1}{e} \ln e^{-1} = -\frac{1}{e}$$

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local minimum at $\left(\frac{1}{e}, -\frac{1}{e}\right)$. i.e., the smallest value of $\frac{\ln x}{x}$ is $\frac{1}{e}$.

7 Consider $f(x) = \frac{\ln x}{x}$ Let $u = \ln x$, $v = x$, $u' = \frac{1}{x}$, $v' = 1$

$$\therefore f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} \quad \{\text{quotient rule}\}$$

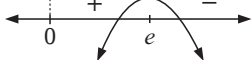
$$= \frac{1 - \ln x}{x^2}$$

Now $f'(x) = 0$ when $1 - \ln x = 0$

$$\therefore \ln x = 1$$

$$\therefore x = e^1, \text{ and } f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Sign diagram of $f'(x)$ is:

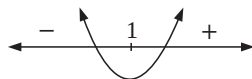


$\therefore$ there is a max. turning point at $\left(e, \frac{1}{e}\right)$, i.e., $f(x) \leq \frac{1}{e}$ for all x , i.e., $\frac{\ln x}{x} \leq \frac{1}{e}$ for all x

8 $f(x) = x - \ln x$

$$\therefore f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$$

and the sign diagram of $f'(x)$ is:



$\therefore f(x)$ has a minimum turning point at $(1, 1 - \ln 1)$, i.e., $(1, 1)$

$\therefore f(x) \geq 1$ for all $x > 0$

$$x - \ln x \geq 1$$

i.e., $\ln x \leq x - 1$ for all $x > 0$

EXERCISE 23D

1 $f(x) = e^{-x}$ $\therefore f(1) = e^{-1}$ i.e., the point of contact is $\left(1, \frac{1}{e}\right)$.

Now $f'(x) = -e^{-x}$

$$f'(1) = -e^{-1} = -\frac{1}{e} \text{ and is the slope of the tangent at } \left(1, \frac{1}{e}\right).$$

$$\therefore \text{tangent has equation } \frac{y - \frac{1}{e}}{x - 1} = -\frac{1}{e} \quad \therefore e\left(y - \frac{1}{e}\right) = -(x - 1)$$

$$\therefore ey - 1 = -x + 1$$

$$\text{i.e., } x + ey = 2 \text{ or } y = -\frac{1}{e}x + \frac{2}{e}$$

- 2** $y = \ln(2 - x)$, so when $x = -1$, $y = \ln 3$ i.e., the point of contact is $(-1, \ln 3)$.

$$\text{Now } \frac{dy}{dx} = \frac{-1}{2-x}$$

$$\therefore \text{ when } x = -1, \frac{dy}{dx} = -\frac{1}{2+1} = -\frac{1}{3} \text{ and is the slope of the tangent at } (-1, \ln 3).$$

$$\therefore \text{ tangent has equation } \frac{y - \ln 3}{x + 1} = -\frac{1}{3} \quad \text{i.e., } 3(y - \ln 3) = -(x + 1)$$

$$\therefore 3y - 3\ln 3 = -x - 1$$

$$\text{i.e., } x + 3y = 3\ln 3 - 1$$

- 3** $y = x^2 e^x$, so when $x = 1$, $y = e$, i.e., the point of contact is $(1, e)$.

$$\text{Now } \frac{dy}{dx} = x^2 e^x + 2xe^x$$

$$\therefore \text{ when } x = 1, \frac{dy}{dx} = e + 2e = 3e \text{ and is the slope of the tangent at } (1, e).$$

$$\therefore \text{ the tangent has equation } \frac{y - e}{x - 1} = 3e \quad \text{i.e., } y - e = 3e(x - 1)$$

$$\therefore y - e = 3ex - 3e$$

$$\therefore y - 3ex = -2e$$

$$\text{i.e., } 3ex - y = 2e$$

$$\text{Cuts the } x\text{-axis when } y = 0 \quad \therefore 3ex = 2e \quad \text{i.e., } x = \frac{2}{3} \quad \text{i.e., at } A(\frac{2}{3}, 0)$$

$$\text{Cuts the } y\text{-axis when } x = 0 \quad \text{i.e., } -y = 2e \quad \text{i.e., at } B(0, -2e)$$

- 4** $y = \ln \sqrt{x}$ $\therefore$ when $y = -1$, $-1 = \frac{1}{2} \ln x$

$$= \ln x^{\frac{1}{2}} \quad \therefore \ln x = -2$$

$$= \frac{1}{2} \ln x \quad \therefore x = e^{-2}$$

$$\therefore x = \frac{1}{e^2} \quad \text{i.e., the point of contact is } \left(\frac{1}{e^2}, -1\right)$$

$$\text{Now } \frac{dy}{dx} = \frac{1}{2} \frac{1}{x} = \frac{1}{2x} \quad \therefore \text{ at } \left(\frac{1}{e^2}, -1\right), \frac{dy}{dx} = \frac{1}{2e^{-2}} = \frac{e^2}{2}$$

$$\therefore \text{ the tangent has slope } = \frac{e^2}{2} \text{ and the normal has slope } = -\frac{2}{e^2}$$

$$\therefore \text{ the normal has equation } \frac{y + 1}{x - \frac{1}{e^2}} = -\frac{2}{e^2} \quad \text{i.e., } e^2(y + 1) = -2\left(x - \frac{1}{e^2}\right)$$

$$\therefore e^2 y + e^2 = -2x + \frac{2}{e^2}$$

$$\therefore 2x + e^2 y = -e^2 + \frac{2}{e^2} \text{ or } y = -\frac{2}{e^2} x + \frac{2}{e^4} - 1$$

- 5** $y = e^x$

When $x = a$, $y = e^a$, i.e., the point of contact is (a, e^a) .

$$\text{Now } \frac{dy}{dx} = e^x \text{ and so at } (a, e^a), \frac{dy}{dx} = e^a \text{ and is the slope of the tangent at } (a, e^a).$$

$$\therefore \text{ the tangent has equation } \frac{y - e^a}{x - a} = e^a \quad \text{i.e., } y - e^a = e^a(x - a) \dots\dots*$$

$$\text{if the tangent is through the origin, } (0, 0) \text{ must satisfy } * \quad \therefore 0 - e^a = e^a(0 - a)$$

$$\therefore -e^a = -ae^a$$

$$\therefore e^a(a - 1)0 = 0$$

$$\therefore a = 1 \quad \{\text{as } e^a > 0\}$$

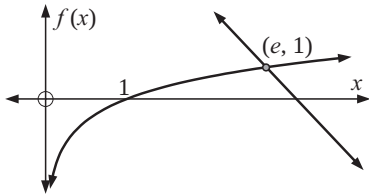
$$\text{So the equation of the tangent is } y - e = ex - e \quad \text{i.e., } y = ex$$

6 a $f(x) = \ln x$ is defined for all $x > 0$.

b $f'(x) = \frac{1}{x}$ which is > 0 for all $x > 0$, $\therefore f(x)$ is increasing on $x > 0$.

$f''(x) = -x^{-2} = \frac{-1}{x^2}$ which is < 0 for all $x > 0$, $\therefore f(x)$ is concave down on $x > 0$.

c



At $y = 1$, $1 = \ln x$

$\therefore x = e^1 = e$,

i.e., the point of contact is $(e, 1)$

and $\frac{dy}{dx} = \frac{1}{x}$

$\therefore$ at $(e, 1)$, $\frac{dy}{dx} = \frac{1}{e}$

$\therefore$ the slope of the tangent is $\frac{1}{e}$, and the slope of the normal is $-e$

$\therefore$ the equation of the normal is $\frac{y-1}{x-e} = -e \quad \therefore y-1 = -e(x-e)$

$\therefore y-1 = -ex + e^2$

$\therefore ex + y = 1 + e^2$

7 $y = 3e^{-x}$ and $y = 2 + e^x$ meet when $3e^{-x} = 2 + e^x$

$\therefore 3 = 2e^x + e^{2x} \quad \{\times \text{ by } e^x\}$

$\therefore e^{2x} + 2e^x - 3 = 0$

$\therefore (e^x + 3)(e^x - 1) = 0$

$\therefore e^x = -3$ or $1 \quad \{\text{as } e^x > 0\}$

$\therefore x = 0 \quad \{\text{as } e^x > 0\}$

and when $x = 0$, $y = 3e^0 = 3$

When $y = 2 + e^x$, $\frac{dy}{dx} = e^x \quad \therefore$ at $(0, 3)$, $\frac{dy}{dx} = e^0 = 1$

i.e., the slope of the tangent is 1

and $1 = \tan \theta$ where θ is the angle between the tangent and the x -axis, $\therefore \theta = 45^\circ$

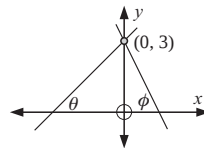
When $y = 3e^{-x}$, $\frac{dy}{dx} = -3e^{-x} \quad \therefore$ at $(0, 3)$, $\frac{dy}{dx} = -3$

and $\tan \phi = 3$ where ϕ is the angle between the tangent and the x -axis

$\therefore \phi = \tan^{-1}(3)$

$\doteq 71.57^\circ$

$\therefore$ angle between the tangents $\doteq (180 - (71.57 + 45))^\circ$
 $\doteq 63.43^\circ$



8 a $W = 200e^{\frac{t}{2}}$

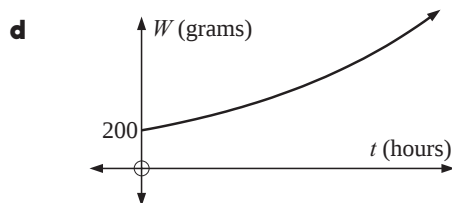
i When $t = 0$,
 $W = 200$ g

ii When $t = \frac{1}{2}$,
 $W = 200e^{\frac{1}{2}}$
 $= 200e^{\frac{1}{4}}$
 $= 200e^{0.25}$
 $= 256.8$ g

iii When $t = 1\frac{1}{2}$,
 $W = 200e^{\frac{3}{2}}$
 $= 200e^{0.75}$
 $\doteq 423.4$ g

b When $W = 1000$, $200e^{\frac{t}{2}} = 1000$
 $\therefore e^{\frac{t}{2}} = 5$
 $\therefore \frac{t}{2} = \ln 5$
 $\therefore t = 2 \ln 5$
 $\therefore t \doteq 3.219$
 i.e., $t \doteq 3$ hours 13 min

c $W = 200e^{\frac{t}{2}}$ **i** when $t = 0$, **ii** when $t = 2$,
 $\therefore \frac{dW}{dt} = \frac{1}{2} \times 200e^{\frac{t}{2}}$ $\frac{dW}{dt} = 100 \times e^0$ $\frac{dW}{dt} = 100 \times e^{\frac{2}{2}}$
 $= 100e^{\frac{t}{2}}$ $= 100 \text{ g/h}$ $= 100e$
 $= 271.8 \text{ g/h}$



9 a $W = 20e^{-kt}$ and when $t = 50$ hours, $W = 10 \text{ g}$
 $\therefore 20e^{-50k} = 10$
 $\therefore e^{-50k} = \frac{1}{2}$
 $\therefore -50k = \ln \frac{1}{2} = -\ln 2$
 $\therefore k = \frac{1}{50} \ln 2 \doteq 0.0139$

b i When $t = 0$, **ii** When $t = 24$, **iii** When $t = 1$ week
 $W = 20e^{-kt}$ $W = 20e^{-24k}$ $= 7 \times 24 \text{ hours}$
 $= 20e^0$ $= 20e^{-24 \frac{\ln 2}{50}}$ $= 168 \text{ hours}$
 $\therefore W = 20 \text{ g}$ $\doteq 14.34 \text{ g}$ $W = 20e^{-168 \frac{\ln 2}{50}}$
 $\doteq 1.948 \text{ g}$

c When $W = 1 \text{ g}$, $20e^{-\frac{\ln 2}{50} \times t} = 1$
 $\therefore e^{-\frac{\ln 2}{50} \times t} = 0.05$
 $\therefore -\frac{\ln 2}{50} \times t = \ln 0.05$
 $\therefore t = \frac{-50 \ln 0.05}{\ln 2} \doteq 216.1 \text{ hours}$

d $\frac{dW}{dt} = 20e^{-kt}(-k)$ **i** When $t = 100$ hours, **ii** When $t = 1000$ hours,
 $= 20e^{-kt}(-k)$ $\frac{dW}{dt} = \left(\frac{-20 \ln 2}{50}\right)e^{-2 \ln 2}$ $\frac{dW}{dt} = \left(\frac{-20 \ln 2}{50}\right)e^{-2 \ln 2}$
 $= \left(-20 \frac{\ln 2}{50}\right) \times e^{-\frac{\ln 2}{50} \times 100}$ $\doteq -0.0693 \text{ g/h}$ $\doteq -2.64 \times 10^{-7} \text{ g/h}$

e $\frac{dW}{dt} = -k(20e^{-kt})$
 $= -kW$ $\therefore \frac{dW}{dt} \propto W$

10 $T = 5 + 95e^{-kt} \text{ } ^\circ\text{C}$

a $T = 20^\circ\text{C}$ when $t = 15$
 $\therefore 20 = 5 + 95e^{-15k}$
 $\therefore 15 = 95e^{-15k}$
 $\therefore \ln\left(\frac{15}{95}\right) = -15k$
 $\therefore k = \frac{\ln\left(\frac{15}{95}\right)}{-15}$
 i.e., $k \doteq 0.1231$

b When $t = 0$,
 $T = 5 + 95e^0$
 $\therefore T = 5 + 95$
 $= 100^\circ\text{C}$

c $\frac{dT}{dt} = 0 + 95e^{-kt}(-k)$
 $= -(95e^{kt})k$
 $= -k(T - 5)$
 $\therefore \frac{dT}{dt} \propto T - 5$

d $\frac{dT}{dt} = -95e^{-kt} \times k$
 $\doteq -11.6902e^{-0.1231t}$

i When $t = 0$,
 $\frac{dT}{dt} \doteq -11.69$
 $\therefore$ temperature is decreasing at 11.69°C/min

ii When $t = 10$,
 $\frac{dT}{dt} \doteq -11.6902e^{-1.231}$
 $\doteq -3.415$
 $\therefore$ temperature is decreasing at 3.415°C/min

iii When $t = 20$,
 $\frac{dT}{dt} \doteq -11.6902e^{-2.462}$
 $\doteq -0.998$
 $\therefore$ temperature is decreasing at 0.998°C/min

11 $H(t) = 20 \ln(3t + 2) + 30 \text{ cm}, t \geq 0$

a Planted when $t = 0$
 $\therefore H(0) = 20 \ln(2) + 30$
 $\doteq 43.86 \text{ cm}$

b When $H = 1 \text{ m} = 100 \text{ cm}$
 $\therefore 20 \ln(3t + 2) + 30 = 100$
 $\therefore 20 \ln(3t + 2) = 70$
 $\therefore \ln(3t + 2) = 3.5$
 $\therefore 3t + 2 = e^{3.5}$
 $\therefore 3t = e^{3.5} - 2$
 $\therefore t = \frac{e^{3.5} - 2}{3} \text{ years}$
 $\therefore t \doteq 10.37 \text{ years}$

c $\frac{dH}{dt} = \frac{20}{(3t + 2)} \times 3$
 $= \frac{60}{3t + 2} \text{ cm/year}$

i When $t = 3$,
 $\frac{dH}{dt} = \frac{60}{11} \doteq 5.454$
 $\therefore$ it is growing at 5.45 cm/year

ii When $t = 10$,
 $\frac{dH}{dt} = \frac{60}{32} = 1.875$
 $\therefore$ it is growing at 1.88 cm/year

12 $A = s(1 - e^{-kt}), t \geq 0$

a When $t = 0$,
 $A = s(1 - e^0)$
 $= s(1 - 1)$
 $= 0$

b When $t = 3$, $A = 5$ and $s = 10$
 $\therefore 5 = 10(1 - e^{-3k})$
 $\therefore 0.5 = 1 - e^{-3k}$
 $\therefore e^{-3k} = 0.5$
 $\therefore e^{3k} = 2$
 $\therefore 3k = \ln 2$
 $\therefore k = \frac{1}{3} \ln 2$
 $\therefore k \doteq 0.231$

c When $t = 5$ and $s = 10$

$$\begin{aligned}\frac{dA}{dt} &= sk e^{-kt} \\ &= 10 \left(\frac{1}{3} \ln 2 \right) \left(e^{-\frac{5}{3} \ln 2} \right) \\ &\doteq 0.7278 \text{ units/hour}\end{aligned}$$

d Now $\frac{dA}{dt} = sk(e^{-kt})$

$$\begin{aligned}&= k(se^{-kt}) \\ &= -k(-se^{-kt}) \\ &= -k(A - s)\end{aligned}$$

$$\therefore \frac{dA}{dt} \propto A - s$$

13 $f(x) = \frac{e^x}{x}$

a No x -intercept since $e^x \neq 0$, i.e., $f(x) \neq 0$, and no y -intercept since $\frac{e^0}{0}$ is undefined.

b as $x \rightarrow +\infty$ $f(x) \rightarrow \infty$, and as $x \rightarrow -\infty$ $f(x) \rightarrow 0$ (below)

c By the quotient rule, $f'(x) = \frac{e^x x - e^x(1)}{x^2} = \frac{e^x(x-1)}{x^2}$ with sign diagram:

and when $x = 1$, $f(x) = \frac{e^1}{1} = e$



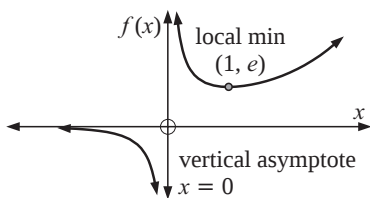
$\therefore$ local minimum when $x = 1$

i.e., local minimum at $(1, e)$.

Note: Vertical asymptote exists at $x = 0$,

i.e., as $x \rightarrow 0$ (from above), $y \rightarrow +\infty$, as $x \rightarrow 0$ (from below), $y \rightarrow -\infty$

d



e

Now $f'(x) = \frac{e^x(x-1)}{x^2}$

$$\therefore f'(-1) = \frac{e^{-1}(-1-1)}{(-1)^2} = -\frac{2}{e}$$

$\therefore$ slope of the tangent is $-\frac{2}{e}$

and when $x = -1$, $y = \frac{e^{-1}}{-1} = -\frac{1}{e}$

$\therefore$ equation of tangent is $\frac{y - \left(-\frac{1}{e}\right)}{x - (-1)} = -\frac{2}{e}$ i.e., $\frac{y + \frac{1}{e}}{x + 1} = -\frac{2}{e}$

$$\therefore e\left(y + \frac{1}{e}\right) = -2(x + 1)$$

$$ey + 1 = -2x - 2$$

$$\therefore 2x + ey = -3$$

14 $f(x) = \frac{x}{e^x}$

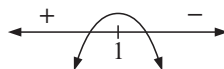
a Cuts x and y -axis at $(0, 0)$

b as $x \rightarrow +\infty$, $f(x) \rightarrow 0$ (above), as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$

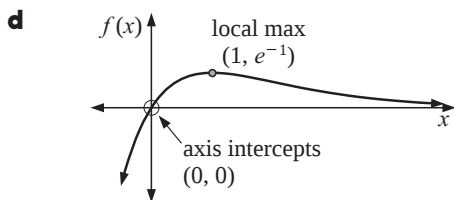
c $f'(x) = \frac{(1)e^x - xe^x}{(e^x)^2}$ {quotient rule}

$$\begin{aligned}&= \frac{e^x(1-x)}{(e^x)^2} \\ &= \frac{1-x}{e^x}\end{aligned}$$

which has sign diagram:



and when $x = 1$, $f(x) = \frac{1}{e^1} = \frac{1}{e}$, $\therefore$ there is a local maximum at $\left(1, \frac{1}{e}\right)$



e When $x = 2$, $f(2) = \frac{2}{e^2}$

and the slope of the tangent $= \frac{1-2}{e^2} = -\frac{1}{e^2}$, $\therefore$ slope of the normal $= e^2$

$\therefore$ equation of the normal is $\frac{y - \frac{2}{e^2}}{x - 2} = e^2$ i.e., $y - \frac{2}{e^2} = e^2(x - 2)$

i.e., $e^2x - y = 2e^2 - \frac{2}{e^2}$

cuts the x -axis when $y = 0$

$\therefore e^2x = 2e^2 - \frac{2}{e^2}$ i.e., at $x = 2 - \frac{2}{e^4}$

15 a $f(t) = 1 - e^{-at}$

$\therefore f'(t) = ae^{-at} = \frac{a}{e^{at}}$

$\therefore$ for all positive a , $f'(t) > 0$

$\therefore$ there are no stationary points.

c when $a = 1$ $f(t) = 1 - e^{-t}$

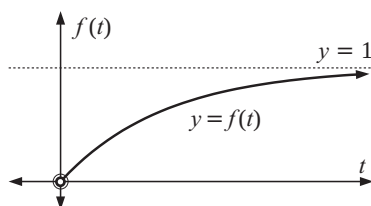
$\therefore f'(t) = e^{-t}$

$\therefore f''(t) = -e^{-t}$

$\therefore$ for all values of t , $f'(t) > 0$ and $f''(t) < 0$ $\{e^{-t} > 0\}$

$\therefore$ the curve is always concave down and $f(t)$ is always increasing.

b when $a = 1$, $f(t) = 1 - e^{-t}$



16 a $x(t) = 100 \left(2 - e^{-\frac{t}{10}} \right) = 200 - 200e^{-\frac{t}{10}}$ cm, $t > 0$

$\therefore v(t) = x'(t) = -100e^{-\frac{t}{10}} \left(-\frac{1}{10} \right) = 10e^{-\frac{t}{10}}$ cm/sec

$\therefore a(t) = v'(t) = -\frac{10}{10}e^{-\frac{t}{10}} \therefore a(t) = -e^{-\frac{t}{10}}$ cm/sec²

b when $t = 0$,

$x(0) = 100(2 - 1) = 100$ cm

$v(0) = 10e^0 = 10$ cm/sec

$a(0) = -e^0 = -1$ cm/sec²

c when $t = 5$,

$x(5) = 100(2 - e^{-\frac{1}{2}}) \doteq 139.3$ cm

$v(5) = 10e^{-\frac{1}{2}} \doteq 6.065$ cm/sec

$a(5) = -e^{-\frac{1}{2}} \doteq -0.6065$ cm/sec²

d when $x(t) = 150$ cm

$100(2 - e^{-\frac{t}{10}}) = 150$

$\therefore 2 - e^{-\frac{t}{10}} = 1.5$

$\therefore e^{-\frac{t}{10}} = 0.5$

$\therefore -\frac{t}{10} = \ln 0.5$

$\therefore t = -10 \ln 0.5$

i.e., $t \doteq 6.93$ seconds

e Now for all $t > 0$,

$v(t) = 10e^{-\frac{t}{10}} > 0$ {as $e^{-\frac{t}{10}} > 0$ }

and $a(t) = -e^{-\frac{t}{10}} < 0$

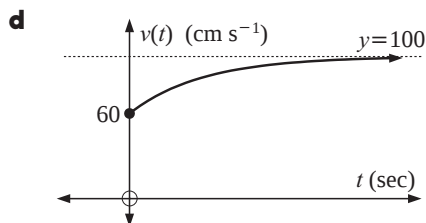
$\therefore$ speed is decreasing for all t , and velocity is decreasing for all t as $a(t) < 0$.

17 a $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, $t \geq 0$
 $v(t) = 100 - 40e^{-\frac{t}{5}}$ cm/sec
 $a(t) = 8e^{-\frac{t}{5}}$ cm/sec²

b When $t = 0$,
 $s(0) = 200$ cm
 $v(0) = 60$ cm/sec
 $a(0) = 8$ cm/sec²

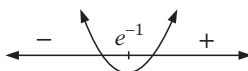
c as $t \rightarrow +\infty$, $e^{-\frac{t}{5}} \rightarrow 0$,
 $\therefore v(t) \rightarrow 100$ cm/sec (below)

e When $v(t) = 80$ cm/sec,
 $100 - 40e^{-\frac{t}{5}} = 80$
 $\therefore -40e^{-\frac{t}{5}} = -20$
 $\therefore e^{-\frac{t}{5}} = 0.5$
 $\therefore -\frac{t}{5} = \ln 0.5$
 $\therefore t = -5 \ln 0.5$
 $\therefore t \doteq 3.47$ seconds

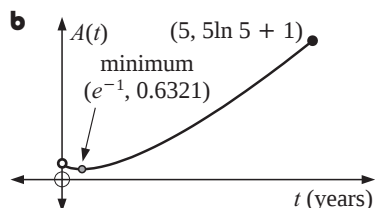


18 a $A(t) = t \ln t + 1$, $0 < t \leq 5$
 $\therefore A'(t) = \ln t + t \times \frac{1}{t} + 0$ {product rule}
 $= \ln t + 1$
 which is 0 when $\ln t = -1$
 i.e., $t = e^{-1}$

and the sign diagram of $A'(t)$ is:



$\therefore A(t)$ is a minimum when $t = \frac{1}{e} \doteq 0.3679$ years
 i.e., at 4.41 months old.



19 a $f(x) = \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$, $\therefore f'(x) = \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}(-x) = \frac{-x}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$

which is 0 only when $x = 0$, and when $x = 0$, $f(0) = \frac{1}{\pi\sqrt{2}}$

and $f'(x)$ has sign diagram: $\therefore$ there is a local maximum at $\left(0, \frac{1}{\pi\sqrt{2}}\right)$

and the function is increasing for $x \leq 0$, and decreasing for $x \geq 0$

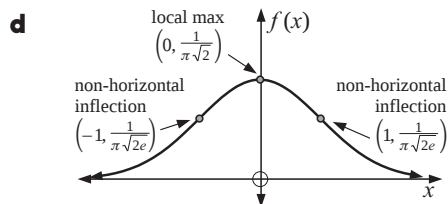
b $f'(x) = \frac{-x}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2} = \frac{1}{\pi\sqrt{2}}(-xe^{-\frac{1}{2}x^2})$

$\therefore f''(x) = \frac{1}{\pi\sqrt{2}}\left((-1)e^{-\frac{1}{2}x^2} + (-x)e^{-\frac{1}{2}x^2}(-x)\right)$ {product rule}
 $= \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}(x^2 - 1)$
 $= \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}(x+1)(x-1)$ which has sign diagram:

When $x = 1$, $f(x) = \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}} = \frac{1}{\pi\sqrt{2e}}$ and also when $x = -1$, $f(x) = \frac{1}{\pi\sqrt{2e}}$

$\therefore$ there are points of inflection at $\left(1, \frac{1}{\pi\sqrt{2e}}\right)$ and $\left(-1, \frac{1}{\pi\sqrt{2e}}\right)$

- c** as $x \rightarrow \infty$, $e^{-\frac{1}{2}x^2} \rightarrow 0$ (above),
 $\therefore f(x) \rightarrow 0$ (above),
 as $x \rightarrow -\infty$, $e^{-\frac{1}{2}x^2} \rightarrow 0$ (above),
 $\therefore f(x) \rightarrow 0$ (above)



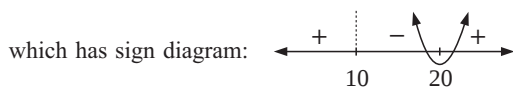
20 $C(x) = 4 \ln x + \left(\frac{30-x}{10}\right)^2$, $x \geq 10$ $\therefore C'(x) = \frac{4}{x} + 2\left(\frac{30-x}{10}\right)\left(-\frac{1}{10}\right)$

$$= \frac{4}{x} - \frac{30-x}{50}$$

$$= \frac{200 - x(30-x)}{50x}$$

$$= \frac{200 - 30x + x^2}{50x}$$

$$= \frac{(x-10)(x-20)}{50x}$$



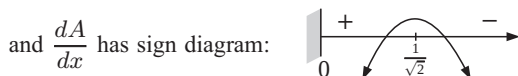
$\therefore$ the minimum cost occurs when $x = 20$, i.e., 20 kettles/day are produced

- 21** Let coordinates of D be $(x, 0)$, where $x > 0$. $\therefore$ the coordinates of C are (x, e^{-x^2})

$\therefore$ area ABCD $= 2xe^{-x^2}$ $\therefore \frac{dA}{dx} = 2e^{-x^2} + 2xe^{-x^2}(-2x)$ {product rule}

$$= 2e^{-x^2}(1 - 2x^2)$$

$$= 2e^{-x^2}(1 + \sqrt{2}x)(1 - \sqrt{2}x)$$



$\therefore$ area is a maximum when $x = \frac{1}{\sqrt{2}}$, and so C is $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

- 22** $P(x) = R(x) - C(x)$

$\therefore P(x) = \left[1000 \ln \left(1 + \frac{x}{400}\right) + 600\right] - [x(1.5) + 300]$

$$= 1000 \ln(1 + 0.0025x) - 1.5x + 300$$

$\therefore P'(x) = 1000 \left(\frac{0.0025}{1 + 0.0025x}\right) - 1.5 = \frac{2.5}{1 + 0.0025x} - 1.5$

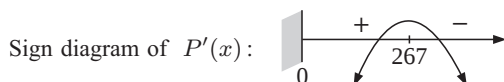
which is 0 when $\frac{2.5}{1 + 0.0025x} = \frac{3}{2}$

$\therefore 3 + 0.0075x = 5$

$\therefore 0.0075x = 2$

$\therefore x = \frac{2}{0.0075} \div 266.7$ and $P(266) \div 410.83$

$P(267) \div 410.83$



$\therefore$ to maximise profit, 266 or 267 torches/day should be produced.

- 23 a** $y = ax^2$ ($a > 0$) touches $y = \ln x$ when $ax^2 = \ln x$

If they touch at $x = b$

$$\text{then } ab^2 = \ln b \quad \dots (1)$$

$$\text{Now if } y = ax^2$$

$$\text{then } \frac{dy}{dx} = 2ax$$

$$\therefore \text{ at } x = b, \frac{dy}{dx} = 2ab$$

$$\text{and if } y = \ln x$$

$$\text{then } \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \text{ at } x = b, \frac{dy}{dx} = \frac{1}{b}$$

and the slope of the tangent at $x = b$ to the curve $y = ax^2$ is also the slope of the tangent to $y = \ln x$ at $x = b$, $\therefore \frac{1}{b} = 2ab \quad \dots (2)$

b Now $ab^2 = \frac{1}{2}$ {from (2)}

$$ab^2 = \ln b \quad \text{{from (1)}}$$

$$\therefore \ln b = \frac{1}{2}$$

$$\therefore b = e^{\frac{1}{2}} \quad \text{i.e., } b = \sqrt{e} \quad \text{and when } x = b = \sqrt{e}, y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$$

$\therefore$ point of contact is $(\sqrt{e}, \frac{1}{2})$

c $a = \frac{1}{2b^2}$ {from (2)} $\therefore a = \frac{1}{2(\sqrt{e})^2} = \frac{1}{2e}$

d The tangent has slope $2ab = 2\left(\frac{1}{2e}\right)\sqrt{e} = \frac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

$$\therefore \text{ tangent is } \frac{y - \frac{1}{2}}{x - \sqrt{e}} = \frac{1}{\sqrt{e}} \quad \therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}(x - \sqrt{e})$$

$$\therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}x - 1$$

$$\therefore y = xe^{-\frac{1}{2}} - \frac{1}{2}$$

24 $P(t) = \frac{50\,000}{1 + 1000e^{-0.5t}}, \quad 0 \leq t \leq 25$

$$= 50\,000(1 + 1000e^{-0.5t})^{-1}$$

$$\therefore P'(t) = -50\,000(1 + 1000e^{-0.5t})^{-2}(-500e^{-0.5t})$$

$$= 2.5 \times 10^7 e^{-0.5t}(1 + 1000e^{-0.5t})^{-2}$$

Now the wasp population is growing the fastest when $\frac{dP}{dt}$ is a maximum.

Using technology, the graph of $P'(t)$ can be drawn and the maximum obtained.

Maximum occurs when $x = 13.8155$, i.e., $x \doteq 13.81$ weeks

25 $f(t) = ate^{bt^2} = (at)e^{bt^2}$

$$\therefore f'(t) = ae^{bt^2} + ate^{bt^2}(2bt) \quad \text{{product rule}}$$

$$= ae^{bt^2}(1 + 2bt^2)$$

$$\therefore f'(t) \text{ is } 0 \text{ when } 1 + 2bt^2 = 0 \quad \text{but } t = 2 \text{ at this point,}$$

$$\therefore 1 + 8b = 0$$

$$b = -\frac{1}{8}$$

$$\therefore f(t) = ate^{-\frac{t^2}{8}} \quad \text{and} \quad f(2) = 1, \quad \therefore 2ae^{-\frac{4}{8}} = 1$$

$$\therefore ae^{-\frac{1}{2}} = \frac{1}{2} \quad \text{and so} \quad a = \frac{\sqrt{e}}{2}$$

REVIEW SET 23A

1 a $y = e^{x^3+2}$
 $= e^u \quad \text{where } u = x^3 + 2$
 $\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$
 $= e^u \times 3x^2$
 $= 3x^2 e^u$
 $= 3x^2 e^{x^3+2}$

b $y = \frac{e^x}{x^2} = \frac{u}{v}$
 Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$
 $\therefore \frac{dy}{dx} = \frac{e^x x^2 - e^x 2x}{x^4}$
 $= \frac{xe^x(x-2)}{x^4}$
 $= \frac{e^x(x-2)}{x^3}$

2 $y = e^{-x^2}$ and when $x = 1$, $y = e^{-x^2} = e^{-1} = \frac{1}{e}$, $\therefore$ point of contact is $\left(1, \frac{1}{e}\right)$

$$\text{Now } \frac{dy}{dx} = -2xe^{-x^2}$$

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = -2e^{-1}$$

$$\therefore \text{ the slope of the tangent} = -\frac{2}{e} \quad \text{and the slope of the normal} = \frac{e}{2}$$

$$\therefore \text{ equation of the normal is } \frac{y - \frac{1}{e}}{x - 1} = \frac{e}{2} \quad \text{i.e., } 2\left(y - \frac{1}{e}\right) = e(x - 1)$$

$$\therefore 2y - \frac{2}{e} = ex - e$$

$$\therefore 2ey - 2 = e^2x - e^2$$

$$\therefore e^2x - 2ey = e^2 - 2 \quad \text{or} \quad y = \frac{e}{2}x - \frac{e}{2} + \frac{1}{e}$$

3 Graphs meet when $e^x + 3 = 9 - e^{-x}$

$$\therefore e^{2x} + 3e^x = 9e^x - 1 \quad \{\times e^x\}$$

$$\therefore e^{2x} - 6e^x + 1 = 0$$

$$\therefore e^x = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

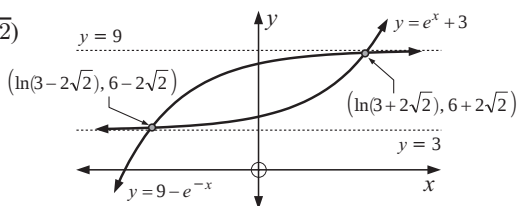
$$\therefore x = \ln(3 + 2\sqrt{2}) \quad \text{or} \quad \ln(3 - 2\sqrt{2})$$

$$\text{and } y = e^{\ln(3+2\sqrt{2})} + 3 = 6 + 2\sqrt{2} \quad \text{or} \quad y = e^{\ln(3-2\sqrt{2})} + 3 = 6 - 2\sqrt{2}$$

$$\text{i.e., graphs meet at } (\ln(3 + 2\sqrt{2}), 6 + 2\sqrt{2})$$

$$\text{and } (\ln(3 - 2\sqrt{2}), 6 - 2\sqrt{2})$$

$$\text{i.e., } (1.76, 8.83) \quad \text{and} \quad (-1.76, 3.17)$$



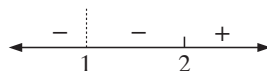
4 a $f(x) = \frac{e^x}{x-1}$ $\therefore$ when $x = 0$, $f(x) = \frac{e^0}{-1} = -1$

$$\text{and so the } y\text{-intercept is at } (0, -1) \quad \text{no } x\text{-intercept exist as } \frac{e^x}{x-1} \neq 0$$

b $f(x)$ is defined for all $x \neq 1$

$$\begin{aligned} \text{c Now } f'(x) &= \frac{e^x(x-1) - e^x(1)}{(x-1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{e^x(x-2)}{(x-1)^2} \end{aligned}$$

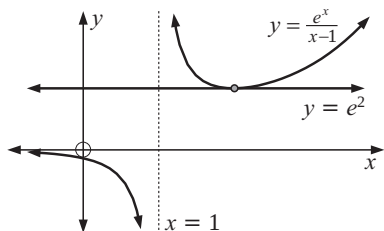
and has sign diagram:


 $\therefore f(x)$ is decreasing for $x < 1$ and $1 < x \leq 2$, and increasing for $x \geq 2$.

$$\begin{aligned} f''(x) &= \frac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1]}{(x-1)^4} \quad \{\text{product and quotient rules}\} \\ &= \frac{[e^x(x-2+1)](x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4} \\ &= \frac{e^x(x-1)(x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4} \\ &= \frac{e^x(x-1)[(x-1)^2 - 2(x-2)]}{(x-1)^4} \\ &= \frac{e^x(x-1)[x^2 - 2x + 1 - 2x + 4]}{(x-1)^4} \\ &= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3} \quad \text{where the quadratic has } \Delta < 0 \end{aligned}$$

 The sign diagram for $f''(x)$ is:

 $\therefore$ concave down for all $x \leq 1$
and concave up for all $x \geq 1$.

d

e Using **c** we have a local minimum at

$$(2, e^2) \quad \{\text{as } f(2) = \frac{e^2}{2-1} = e^2\}$$

 $\therefore$ the tangent at $x = 2$ is horizontal
and is $y = e^2$

5 $H(t) = 60 + 40 \ln(2t + 1)$ cm, $t \geq 0$

a when planted $t = 0$ $\therefore H(0) = 60 + 40 \ln(1) = 60 + 40(0) = 60$ cm

b i When $H(t) = 150$ cm,

$$\therefore 60 + 40 \ln(2t + 1) = 150$$

$$\therefore 40 \ln(2t + 1) = 90$$

$$\therefore \ln(2t + 1) = \frac{90}{40} = 2.25$$

$$\therefore 2t + 1 = e^{2.25}$$

$$\therefore 2t = e^{2.25} - 1$$

$$\therefore t = \frac{1}{2}(e^{2.25} - 1)$$

$$\therefore t \div 4.244 \text{ years}$$

ii When $H(t) = 300$ cm,

$$\therefore 60 + 40 \ln(2t + 1) = 300$$

$$\therefore 40 \ln(2t + 1) = 240$$

$$\therefore \ln(2t + 1) = 6$$

$$\therefore 2t + 1 = e^6$$

$$\therefore 2t = e^6 - 1$$

$$\therefore t = \frac{1}{2}(e^6 - 1)$$

$$\therefore t \div 201.2 \text{ years}$$

c $H'(t) = 40 \left(\frac{2}{2t + 1} \right) = \frac{80}{2t + 1}$ cm/year

i When $t = 2$, $H'(2) = \frac{80}{5} = 16$ cm/year

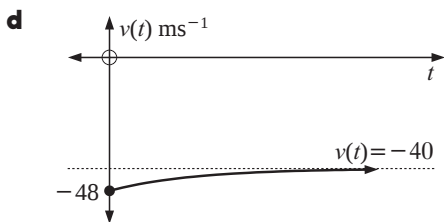
ii When $t = 20$, $H'(20) = \frac{80}{41} \div 1.95$ cm/year

6 $s(t) = 80e^{-\frac{t}{10}} - 40t$ metres, $t \geq 0$

a $\therefore v(t) = -8e^{-\frac{t}{10}} - 40$ m/sec and $a(t) = 0.8e^{-\frac{t}{10}}$ m/sec²

b When $t = 0$, $s(0) = 80$ m $v(0) = -48$ m/sec $a(0) = 0.8$ m/sec²

c as $t \rightarrow \infty$, $e^{-\frac{t}{10}} \rightarrow 0$, $\therefore v(t) \rightarrow -40$ m/sec (below)



e when $v(t) = -44$ m/sec

$$-8e^{-\frac{t}{10}} - 40 = -44$$

$$\therefore -8e^{-\frac{t}{10}} = -4$$

$$\therefore e^{-\frac{t}{10}} = 0.5$$

$$\therefore -\frac{t}{10} = \ln 0.5$$

$$\therefore t = -10 \ln 0.5$$

$$\therefore t \doteq 6.931 \text{ seconds}$$

7 Let the coordinates of B be $(x, 0)$

$\therefore$ the coordinates of A are (x, e^{-x})

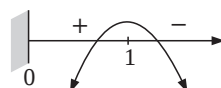
$$\therefore \text{area OABC} = xe^{-x}$$

$$\therefore \frac{dA}{dx} = (1)e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\}$$

$$= e^{-x}(1 - x)$$

$$= \frac{1 - x}{e^x}$$

and has sign diagram:



$\therefore$ has a local maximum when $x = 1$

$$\text{and when } x = 1, y = 1e^{-1} = \frac{1}{e}$$

$\therefore$ the coordinates of A are $\left(1, \frac{1}{e}\right)$

8 $P(x) = R(x) - C(x)$

$$= \left[200 \ln \left(1 + \frac{x}{100}\right) + 1000\right] - [(x - 100)^2 + 200]$$

$$= 200 \ln(1 + 0.01x) - (x - 100)^2 + 800$$

$$\frac{dP}{dx} = 200 \left(\frac{0.01}{1 + 0.01x} \right) - 2(x - 100)^1$$

$$= \frac{2}{1 + 0.01x} - \frac{2(x - 100)}{1}$$

$$= \frac{2 - 2(x - 100)(1 + 0.01x)}{1 + 0.01x}$$

$$= \frac{2 - 2(x + 0.01x^2 - 100 - x)}{1 + 0.01x}$$

$$= \frac{2 - 0.02x^2 + 200}{1 + 0.01x}$$

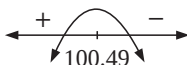
$$= \frac{202 - 0.02x^2}{1 + 0.01x} \quad \text{which is 0 when } 0.02x^2 = 202$$

$$\therefore x^2 = 10100$$

$$\therefore x = \sqrt{10100} \quad \{\text{as } x > 0\}$$

$$\therefore x \doteq 100.49$$

and sign diagram of $\frac{dP}{dx}$ is:



$\therefore$ maximum profit when $x \doteq 100.49$

Now $P(100) = \$938.63$ and $P(101) = \$938.63$

$\therefore$ maximum profit of \$938.63 when 100 or 101 shirts are made.

REVIEW SET 23B

1 a

$$y = \ln(x^3 - 3x)$$

$$\therefore y = \ln u \quad \text{where} \quad u = x^3 - 3x$$

$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= \frac{1}{u}(3x^2 - 3)$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - 3}{x^3 - 3x}$$

b

$$y = \ln\left(\frac{x+3}{x^2}\right)$$

$$= \ln(x+3) - \ln x^2$$

$$= \ln(x+3) - 2 \ln x$$

$$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{2}{x}$$

2

$$y = \ln(x^2 + 3)$$

$$\therefore y = \ln u \quad \text{where} \quad u = x^2 + 3$$

$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \times 2x$$

$$\therefore \frac{dy}{dx} = \frac{2x}{x^2 + 3} \quad \text{and when } x = 0, \quad \frac{dy}{dx} = 0, \quad \text{i.e., the slope of the tangent is 0.}$$

But at $x = 0$, $y = \ln(0 + 3) = \ln 3$

$\therefore$ tangent is $y = \ln 3$ which does not cut the x -axis.

3 a

$$e^{2x} = 3e^x$$

$$\therefore e^{2x} - 3e^x = 0$$

$$\therefore e^x(e^x - 3) = 0$$

$$\therefore e^x = 0 \text{ or } 3$$

$$\therefore e^x = 3 \quad \{\text{as } e^x > 0\}$$

$$\therefore x = \ln 3$$

b

$$e^{2x} - 7e^x + 12 = 0$$

$$\therefore (e^x - 3)(e^x - 4) = 0$$

$$\therefore e^x = 3 \text{ or } 4$$

$$\therefore x = \ln 3 \text{ or } \ln 4$$

4 $f(x) = e^x - x$

a $f'(x) = e^x - 1$

which is 0 when $e^x = 1$
i.e., $x = 0$

Sign diagram of $f'(x)$ is:



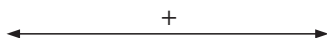
and at $x = 0$, $f(0) = e^0 - 0 = 1$

$\therefore$ there is a local minimum at $(0, 1)$

c $f''(x) = e^x$

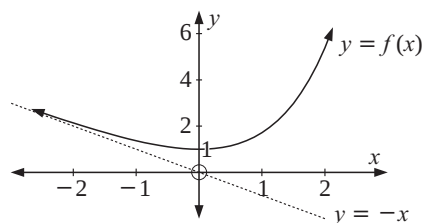
$$\therefore f''(x) > 0 \quad \text{for all } x$$

$\therefore f(x)$ is concave up for all x



b as $x \rightarrow +\infty$, $e^x \rightarrow \infty$ faster
than $x \therefore f(x) \rightarrow +\infty$,
as $x \rightarrow -\infty$, $e^x \rightarrow 0$
 $\therefore f(x) \rightarrow -x$ (from above)

d



e Since a local minimum exists at $(0, 1)$, $f(x) \geq 1$ for all x .

$$\therefore e^x - x \geq 1$$

$$\therefore e^x \geq x + 1 \quad \text{for all } x$$

5 a $f(x) = \ln(e^x + 3)$

$$\therefore y = \ln u \quad \text{where } u = e^x + 3$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= \frac{1}{u} \times e^x$$

$$= \frac{e^x}{u}$$

$$\therefore f'(x) = \frac{e^x}{e^x + 3}$$

b $f(x) = \ln\left(\frac{(x+2)^3}{x}\right)$

$$= \ln(x+2)^3 - \ln x$$

$$= 3 \ln(x+2) - \ln x$$

$$\therefore f'(x) = \frac{3}{x+2} - \frac{1}{x}$$

$$= \frac{3x - (x+2)}{x(x+2)}$$

$$\therefore f'(x) = \frac{2x-2}{x(x+2)}$$

6 a $3e^x - 5 = -2e^{-x}$

$$\therefore 3e^{2x} - 5e^x = -2 \quad \{\times \text{ by } e^x\}$$

$$\therefore 3e^{2x} - 5e^x + 2 = 0$$

$$\therefore (3e^x - 2)(e^x - 1) = 0$$

$$\therefore e^x = \frac{2}{3} \text{ or } 1$$

$$\therefore x = \ln \frac{2}{3} \text{ or } 0$$

b $2 \ln x - 3 \ln\left(\frac{1}{x}\right) = 10$

$$\therefore 2 \ln x - 3 \ln(x^{-1}) = 10$$

$$\therefore 2 \ln x + 3 \ln x = 10$$

$$\therefore 5 \ln x = 10$$

$$\therefore \ln x = 2$$

$$\therefore x = e^2$$

7 $s(t) = 25t - 10 \ln t \text{ cm} \quad t \geq 1$

a $v(t) = 25 - \frac{10}{t} \text{ cm/min}$

$$\therefore a(t) = 10t^{-2}$$

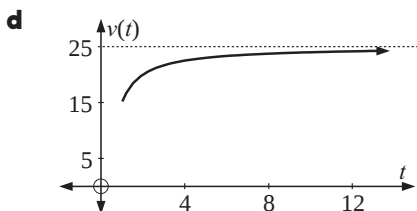
$$= \frac{10}{t^2} \text{ cm/min}^2$$

b when $t = e$, $s(e) = 25e - 10 \ln e = 25e - 10 \text{ cm} \div 57.96 \text{ cm}$

$$v(e) = 25 - \frac{10}{e} \text{ cm/min} \div 21.32 \text{ cm/min}$$

$$a(e) = \frac{10}{e^2} \text{ cm/min}^2 \div 1.35 \text{ cm/min}^2$$

c as $t \rightarrow +\infty$, $\frac{10}{t} \rightarrow 0$, $\therefore v(t) \rightarrow 25 \text{ cm/min}$ (below)



e when $v(t) = 12 \text{ cm/min}$,

$$25 - \frac{10}{t} = 12$$

$$\therefore \frac{10}{t} = 13$$

$$\therefore t = \frac{10}{13} \text{ minutes}$$

$$\mathbf{8} \quad C(x) = 10 \ln x + \left(20 - \frac{x}{10}\right)^2 = 10 \ln x + 400 - 4x + \frac{x^2}{100}$$

$$\begin{aligned} \therefore C'(x) &= \frac{10}{x} - 4 + \frac{x}{50} \\ &= \frac{500 - 200x + x^2}{50x} \end{aligned}$$

$$\text{which is } 0 \text{ when } x^2 - 200x + 500 = 0$$

$$\text{i.e., when } x = \frac{200 \pm \sqrt{38\,000}}{2} \doteq 2.53 \text{ or } 197.47$$

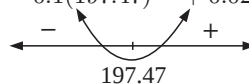
$$\text{But } x \geq 50, \quad x \doteq 197.47 \quad \text{and} \quad C(197) \doteq 52.92$$

$$C(198) \doteq 52.92$$

$$\text{Now } C''(x) = -10x^{-2} + \frac{1}{50}, \quad \therefore C''(197.47) \doteq -0.1(197.47)^{-2} + 0.02 \doteq 0.02 \text{ which is } > 0$$

$$\therefore \text{ minimum cost when } x \doteq 197.46$$

i.e., need to produce 197 or 198 clocks/day



Chapter 24

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

EXERCISE 24A

1 a $y = \sin(2x)$
 $\therefore \frac{dy}{dx} = \cos(2x) \frac{d}{dx}(2x)$
 $= 2 \cos(2x)$

c $y = \cos(3x) - \sin x$
 $\therefore \frac{dy}{dx} = -\sin(3x) \times 3 - \cos x$
 $= -3 \sin(3x) - \cos x$

e $y = \cos(3 - 2x)$
 $\therefore \frac{dy}{dx} = -\sin(3 - 2x) \times -2$
 $= 2 \sin(3 - 2x)$

g $y = \sin\left(\frac{x}{2}\right) - 3 \cos x$
 $\therefore \frac{dy}{dx} = \frac{1}{2} \cos\left(\frac{x}{2}\right) + 3 \sin x$

i $y = 4 \sin x - \cos(2x)$
 $\therefore \frac{dy}{dx} = 4 \cos x + \sin(2x) \times 2$
 $= 4 \cos x + 2 \sin(2x)$

2 a $y = x^2 + \cos x$
 $\therefore \frac{dy}{dx} = 2x - \sin x$

c $y = e^x \cos x$
 $\therefore \frac{dy}{dx} = e^x(-\sin x) + e^x \cos x$
 $= e^x \cos x - e^x \sin x$

e $y = \ln(\sin x)$
 $\therefore \frac{dy}{dx} = \frac{\cos x}{\sin x}$

g $y = \sin(3x)$
 $\therefore \frac{dy}{dx} = 3 \cos(3x)$

b $y = \sin x + \cos x$
 $\therefore \frac{dy}{dx} = \cos x - \sin x$

d $y = \sin(x + 1)$
 $\therefore \frac{dy}{dx} = \cos(x + 1) \frac{d}{dx}(x + 1)$
 $= 1 \cos(x + 1)$
 $= \cos(x + 1)$

f $y = \tan(5x)$
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2(5x)} \times 5$
 $= \frac{5}{\cos^2(5x)}$

h $y = 3 \tan(\pi x)$
 $\therefore \frac{dy}{dx} = 3 \times \frac{1}{\cos^2(\pi x)} \times \pi$
 $= \frac{3\pi}{\cos^2(\pi x)}$

b $y = \tan x - 3 \sin x$
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} - 3 \cos x$

d $y = e^{-x} \sin x$
 $\therefore \frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$

f $y = e^{2x} \tan x$
 $\therefore \frac{dy}{dx} = 2e^{2x} \tan x + e^{2x} \times \frac{1}{\cos^2 x}$
 $\therefore \frac{dy}{dx} = 2e^{2x} \tan x + \frac{e^{2x}}{\cos^2 x}$

h $y = \cos\left(\frac{x}{2}\right)$
 $\therefore \frac{dy}{dx} = -\frac{1}{2} \sin\left(\frac{x}{2}\right)$

$$\begin{aligned}\mathbf{i} \quad y &= 3 \tan(2x) \\ \therefore \frac{dy}{dx} &= 3 \times \frac{1}{\cos^2(2x)} \times 2 \\ &= \frac{6}{\cos^2(2x)}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad y &= x \cos x \\ \therefore \frac{dy}{dx} &= 1 \times \cos x + x(-\sin x) \\ &= \cos x - x \sin x\end{aligned}$$

$$\begin{aligned}\mathbf{k} \quad y &= \frac{\sin x}{x} \\ \therefore \frac{dy}{dx} &= \frac{x(\cos x) - \sin x \times 1}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2}\end{aligned}$$

$$\begin{aligned}\mathbf{l} \quad y &= x \tan x \\ \therefore \frac{dy}{dx} &= 1 \times \tan x + x \times \frac{1}{\cos^2 x} \\ \therefore \frac{dy}{dx} &= \tan x + \frac{x}{\cos^2 x}\end{aligned}$$

$$\begin{aligned}\mathbf{3} \quad \mathbf{a} \quad y &= \sin(x^2) \\ \therefore \frac{dy}{dx} &= 2x \cos(x^2)\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad y &= \cos(\sqrt{x}) \\ &= \cos(x^{\frac{1}{2}}) \\ \therefore \frac{dy}{dx} &= -\sin(x^{\frac{1}{2}}) \times \frac{1}{2}x^{-\frac{1}{2}} \\ &= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x})\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad y &= \sqrt{\cos x} \\ &= (\cos x)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \frac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x) \\ &= -\frac{\sin x}{2\sqrt{\cos x}}\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad y &= \sin^2 x \\ &= (\sin x)^2 \\ \therefore \frac{dy}{dx} &= 2 \sin x \cos x\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad y &= \cos^3 x \\ &= (\cos x)^3 \\ \therefore \frac{dy}{dx} &= 3 \cos^2 x \times (-\sin x) \\ &= -3 \sin x \cos^2 x\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad y &= \cos x \sin(2x) \\ \therefore \frac{dy}{dx} &= (-\sin x) \sin(2x) + \cos x(2 \cos(2x)) \\ &= -\sin x \sin(2x) + 2 \cos x \cos(2x)\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad y &= \cos(\cos x) \\ \therefore \frac{dy}{dx} &= -\sin(\cos x) \times (-\sin x) \\ &= \sin x \times \sin(\cos x)\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad y &= \cos^3(4x) \\ &= (\cos(4x))^3 \\ \therefore \frac{dy}{dx} &= 3(\cos(4x))^2 \times (-4 \sin(4x)) \\ &= -12 \sin(4x) \cos^2(4x)\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad y &= \frac{1}{\sin x} \\ &= (\sin x)^{-1} \\ \therefore \frac{dy}{dx} &= -1(\sin x)^{-2} \times \cos x \\ &= -\frac{\cos x}{\sin^2 x}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad y &= \frac{1}{\cos(2x)} \\ &= (\cos(2x))^{-1} \\ \therefore \frac{dy}{dx} &= -1(\cos(2x))^{-2} \times (-2 \sin(2x)) \\ &= \frac{2 \sin(2x)}{\cos^2(2x)}\end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad y &= \frac{2}{\sin^2(2x)} \\ &= 2(\sin(2x))^{-2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -4(\sin(2x))^{-3} \times 2 \cos(2x) \\ &= -\frac{8 \cos(2x)}{\sin^3(2x)} \end{aligned}$$

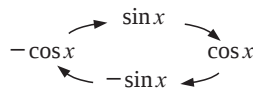
$$\mathbf{l} \quad y = \frac{8}{\tan^3\left(\frac{x}{2}\right)} = 8 \left[\tan\left(\frac{x}{2}\right) \right]^{-3}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -24 \left[\tan\left(\frac{x}{2}\right) \right]^{-4} \times \frac{1}{2} \times \frac{1}{\cos^2\left(\frac{x}{2}\right)} \\ &= \frac{-12}{\cos^2\left(\frac{x}{2}\right) \tan^4\left(\frac{x}{2}\right)} \end{aligned}$$

$$\mathbf{4} \quad \mathbf{a} \quad \text{If } y = \sin x, \text{ then } \frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x \quad \text{and} \quad \frac{d^4y}{dx^4} = \sin x$$

b Successive derivatives will cycle through the pattern,

so $\frac{d^n y}{dx^n}$ may only take the values found in **a**.



$$\mathbf{5} \quad \mathbf{a} \quad \text{If } y = \sin(2x + 3), \text{ then } \frac{dy}{dx} = 2 \cos(2x + 3) \quad \text{and} \quad \frac{d^2y}{dx^2} = -4 \sin(2x + 3)$$

$$\therefore \frac{d^2y}{dx^2} + 4y = -4 \sin(2x + 3) + 4 \sin(2x + 3) = 0$$

$$\mathbf{b} \quad \text{If } y = 2 \sin x + 3 \cos x$$

$$\text{then } y' = 2 \cos x - 3 \sin x$$

$$\text{and } y'' = -2 \sin x - 3 \cos x$$

$$\therefore y'' + y = -2 \sin x - 3 \cos x + 2 \sin x + 3 \cos x = 0$$

$$\mathbf{c} \quad y = \frac{\cos x}{1 + \sin x}$$

$$\therefore \frac{dy}{dx} = \frac{(-\sin x)(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$

$$= \frac{-1 - \sin x}{(1 + \sin x)^2} \quad \{\text{as } \sin^2 x + \cos^2 x = 1\}$$

$$= -\frac{(1 + \sin x)}{(1 + \sin x)^2}$$

$$= \frac{-1}{1 + \sin x}$$

Since $\frac{-1}{1 + \sin x}$ never equals 0, there are no horizontal tangents.

$$\mathbf{6} \quad \mathbf{a} \quad y = \sin x$$

$$\therefore \frac{dy}{dx} = \cos x$$

$\therefore$ At $x = 0$, the tangent has slope

$$\cos 0 = 1$$

$$\therefore \text{the equation is } \frac{y - 0}{x - 0} = 1$$

$$\text{i.e., } y = x$$

$$\mathbf{b} \quad y = \tan x$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x}$$

$\therefore$ at $x = 0$, the tangent has slope

$$\frac{1}{\cos^2 0} = 1$$

$$\text{so the equation is } \frac{y - 0}{x - 0} = 1$$

$$\text{i.e., } y = x$$

c $y = \cos x$ and at $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$

$$\therefore \frac{dy}{dx} = -\sin x$$

At $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$ and the tangent has slope $-\sin \frac{\pi}{6} = -\frac{1}{2}$

$\therefore$ the normal has slope 2,

so its equation is $\frac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2$

$$\text{i.e. } y - \frac{\sqrt{3}}{2} = 2x - \frac{\pi}{3}$$

$$\text{i.e. } 2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

d $y = \frac{1}{\sin(2x)} = (\sin(2x))^{-1}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -1(\sin(2x))^{-2} \times 2 \cos(2x) \\ &= -\frac{2 \cos(2x)}{(\sin(2x))^2} \end{aligned}$$

At $x = \frac{\pi}{4}$, $y = 1$ and the tangent

$$\text{has slope } -\frac{2 \cos \frac{\pi}{2}}{(\sin \frac{\pi}{2})^2} = 0$$

$\therefore$ the slope of the normal is undefined, so the normal is $x = \frac{\pi}{4}$.

7 $d = 9.3 + 6.8 \cos(0.507t)$ m $\therefore \frac{dd}{dt} = -6.8 \sin(0.507t) \times 0.507$
 $= -3.4476 \sin(0.507t)$

a When $t = 8$, $\frac{dd}{dt} \div 2.731 > 0$
 $\therefore$ the tide is rising.

b When $t = 8$, the tide is rising at the rate of 2.731 m per hour.

8 a $V(t) = 340 \sin(100\pi t)$
 $\therefore \frac{dV}{dt} = 340 \cos(100\pi t) \times 100\pi$
 $= 34\,000\pi \cos(100\pi t)$

When $t = 0.01$,

$$\begin{aligned} \frac{dV}{dt} &= 34\,000\pi \times \cos \pi \\ &= -34\,000\pi \text{ units/second} \end{aligned}$$

b $V(t)$ is a maximum when $\sin(100\pi t) = 1$.

This occurs when $100\pi t = \frac{\pi}{2}$, and at this time,

$$\begin{aligned} \frac{dV}{dt} &= 34\,000\pi \cos(100\pi t) \\ &= 34\,000\pi \cos\left(\frac{\pi}{2}\right) \\ &= 0 \text{ units/second} \end{aligned}$$

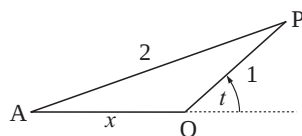
9 a The distance from $A(-x, 0)$ to $P(\cos t, \sin t)$ is fixed at 2 m.

$$\therefore (\cos t + x)^2 + \sin^2 t = 2^2$$

$$\therefore (\cos t + x)^2 = 4 - \sin^2 t$$

$$\therefore x + \cos t = \pm \sqrt{4 - \sin^2 t}$$

$$\therefore \text{since } x > 0, x = \sqrt{4 - \sin^2 t} - \cos t$$



b Now $\frac{dx}{dt} = \frac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2 \sin t \cos t) + \sin t$
 $= \frac{-\sin t \cos t}{\sqrt{4 - \sin^2 t}} + \sin t$

i When $t = 0$,
 $\sin t = 0$ and $\cos t = 1$

$$\therefore \frac{dx}{dt} = 0 + 0 = 0$$

iii When $t = \frac{2\pi}{3}$,

$$\sin t = \frac{\sqrt{3}}{2} \text{ and } \cos t = -\frac{1}{2}$$

$$\therefore \frac{dx}{dt} = \frac{-\frac{\sqrt{3}}{2}(-\frac{1}{2})}{\sqrt{4 - \frac{3}{4}}} + \frac{\sqrt{3}}{2} \div 1.106$$

ii When $t = \frac{\pi}{2}$,

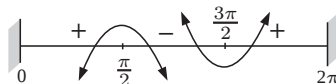
$$\sin t = 1 \text{ and } \cos t = 0$$

$$\therefore \frac{dx}{dt} = 0 + \sin \frac{\pi}{2} = 1$$

10 a If $y = \sin x$ then $\frac{dy}{dx} = \cos x$

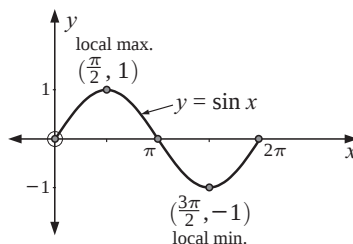
Stationary points occur when $\frac{dy}{dx} = 0$, i.e., when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Sign diagram for $\frac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{2}, 1)$.

Local minimum at $(\frac{3\pi}{2}, -1)$.



b If $y = \cos(2x)$ then $\frac{dy}{dx} = -2 \sin(2x)$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad -2 \sin(2x) = 0$$

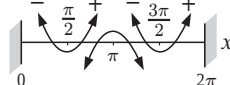
$$\therefore \sin(2x) = 0$$

$$\therefore 2x = k\pi \quad \text{for any integer } k$$

$$\therefore x = \frac{k\pi}{2}$$

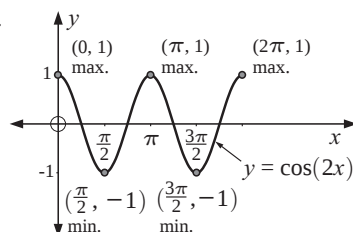
$$\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ or } 2\pi \quad \text{for } 0 \leq x \leq 2\pi$$

Sign diagram for $\frac{dy}{dx}$ is:



Local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$.

Local minima at $(\frac{\pi}{2}, -1)$, $(\frac{3\pi}{2}, -1)$.



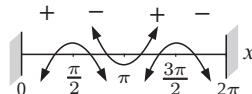
c If $y = \sin^2 x$ then $\frac{dy}{dx} = 2 \sin x \cos x = \sin(2x)$

$$\therefore \frac{dy}{dx} = 0 \quad \text{when} \quad \sin(2x) = 0$$

$$\text{i.e., when } x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ or } 2\pi \quad \text{for } 0 \leq x \leq 2\pi$$

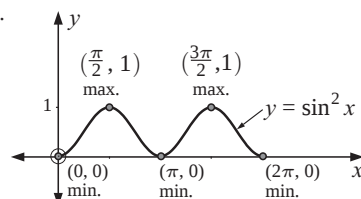
{using **b**}

Sign diagram for $\frac{dy}{dx}$ is:



Local minima at $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$.

Local maxima at $(\frac{\pi}{2}, 1)$, $(\frac{3\pi}{2}, 1)$.



11 a $f(x) = \frac{1}{\cos x}$ for $0 \leq x \leq 2\pi$
 $\therefore f(x)$ is undefined whenever
 $\cos x = 0$
 i.e., $x = \frac{\pi}{2}, \frac{3\pi}{2}$

b $f(x) = (\cos x)^{-1}$
 $\therefore f'(x) = -1(\cos x)^{-2}(-\sin x)$
 $= \frac{\sin x}{\cos^2 x}$
 $\therefore f'(x) = 0$ when $\sin x = 0$
 i.e., $x = 0, \pi, 2\pi$

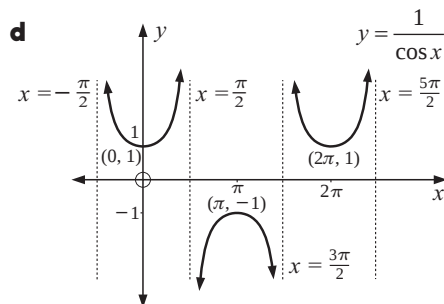
Sign diagram for $\frac{dy}{dx}$ is:



Local minima at $(0, 1)$, $(2\pi, 1)$

Local maximum at $(\pi, -1)$

c $f(x) = \frac{1}{\cos x}$
 $\therefore f(x + 2\pi) = \frac{1}{\cos(x + 2\pi)}$
 $= \frac{1}{\cos x}$
 $= f(x)$
 i.e., $f(x)$ has a period of 2π .

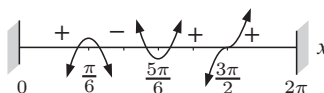


12 If $y = \sin(2x) + 2\cos x$
 then $\frac{dy}{dx} = 2\cos(2x) + (-2\sin x)$
 $= 2[1 - 2\sin^2 x] - 2\sin x$
 $= 2 - 4\sin^2 x - 2\sin x$

At the stationary points, $\frac{dy}{dx} = 0$.

$$\begin{aligned}\therefore -4\sin^2 x - 2\sin x + 2 &= 0 \\ \therefore 2\sin^2 x + \sin x - 1 &= 0 \\ \therefore (2\sin x - 1)(\sin x + 1) &= 0 \\ \therefore \sin x = \frac{1}{2} \text{ or } \sin x &= -1 \\ \therefore x = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } x &= \frac{3\pi}{2}\end{aligned}$$

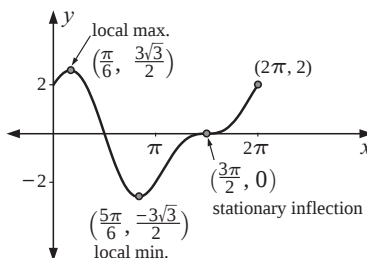
Sign diagram for $\frac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$.

Local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$.

Stationary inflection at $(\frac{3\pi}{2}, 0)$.



13 $x(t) = 1 - 2 \cos t$ cm

$\therefore v(t) = x'(t) = 2 \sin t$ and $a(t) = v'(t) = 2 \cos t$

a When $t = 0$

$$x(0) = 1 - 2 \cos(0)$$

$$= -1 \text{ cm}$$

$$v(0) = 2 \sin(0)$$

$$= 0 \text{ cm/s}$$

$$a(0) = 2 \cos(0)$$

$$= 2 \text{ cm/s}^2$$

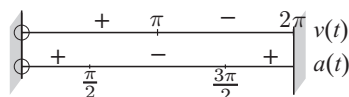
c The particle stops momentarily before it reverses direction. Therefore we need to look for the points where the velocity equals zero, i.e., when

$$v(t) = 2 \sin t = 0$$

$$\therefore \sin t = 0$$

$$\therefore t = 0, \pi, 2\pi \quad (0 \leq t \leq 2\pi)$$

Sign diagrams for $v(t)$ and $a(t)$ are:



The particle reverses direction at $t = 0, \pi, 2\pi$ ($0 \leq t \leq 2\pi$)

At $t = 0$, $x(0) = -1$ cm At $t = \pi$, $x(\pi) = 3$ cm At $t = 2\pi$, $x(2\pi) = -1$ cm

b When $t = \frac{\pi}{4}$

$$x\left(\frac{\pi}{4}\right) = 1 - \frac{2}{\sqrt{2}} = 1 - \sqrt{2} \text{ cm}$$

$$v\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s}$$

$$a\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s}^2$$

The particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm/s with increasing speed.

d The particle's speed is increasing when $v(t) = 2 \sin t$ and $a(t) = 2 \cos t$ have the same sign.

Considering $a(t) = 0$

i.e., $2 \cos t = 0$

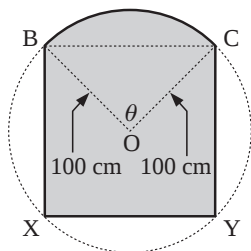
$$\cos t = 0$$

$$t = \frac{\pi}{2}, \frac{3\pi}{2}$$

i.e., $0 \leq t \leq \frac{\pi}{2}$ and $\pi \leq t \leq \frac{3\pi}{2}$

EXERCISE 24B

1



Using the cosine rule in $\triangle BCO$,

$$BC^2 = 10^2 + 10^2 - 2 \times 10 \times 10 \cos \theta$$

$$\therefore BC = \sqrt{200 - 200 \cos \theta}$$

and $XY = BC$

Now $BY^2 = BX^2 + XY^2$ {Pythagoras}

$$\therefore 400 = BX^2 + (200 - 200 \cos \theta)$$

$$\therefore BX^2 = 200 + 200 \cos \theta$$

$$\therefore BX = \sqrt{200 + 200 \cos \theta}$$

The shaded area is equal to the area of the sector plus $\frac{3}{4}$ of the area of $BCYX$

$$\therefore A = \frac{1}{2} (10)^2 \theta + \frac{3}{4} [BX \times BC]$$

$$= 50\theta + \frac{3}{4} \sqrt{200 + 200 \cos \theta} \sqrt{200 - 200 \cos \theta}$$

$$= 50\theta + \frac{3}{4} \times 200 \sqrt{1 + \cos \theta} \sqrt{1 - \cos \theta}$$

$$= 50\theta + 150 \sqrt{1 - \cos^2 \theta}$$

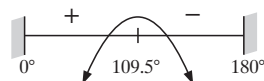
$$= 50\theta + 150 \sin \theta$$

$$= 50(\theta + 3 \sin \theta) \text{ as required}$$

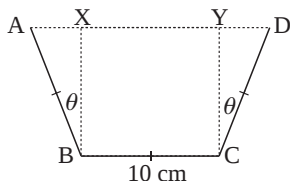
$$\therefore \frac{dA}{d\theta} = 50 + 150 \cos \theta = 50(1 + 3 \cos \theta),$$

which is zero when $\cos \theta = -\frac{1}{3}$

The sign diagram of $\frac{dA}{d\theta}$ is:



$\therefore$ since $0 < \theta < 180$, max A is when $\theta \doteq 109.5^\circ$.

2 aThe triangles have height $10 \cos \theta$ and width $10 \sin \theta$.

$$\begin{aligned}
 \therefore \text{area } A &= \text{area of } \Delta s + \text{area of rectangle} \\
 &= 2 \times \frac{1}{2} \times 10 \cos \theta \times 10 \sin \theta + 10 \times 10 \cos \theta \\
 &= 100 \sin \theta \cos \theta + 100 \cos \theta \\
 &= 100 \cos \theta (1 + \sin \theta)
 \end{aligned}$$

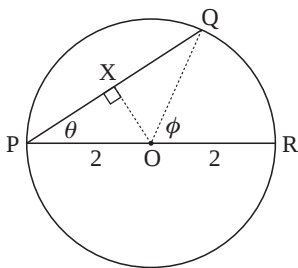
$$\begin{aligned}
 \mathbf{b} \quad \frac{dA}{d\theta} &= -100(2 \sin \theta - 1)(\sin \theta + 1) \\
 &= 100(-\sin \theta(1 + \sin \theta) + \cos \theta \times \cos \theta) \\
 &= 100(-\sin \theta - \sin^2 \theta + \cos^2 \theta) \\
 &= 100(-\sin \theta - \sin^2 \theta + 1 - \sin^2 \theta) \\
 &= -100(2 \sin^2 \theta + \sin \theta - 1) \\
 &= -100(2 \sin \theta - 1)(\sin \theta + 1)
 \end{aligned}$$

$$\therefore \frac{dA}{d\theta} = 0 \quad \text{when} \quad 2 \sin \theta - 1 = 0 \quad \text{or} \quad \sin \theta + 1 = 0$$

$$\text{i.e., } \sin \theta = \frac{1}{2} \quad \text{or} \quad \sin \theta = -1$$

$$\mathbf{c} \quad \text{Using } \mathbf{b}, \quad \frac{dA}{d\theta} = 0 \quad \text{when} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}.$$

Sign diagram for $\frac{dA}{d\theta}$ is: so the maximum area is when $\theta = \frac{\pi}{6} = 30^\circ$

3

$$\frac{PX}{2} = \cos \theta, \quad \therefore PQ = 2PX = 4 \cos \theta$$

 $\therefore$ the time taken to row from P to Q is

$$\frac{4 \cos \theta}{3} \text{ hours}$$

Now $\phi = 2\theta$ {angle at the centre}But, arc length $QR_{\text{arc}} = 2\phi$,

$$\therefore QR_{\text{arc}} = 4\theta,$$

and the time taken to walk from Q to R is $\frac{4\theta}{5}$

$$\therefore \text{total time from P to R, } T = \frac{4}{3} \cos \theta + \frac{4\theta}{5}$$

$$\therefore \frac{dT}{d\theta} = -\frac{4}{3} \sin \theta + \frac{4}{5}$$

$$\therefore \frac{dT}{d\theta} = 0 \quad \text{when} \quad -\frac{4}{3} \sin \theta = -\frac{4}{5}$$

$$\therefore \sin \theta = \frac{3}{5}$$

$$\therefore \theta \doteq 0.6435 \text{ radians}$$

$$\text{i.e., } \theta \doteq 36.87^\circ$$

sign diagram of $\frac{dT}{d\theta}$ is: i.e., the maximum time is when $\theta \doteq 36.87^\circ$

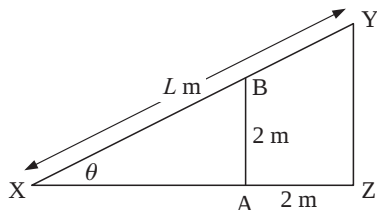
$$\text{and the maximum time is } \frac{4}{3} \cos 0.6435 + \frac{4}{5} \times 0.6435$$

$$\doteq 1.581 \text{ hours}$$

$$\doteq 1 \text{ hour } 34 \text{ min } 53 \text{ sec}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \tan \theta &= \frac{2}{AX} & \text{and} \quad \sin \theta &= \frac{2}{BX} \\ \therefore AX &= \frac{2}{\tan \theta} = \frac{2 \cos \theta}{\sin \theta} & \text{and} \quad BX &= \frac{2}{\sin \theta} \end{aligned}$$

$$\text{From the similar } \Delta's, \quad \frac{L}{BX} = \frac{AX+2}{AX} = 1 + \frac{2}{AX}$$



$$\therefore L = BX + \frac{2BX}{AX}$$

$$\therefore L = \frac{2}{\sin \theta} + \frac{2 \left(\frac{2}{\sin \theta} \right)}{\left(\frac{2 \cos \theta}{\sin \theta} \right)}$$

$$\therefore L = \frac{2}{\sin \theta} + 2 \left(\frac{2}{\sin \theta} \right) \left(\frac{\sin \theta}{2 \cos \theta} \right)$$

$$\therefore L = \frac{2}{\sin \theta} + \frac{2}{\cos \theta}, \text{ as required.}$$

$$\mathbf{b} \quad \text{Now } L = 2(\cos \theta)^{-1} + 2(\sin \theta)^{-1}$$

$$\begin{aligned} \therefore \frac{dL}{d\theta} &= -2(\cos \theta)^{-2}(-\sin \theta) + -2(\sin \theta)^{-2}(\cos \theta) \\ &= \frac{2 \sin \theta}{\cos^2 \theta} - \frac{2 \cos \theta}{\sin^2 \theta} \\ &= \frac{2 \sin^3 \theta - 2 \cos^3 \theta}{\sin^2 \theta \cos^2 \theta} \end{aligned}$$

$$\mathbf{c} \quad \text{Now } \frac{dL}{d\theta} = 0 \quad \text{when } 2 \sin^3 \theta - 2 \cos^3 \theta = 0$$

$$\therefore 2 \sin^3 \theta = 2 \cos^3 \theta$$

$$\therefore \tan^3 \theta = 1$$

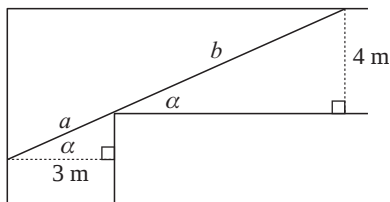
$$\therefore \tan \theta = 1$$

$$\therefore \theta = 45^\circ \quad \text{since } 0 < \theta < 90^\circ$$

Sign diagram of $\frac{dL}{d\theta}$ is: $\therefore$ the shortest ladder is required when $\theta = 45^\circ$

$$\text{and } L_{\min} = \frac{2}{\frac{1}{\sqrt{2}}} + \frac{2}{\frac{1}{\sqrt{2}}} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2} \text{ m}$$

5



$$\cos \alpha = \frac{3}{a} \quad \text{and} \quad \sin \alpha = \frac{4}{b}$$

$$\therefore a = \frac{3}{\cos \alpha} \quad \text{and} \quad b = \frac{4}{\sin \alpha}$$

$$\text{Now } L = a + b$$

$$\therefore L = \frac{3}{\cos \alpha} + \frac{4}{\sin \alpha}$$

$$\therefore L = 3(\cos \alpha)^{-1} + 4(\sin \alpha)^{-1}$$

$$\begin{aligned} \therefore \frac{dL}{d\alpha} &= -3(\cos \alpha)^{-2}(-\sin \alpha) + 4(-(\sin \alpha)^{-2}) \cos \alpha \\ &= \frac{3 \sin \alpha}{\cos^2 \alpha} - \frac{4 \cos \alpha}{\sin^2 \alpha} \\ &= \frac{3 \sin^3 \alpha - 4 \cos^3 \alpha}{\cos^2 \alpha \sin^2 \alpha} \end{aligned}$$

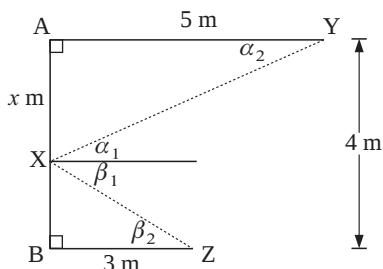
$$\begin{aligned}\therefore \frac{dL}{d\alpha} &= 0 \quad \text{when} \quad 3\sin^3\alpha - 4\cos^3\alpha = 0 \\ &\quad \text{i.e.,} \quad 3\sin^3\alpha = 4\cos^3\alpha \\ &\quad \tan^3\alpha = \frac{4}{3}\end{aligned}$$

$$\therefore \tan\alpha = \sqrt[3]{\frac{4}{3}} \quad \text{and so} \quad \alpha \doteq 47.74^\circ$$

Sign diagram of $\frac{dL}{d\alpha}$ is:

$$\therefore \text{AB is minimised when } \alpha = 47.74^\circ \quad \text{and} \quad L = \frac{3}{\cos\alpha} + \frac{4}{\sin\alpha} \doteq 9.866 \text{ m}$$

6



Now $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {alternate angles}

$$\therefore \theta = \alpha + \beta$$

Let $AX = x \text{ m}$

$$\therefore XB = (4 - x) \text{ m}$$

$$\therefore \tan\alpha = \frac{x}{5} \quad \therefore x = 5\tan\alpha$$

$$\text{and } \tan\beta = \frac{4-x}{3}$$

$$\text{Now } \tan\theta = \tan(\alpha + \beta)$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\text{i.e., } \tan\theta = \frac{\frac{x}{5} + \frac{4-x}{3}}{1 - \frac{x}{5}\left(\frac{4-x}{3}\right)}$$

$$\text{i.e., } \tan\theta = \frac{3x + 20 - 5x}{15 - x(4-x)} \quad \{\text{multiplying top and bottom by 15}\}$$

$$\therefore \tan\theta = \frac{20 - 2x}{x^2 - 4x + 15}$$

Differentiating both sides with respect to x

$$\frac{1}{\cos^2\theta} \frac{d\theta}{dx} = \frac{-2(x^2 - 4x + 15) - (20 - 2x)(2x - 4)}{(x^2 - 4x + 15)^2} \quad \{\text{Chain and Quotient rules}\}$$

$$\therefore \frac{1}{\cos^2\theta} \frac{d\theta}{dx} = \frac{[-2x^2 + 8x - 30 - 40x + 80 + 4x^2 - 8x]}{(x^2 - 4x + 15)^2}$$

$$\therefore \frac{1}{\cos^2\theta} \frac{d\theta}{dx} = \frac{[2x^2 - 40x + 50]}{(x^2 - 4x + 15)^2}$$

$$\therefore \frac{d\theta}{dx} = 2\cos^2\theta \frac{[x^2 - 20x + 25]}{(x^2 - 4x + 15)^2}$$

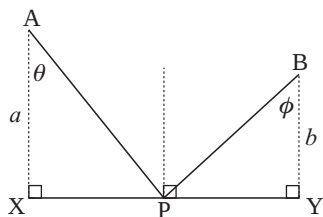
$$\text{Now by inspection, } \theta < 90^\circ, \quad \text{so } \frac{d\theta}{dx} = 0 \quad \text{when } x^2 - 20x + 25 = 0$$

$$\therefore x \doteq 1.3397 \quad \text{or} \quad x \doteq 18.660 \quad (\text{where } 18.660 \text{ is not physically possible})$$

$$\therefore x \doteq 1.340 \text{ m from A}$$

Sign diagram for $\frac{d\theta}{dx}$ is:

$$\therefore \theta \text{ is a maximum when } x \doteq 1.340 \text{ m from A}$$

7 a

$$\frac{a}{AP} = \cos \theta \quad \text{and} \quad \frac{b}{BP} = \cos \phi$$

$$AP = \frac{a}{\cos \theta} \quad \text{and} \quad BP = \frac{b}{\cos \phi}$$

$$\text{Now } L = AP + BP$$

$$\therefore L = \frac{a}{\cos \theta} + \frac{b}{\cos \phi}$$

$$\mathbf{b} \quad L = a(\cos \theta)^{-1} + b(\cos \phi)^{-1}$$

$$\frac{dL}{d\theta} = -a(\cos \theta)^{-2}(-\sin \theta) - b(\cos \phi)^{-2}(-\sin \phi) \frac{d\phi}{d\theta}$$

$$= \frac{a \sin \theta}{\cos^2 \theta} + \frac{b \sin \phi}{\cos^2 \phi} \frac{d\phi}{d\theta}, \quad \text{as required.}$$

$$\mathbf{c} \quad \text{Now } \frac{XP}{a} = \tan \theta \quad \text{and} \quad \frac{YP}{b} = \tan \phi$$

$$\therefore XP = a \tan \theta \quad \text{and} \quad YP = b \tan \phi$$

$$\therefore XY = a \tan \theta + b \tan \phi$$

But XY is a fixed distance, so $a \tan \theta + b \tan \phi$ is a constant,

$$\text{i.e., } a \tan \theta + b \tan \phi = c$$

$$\therefore \frac{a}{\cos^2 \theta} + \frac{b}{\cos^2 \phi} \frac{d\phi}{d\theta} = 0 \quad \{\text{differentiating with respect to } \theta\}$$

$$\therefore \frac{b}{\cos^2 \phi} \frac{d\phi}{d\theta} = -\frac{a}{\cos^2 \theta}$$

$$\therefore \frac{d\phi}{d\theta} = -\frac{a \cos^2 \phi}{b \cos^2 \theta}, \quad \text{as required.}$$

$$\mathbf{d} \quad \frac{dL}{d\theta} = \frac{a \sin \theta}{\cos^2 \theta} + \frac{b \sin \phi}{\cos^2 \phi} \left(\frac{-a \cos^2 \phi}{b \cos^2 \theta} \right) \quad \{\text{using } \mathbf{b} \text{ and } \mathbf{c}\}$$

$$= \frac{a \sin \theta}{\cos^2 \theta} - \frac{a \sin \phi}{\cos^2 \theta}$$

$$= \frac{a(\sin \theta - \sin \phi)}{\cos^2 \theta}$$

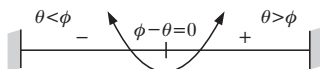
$$\therefore \frac{dL}{d\theta} = 0 \quad \text{when} \quad \sin \theta - \sin \phi = 0$$

$$\text{i.e., when } \sin \phi = \sin \theta$$

$$\mathbf{e} \quad \text{Since } \frac{dL}{d\theta} = 0 \quad \text{when} \quad \sin \phi = \sin \theta,$$

$L = AP + PB$ is either a maximum or minimum when $\phi = \theta$

Sign diagram of $\frac{dL}{d\theta}$ is:



$$\therefore AP + PB \quad \text{is a minimum} \quad \text{when } \theta - \phi = 0$$

$$\text{i.e., when } \theta = \phi$$

$\therefore$ it will be cheapest for the pump house to be located at the point such that $\theta = \phi$.

REVIEW SET 24

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & y = 10x - \sin(10x) \\ \therefore \frac{dy}{dx} &= 10 - 10 \cos(10x) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & y = \sin(3x) \cos(2x) \\ \therefore \frac{dy}{dx} &= 3 \cos(3x) \cos(2x) + \sin(3x) \times -2 \sin(2x) \\ &= 3 \cos(3x) \cos(2x) - 2 \sin(3x) \sin(2x) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \frac{d}{dx} (e^{-2x} \sin x) \\ &= -2e^{-2x} \sin x + e^{-2x} \cos x \\ &= e^{-2x} (\cos x - 2 \sin x) \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \frac{d}{dx} \left(\ln \left(\frac{1}{\cos x} \right) \right) \\ &= \frac{d}{dx} (-\ln(\cos x)) \\ &= -\frac{1}{\cos x} \times (-\sin x) \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & y = \sin(5x) \ln(2x) \\ \therefore \frac{dy}{dx} &= 5 \cos(5x) \ln(2x) + \sin(5x) \times \frac{2}{2x} \\ &= 5 \cos(5x) \ln(2x) + \frac{\sin(5x)}{x} \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad & y = x \tan x \\ \therefore \frac{dy}{dx} &= 1 \times \tan x + x \times \frac{1}{\cos^2 x} \\ \text{Now } \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} = \quad \text{and } \tan \frac{\pi}{4} = 1 \end{aligned}$$

$$\therefore \text{ at } x = \frac{\pi}{4}, \quad y = \frac{\pi}{4} \quad \text{and} \quad \frac{dy}{dx} = 1 + \frac{\pi}{2}$$

$$\begin{aligned} \therefore \text{ the equation of the tangent is } & \frac{y - \frac{\pi}{4}}{x - \frac{\pi}{4}} = 1 + \frac{\pi}{2} \\ \therefore y - \frac{\pi}{4} &= (1 + \frac{\pi}{2})(x - \frac{\pi}{4}) \\ &= x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8} \\ \therefore y &= (1 + \frac{\pi}{2})x - \frac{\pi^2}{8} \\ \therefore 2y &= (2 + \pi)x - \frac{\pi^2}{4} \\ \therefore (2 + \pi)x - 2y &= \frac{\pi^2}{4} \quad \text{as required} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & f(x) = 3 \sin x - 4 \cos(2x) \\ \therefore f'(x) &= 3 \cos x + 8 \sin(2x) \quad \text{and} \quad f''(x) = -3 \sin x + 16 \cos(2x) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & f(x) = x^{\frac{1}{2}} \cos(4x) \\ \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \cos(4x) + x^{\frac{1}{2}}(-4 \sin(4x)) \\ &= \frac{1}{2}x^{-\frac{1}{2}} \cos(4x) - 4x^{\frac{1}{2}} \sin(4x) \\ \text{and } f''(x) &= -\frac{1}{4}x^{-\frac{3}{2}} \cos(4x) + \frac{1}{2}x^{-\frac{1}{2}}(-4 \sin(4x)) - \left[2x^{-\frac{1}{2}} \sin(4x) + 4x^{\frac{1}{2}} \times 4 \cos(4x) \right] \\ &= -\frac{1}{4}x^{-\frac{3}{2}} \cos(4x) - 4x^{-\frac{1}{2}} \sin(4x) - 16x^{\frac{1}{2}} \cos(4x) \end{aligned}$$

$$\mathbf{4} \quad x(t) = 3 + \sin(2t) \text{ cm, } t \geq 0 \text{ sec}$$

$$\begin{aligned} \mathbf{a} \quad & x'(t) = 0 + 2 \cos(2t) & \therefore x(0) = 3 \text{ cm} \\ & x''(t) = -4 \sin(2t) & x'(0) = 2 \text{ cm/sec} \\ & & x''(0) = 0 \text{ cm/sec}^2 \end{aligned}$$

$\therefore$ initially the particle is 3 cm right of O, moving right at a speed of 2 cm/sec.

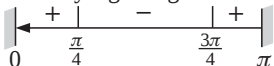
b $x'(t) = 0$ when $2\cos(2t) = 0$

$$\therefore \cos(2t) = 0$$

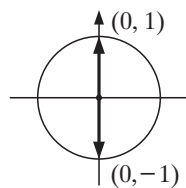
$$\therefore 2t = \frac{\pi}{2} + k\pi$$

$$\therefore t = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \quad \{\text{as } 0 \leq t \leq \pi\}$$

velocity sign diagram



$\therefore$ reverses direction at $t = \frac{\pi}{4}, \frac{3\pi}{4}$



c $x(0) = 3, \quad x\left(\frac{\pi}{4}\right) = 3 + \sin\left(\frac{\pi}{2}\right) = 4$

$$x\left(\frac{3\pi}{4}\right) = 3 + \sin\left(\frac{3\pi}{2}\right) = 3 - 1 = 2 \quad x(\pi) = 3 + \sin(2\pi) = 3$$

$\therefore$ total distance travelled $= 1 + 2 + 1 = 4$ cm.



5 a $f(x) = \sqrt{\cos x}, \quad 0 \leq x \leq 2\pi$

$\therefore f(x)$ is meaningful when $\cos x \geq 0$,

i.e., when $0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$

b $f(x) = (\cos x)^{\frac{1}{2}}$

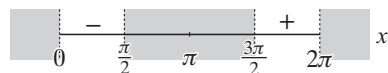
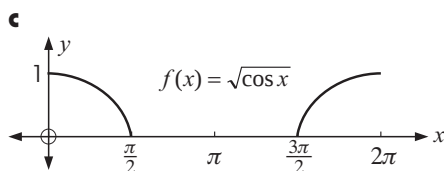
$$\therefore f'(x) = \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$$

$$= \frac{-\sin x}{2\sqrt{\cos x}}$$

$$\therefore f'(x) = 0 \text{ when } -\sin x = 0$$

i.e., $x = 0, \pi, 2\pi$ etc.

Sign diagram for $f'(x)$ is:



$f(x)$ is increasing for $\frac{3\pi}{2} < x \leq 2\pi$ $f(x)$ is decreasing for $0 \leq x < \frac{\pi}{2}$

6 a $y = x \ln(\sin x)$

$$\therefore \frac{dy}{dx} = 1 \ln(\sin x) + x \frac{\cos x}{\sin x}$$

$$= \ln(\sin x) + \frac{x \cos x}{\sin x}$$

b $y = (e^{\tan x})^{\frac{1}{2}} = e^{\frac{1}{2} \tan x}$

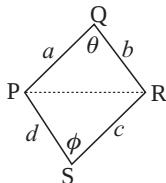
$$\therefore \frac{dy}{dx} = e^{\frac{1}{2} \tan x} \times \frac{1}{2} \left(\frac{1}{\cos^2 x} \right)$$

$$= \frac{e^{\frac{1}{2} \tan x}}{2 \cos^2 x}$$

c $y = \frac{\cos(3x)}{x^{\frac{1}{2}}} \quad \therefore \frac{dy}{dx} = \frac{-3 \sin(3x)x^{\frac{1}{2}} - \cos(3x)\frac{1}{2}x^{-\frac{1}{2}}}{x^1}$

$$= \frac{-3\sqrt{x} \sin(3x) - \frac{\cos(3x)}{2\sqrt{x}}}{x}$$

7 a



Using the cosine rule,

$$\text{in } \triangle PQR, \quad PR^2 = a^2 + b^2 - 2ab \cos \theta$$

$$\text{in } \triangle PSR, \quad PR^2 = c^2 + d^2 - 2cd \cos \phi$$

$$\therefore a^2 + b^2 - 2ab \cos \theta = c^2 + d^2 - 2cd \cos \phi$$

Now a, b, c and d are constants, so differentiating with respect to ϕ ,

$$\therefore 2ab \sin \theta \frac{d\theta}{d\phi} = 2cd \sin \phi$$

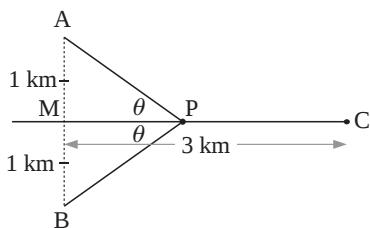
$$\therefore \frac{d\theta}{d\phi} = \frac{2cd \sin \phi}{2ab \sin \theta} = \frac{cd \sin \phi}{ab \sin \theta} \text{ as required}$$

- b** Area of quadrilateral, $A = \text{area of } \triangle PQR + \text{area of } \triangle PSR$

$$\begin{aligned}
 &= \frac{1}{2}ab \sin \theta + \frac{1}{2}cd \sin \phi \\
 \therefore \frac{dA}{d\phi} &= \frac{1}{2}ab \cos \theta \frac{d\theta}{d\phi} + \frac{1}{2}cd \cos \phi \\
 &= \frac{1}{2}ab \cos \theta \left(\frac{cd \sin \phi}{ab \sin \theta} \right) + \frac{1}{2}cd \cos \phi \quad \{\text{using a}\} \\
 &= \frac{1}{2}cd \left[\frac{\cos \theta \sin \phi}{\sin \theta} + \cos \phi \right] \\
 &= \frac{cd}{2 \sin \theta} (\sin \phi \cos \theta + \cos \phi \sin \theta) \\
 &= \frac{cd}{2 \sin \theta} \sin(\phi + \theta) \\
 \therefore \frac{dA}{d\phi} &= 0 \quad \text{when} \quad \sin(\phi + \theta) = 0 \quad \text{i.e., when} \quad \phi + \theta = 2\pi
 \end{aligned}$$

$\therefore$ the area of PQRS is a maximum when the opposite angles are supplementary,
i.e., when PQRS is a cyclic quadrilateral.

8



- a** The length of cable required
= $(PA + PB + PC)$ km

- i** If P is at M, then $PA = PB = 1$ km
and $PC = 3$ km
 $\therefore$ 5 km of cable is required
- ii** If P is at C, then PA
= $PB = \sqrt{1^2 + 3^2} = \sqrt{10}$ km
and $PC = 0$ km
 $\therefore$ $2\sqrt{10}$ km of cable is required

b

$$\text{Now } \sin \theta = \frac{1}{AP} = \frac{1}{BP} \quad \text{and} \quad \tan \theta = \frac{1}{MP}$$

$$\therefore AP = BP = \frac{1}{\sin \theta} \quad \text{and} \quad MP = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

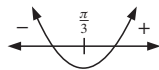
$$\therefore AP + BP + CP = \frac{2}{\sin \theta} + (CM - MP)$$

$$L = \frac{2}{\sin \theta} + 3 - \frac{\cos \theta}{\sin \theta} \quad \text{as required}$$

- c** Since $L = 2(\sin \theta)^{-1} + 3 - (\tan \theta)^{-1}$,

$$\begin{aligned}
 \frac{dL}{d\theta} &= -2 \cos \theta (\sin \theta)^{-2} + (\tan \theta)^{-2} \times \frac{1}{\cos^2 \theta} \\
 &= \frac{-2 \cos \theta}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} \frac{\cos^2 \theta}{\sin^2 \theta} \\
 &= \frac{1 - 2 \cos \theta}{\sin^2 \theta} \quad \text{as required}
 \end{aligned}$$

$$\therefore \frac{dL}{d\theta} = 0 \quad \text{if} \quad \cos \theta = \frac{1}{2}, \quad \text{i.e., } \theta = \frac{\pi}{3} \quad \text{and sign diagram of } \frac{dL}{d\theta} \text{ is:}$$

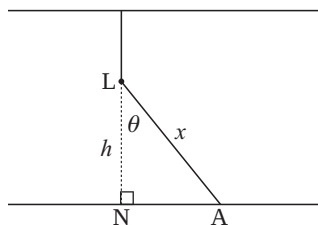


$\therefore$ the minimum length of cable is required when $\theta = \frac{\pi}{3}$

$$\text{Then } \sin \theta = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan \theta = \sqrt{3},$$

$$\text{so } L_{\min} = \frac{4}{\sqrt{3}} + 3 - \frac{1}{\sqrt{3}} = (3 + \sqrt{3}) \text{ km} \quad \text{as required}$$

9



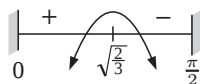
a

$$\begin{aligned}\sin \theta &= \frac{NA}{x} = \frac{1}{x} \\ \therefore \frac{1}{x^2} &= \sin^2 \theta \\ \therefore \text{ at A, } I &= \frac{\sqrt{8} \cos \theta}{x^2} \\ &= \sqrt{8} \cos \theta \sin^2 \theta\end{aligned}$$

$$\begin{aligned}\text{b } \frac{dI}{d\theta} &= \sqrt{8}(-\sin \theta) \sin^2 \theta + \sqrt{8} \cos \theta (2 \sin \theta \cos \theta) \\ &= \sqrt{8} \sin \theta [2 \cos^2 \theta - \sin^2 \theta] \\ &= \sqrt{8} \sin \theta [2(1 - \sin^2 \theta) - \sin^2 \theta] \\ &= \sqrt{8} \sin \theta [2 - 3 \sin^2 \theta]\end{aligned}$$

$$\frac{dI}{d\theta} = 0 \quad \text{when } \sin \theta = \sqrt{\frac{2}{3}}, \quad 0 < \theta < \frac{\pi}{2}$$

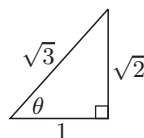
and the sign diagram of $\frac{dI}{d\theta}$ is:



$\therefore$ the maximum illumination at A is obtained when $\sin \theta = \sqrt{\frac{2}{3}}$.

$$\therefore x = \frac{1}{\sin \theta} = \sqrt{\frac{3}{2}}$$

$$\therefore h = \sqrt{x^2 - NA^2} = \sqrt{\frac{3}{2} - 1} = \frac{1}{\sqrt{2}} \quad \text{i.e., the bulb is } \frac{1}{\sqrt{2}} \text{ m above the floor.}$$



$$\text{c } \cos \theta = \frac{h}{x}$$

$$\therefore I = \frac{\sqrt{8} \cos \theta}{x^2} = \frac{\sqrt{8} \times \frac{h}{x}}{x^2} = \frac{\sqrt{8}h}{x^3}$$

$$\text{Now } x^2 = h^2 + 1$$

$$x^3 = (h^2 + 1)^{\frac{3}{2}}$$

$$\therefore I = \sqrt{8}h(h^2 + 1)^{-\frac{3}{2}}$$

$$\begin{aligned}\therefore \frac{dI}{dt} &= \sqrt{8} \frac{dh}{dt} (h^2 + 1)^{-\frac{3}{2}} - \frac{3}{2} \sqrt{8}h (h^2 + 1)^{-\frac{5}{2}} \left(2h \frac{dh}{dt} \right) \\ &= \sqrt{8} \frac{dh}{dt} \left[(h^2 + 1)^{-\frac{3}{2}} - 3h^2 (h^2 + 1)^{-\frac{5}{2}} \right]\end{aligned}$$

$$\text{Particular case: } \frac{dh}{dt} = -0.1 \text{ ms}^{-1}, \quad \text{and when } h = 1 \text{ m}$$

$$\frac{dI}{dt} = \sqrt{8}(-0.1) \left(2^{-\frac{3}{2}} - 3 \times 2^{-\frac{5}{2}} \right) = 0.05$$

$\therefore$ the illumination is increasing at 0.05 units/second.

10 a

$$s(t) = 30 + \cos(\pi t) \text{ cm, } t \geq 0$$

$$\therefore v(t) = -\pi \sin(\pi t)$$

$$\text{So, } v(0) = 0 \text{ cms}^{-1}$$

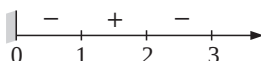
$$v\left(\frac{1}{2}\right) = -\pi \text{ cms}^{-1}$$

$$v(1) = 0 \text{ cms}^{-1}$$

$$v\left(1\frac{1}{2}\right) = \pi \text{ cms}^{-1}$$

$$v(2) = 0 \text{ cms}^{-1}$$

sign diagram of $v(t)$:



b

The cork is falling when $v(t) \leq 0$

i.e., $0 \leq t \leq 1$, $2 \leq t \leq 3$, etc.

i.e., $2n \leq t \leq 2n + 1$ $n = 0, 1, 2, 3, \dots$

Chapter 25

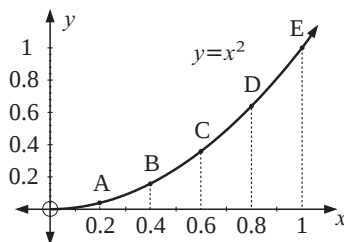
AREAS WITHIN CURVED BOUNDARIES

EXERCISE 25A.1

- 1 a** Divide the region into strips of width $\frac{1}{5}$ unit.

$$O(0, 0) \quad A(\frac{1}{5}, \frac{1}{25}) \quad B(\frac{2}{5}, \frac{4}{25})$$

$$C(\frac{3}{5}, \frac{9}{25}) \quad D(\frac{4}{5}, \frac{16}{25}) \quad E(1, 1)$$



$$\text{Now } A < A_1 + A_2 + A_3 + A_4 + A_5$$

$$\text{i.e., } A < \frac{(0 + \frac{1}{25})}{2} \cdot \frac{1}{5} + \frac{(\frac{1}{25} + \frac{4}{25})}{2} \cdot \frac{1}{5} + \dots + \frac{(\frac{16}{25} + \frac{25}{25})}{2} \cdot \frac{1}{5}$$

$$\therefore A < \frac{1}{10} \left(\frac{85}{25} \right)$$

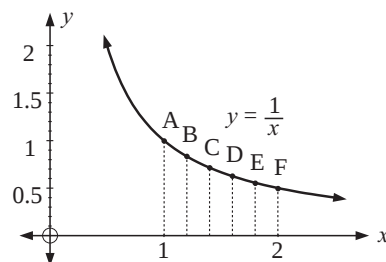
$$\therefore A < 0.34$$

and $A \div 0.34$ is an estimate of the area

- b** Divide the region into strips of width $\frac{1}{5}$ unit.

$$A(1, 1) \quad B(\frac{6}{5}, \frac{5}{6}) \quad C(\frac{7}{5}, \frac{5}{7})$$

$$D(\frac{8}{5}, \frac{5}{8}) \quad E(\frac{9}{5}, \frac{5}{9}) \quad F(2, \frac{1}{2})$$



$$\text{Now } A > A_1 + A_2 + A_3 + A_4 + A_5$$

$$\text{i.e., } A > \frac{(1 + \frac{5}{6})}{2} \cdot \frac{1}{5} + \frac{(\frac{5}{6} + \frac{5}{7})}{2} \cdot \frac{1}{5} + \dots + \frac{(\frac{5}{9} + \frac{1}{2})}{2} \cdot \frac{1}{5}$$

$$\therefore A > \frac{1}{10} \left(\frac{11}{6} + \frac{65}{42} + \frac{75}{56} + \frac{85}{72} + \frac{19}{18} \right)$$

$$\therefore A > 0.6956$$

and $A \div 0.70$ is an estimate of the area

- 2 a** Using 10 vertical strips in question 1 a

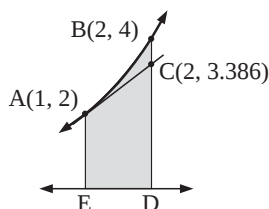
$$\begin{aligned} A &< \frac{(\frac{0}{100} + \frac{1}{100})}{2} \cdot \frac{1}{10} + \frac{(\frac{1}{100} + \frac{4}{100})}{2} \cdot \frac{1}{10} + \dots + \frac{(\frac{81}{100} + \frac{100}{100})}{2} \cdot \frac{1}{10} \\ &< \frac{1}{20} \left(\frac{1}{100} + \frac{5}{100} + \frac{13}{100} + \frac{25}{100} + \frac{41}{100} + \frac{61}{100} + \frac{85}{100} + \frac{113}{100} + \frac{145}{100} + \frac{181}{100} \right) \\ &< \frac{1}{20} \left(\frac{670}{100} \right) \\ &< 0.335 \end{aligned}$$

and $A \div 0.335$ is an estimate of the area

- b** Using 10 vertical strips in question 1 b

$$A > \frac{(\frac{10}{10} + \frac{10}{11})}{2} \cdot \frac{1}{10} + \frac{(\frac{10}{11} + \frac{10}{12})}{2} \cdot \frac{1}{10} + \dots + \frac{(\frac{10}{19} + \frac{10}{20})}{2} \cdot \frac{1}{10}$$

$$\begin{aligned}
 \therefore A &> \frac{10}{20} \left(\left(\frac{1}{10} + \frac{1}{11} \right) + \left(\frac{1}{11} + \frac{1}{12} \right) + \dots + \left(\frac{1}{19} + \frac{1}{20} \right) \right) \\
 &> \frac{1}{2} \left(\frac{21}{110} + \frac{23}{132} + \frac{25}{156} + \frac{27}{182} + \frac{29}{210} + \frac{31}{240} + \frac{33}{272} + \frac{35}{306} + \frac{37}{342} + \frac{39}{380} \right) \\
 &> 0.693771 \\
 \text{and } A &\doteq 0.6938 \text{ is an estimate of the area}
 \end{aligned}$$

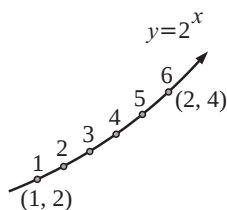
EXERCISE 25A.2**1 a**

The lower bound is
the area ACDE

$$\begin{aligned}
 &= \frac{2 + 3.386}{2} \\
 &= 2.693
 \end{aligned}$$

The upper bound is
the area ABDE

$$\begin{aligned}
 &= \frac{2 + 4}{2} \\
 &= 3
 \end{aligned}$$

b Without the tangent and using 5 subdivisions on $[1, 2]$ $n = 5$ 

Point	Coordinates
1	(1, 2)
2	(1.2, 2 ^{1.2})
3	(1.4, 2 ^{1.4})
4	(1.6, 2 ^{1.6})
5	(1.8, 2 ^{1.8})
6	(2, 4)

$$\begin{aligned}
 A_L &= \frac{1}{5}(2^1) + \frac{1}{5}(2^{1.2}) + \frac{1}{5}(2^{1.4}) + \frac{1}{5}(2^{1.6}) + \frac{1}{5}(2^{1.8}) \\
 &= \frac{1}{5}(2 + 2^{1.2} + 2^{1.4} + 2^{1.6} + 2^{1.8}) \\
 &\doteq 2.690
 \end{aligned}$$

$$\begin{aligned}
 A_U &= \frac{1}{5}(2^{1.2} + 2^{1.4} + 2^{1.6} + 2^{1.8} + 2^2) \\
 &\doteq 3.090
 \end{aligned}$$

c Using the provided software:

n	A_L	A_U
10	2.786 55	2.986 55
50	2.865 44	2.905 44
100	2.875 40	2.895 40
500	2.883 39	2.887 39
5000	2.885 19	2.885 59

The upper and lower sums converge to 2.885 (approx.)

2 a $y = 1 + e^x$ $[0, 1]$ choosing 5 subdivisions on this $[0, 1]$.

Point	Coordinates
1	(0, 2)
2	(0.2, 1 + e ^{0.2})
3	(0.4, 1 + e ^{0.4})
4	(0.6, 1 + e ^{0.6})
5	(0.8, 1 + e ^{0.8})
6	(1, 1 + e ¹)

$$\begin{aligned}
 A_L &= 0.2(1 + e^0) + 0.2(1 + e^{0.2}) + \dots + 0.2(1 + e^{0.8}) \\
 &= 0.2(5 + (e^0 + e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8})) \\
 &\doteq 2.552
 \end{aligned}$$

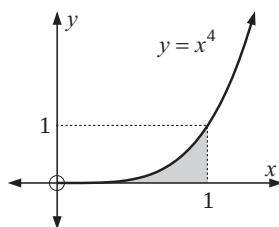
$$\begin{aligned}
 \therefore A_U &= 0.2(1 + e^{0.2}) + 0.2(1 + e^{0.4}) + \dots + 0.2(1 + e^1) \\
 &= 0.2(5 + (e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8} + e^1)) \\
 &\doteq 2.896
 \end{aligned}$$

b

n	A_L	A_U
100	2.709 70	2.726 89
1000	2.717 42	2.719 14
10 000	2.718 20	2.718 37
100 000	2.718 27	2.718 29

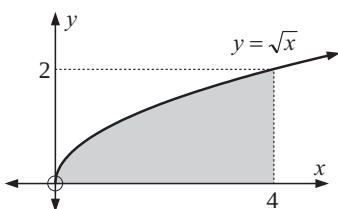
c It appears that as $t \rightarrow \infty$ both upper and lower sums converge to e , where $e \doteq 2.718\,28$

EXERCISE 25B.1

1 a

b

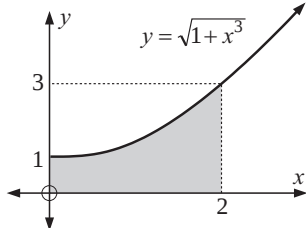
n	A_L	A_U
10	0.153 33	0.253 33
100	0.195 03	0.205 03
1000	0.199 50	0.200 50
10 000	0.199 95	0.200 05

c Guess: Area = $\int_0^1 x^4 dx = 0.2$

2 a

b

n	A_L	A_U
100	5.291 70	5.371 70
10 000	5.332 93	5.333 73

c Guess: Area = $\int_0^4 \sqrt{x} dx = 5.\bar{3}$
 $= 5\frac{1}{3}$

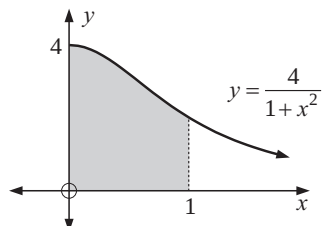
3 a

b

n	A_L	A_U
100	3.221 38	3.261 38
10 000	3.241 11	3.241 51

c A good estimate would be the average

$$\text{i.e., } \frac{3.241\,11 + 3.241\,51}{2}$$

$$\therefore \int_0^2 \sqrt{1+x^3} dx \doteq 3.2413$$

4 a

b

n	A_L	A_U
100	3.131 58	3.151 58
1000	3.140 59	3.142 59
10 000	3.141 49	3.141 69

c Area = $\int_0^1 \frac{4}{1+x^2} dx = \frac{3.141\,69 + 3.141\,49}{2}$
 $\doteq 3.141\,59$
 $\doteq 3.1416$
 (We suspect this is π)

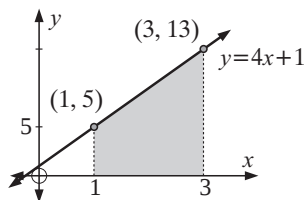
- 5 For the curve $y = \sqrt[3]{x^2 + 2}$, $y > 0$ for $0 \leq x \leq 5$.

Using the software provided

n	A_L	A_U
100	10.159 79	10.246 80
1000	10.198 87	10.207 57
10 000	10.202 78	10.203 65

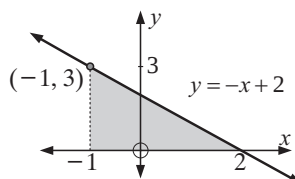
A good estimate is $\int_0^5 \sqrt[3]{x^2 + 2} dx = \frac{10.202\,78 + 10.203\,65}{2} \div 10.203\,22$

6 a



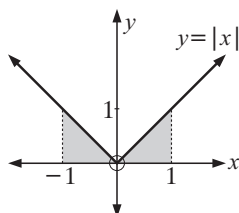
$$\begin{aligned} \int_1^3 (1 + 4x) dx \\ &= \text{area of the shaded trapezium} \\ &= \left(\frac{5 + 13}{2} \right) \times 2 \\ &= 18 \end{aligned}$$

b



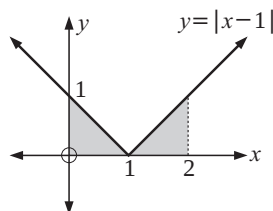
$$\begin{aligned} \int_{-1}^2 (2 - x) dx \\ &= \text{area of shaded triangle} \\ &= \frac{1}{2}(3 \times 3) \\ &= 4.5 \end{aligned}$$

c



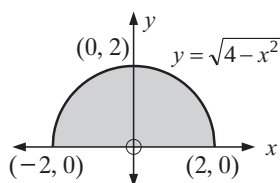
$$\begin{aligned} \int_{-1}^1 |x| dx &= \text{shaded area shown} \\ \text{Due to symmetry we need to only} \\ &\text{consider the RHS.} \\ \therefore \int_0^1 |x| dx &= \text{area of a shaded } \Delta \\ &= \frac{1}{2}(1 \times 1) \\ &= 0.5 \\ \therefore \int_{-1}^1 |x| dx &= 2 \times 0.5 = 1 \end{aligned}$$

d



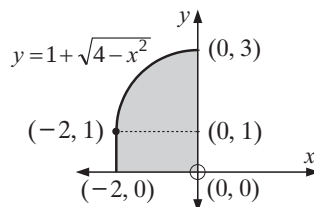
$$\begin{aligned} y &= \sqrt{4 - x^2} \\ \therefore \int_0^2 |x - 1| dx &= 2 \int_1^2 |x - 1| dx \\ &\quad \{\text{by symmetry}\} \\ &= 2 \times \left[\frac{1}{2}(1 \times 1) \right] \\ &= 1 \end{aligned}$$

e



$$\begin{aligned} \int_{-2}^2 \sqrt{4 - x^2} dx \\ &= \text{area of semi-circle radius 2} \\ &= \frac{1}{2}(\pi \times 2^2) \\ &= 2\pi \end{aligned}$$

f



$$\begin{aligned} \int_{-2}^0 (1 + \sqrt{4 - x^2}) dx \\ &= \text{area of rectangle} \\ &\quad + \text{area of quarter of circle with radius 2} \\ &= 2 \times 1 + \frac{1}{4}(\pi \times 2^2) \\ &= 2 + \pi \end{aligned}$$

EXERCISE 25B.2

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int_1^4 \sqrt{x} \, dx &= 4.667 \\ \int_1^4 (-\sqrt{x}) \, dx &= -4.667 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_0^1 x^7 \, dx &= 0.125 = \frac{1}{8} \\ \int_0^1 (-x^7) \, dx &= -0.125 = -\frac{1}{8} \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \int_0^1 x^2 \, dx = \frac{1}{3} \quad \mathbf{b} \quad \int_1^2 x^2 \, dx = \frac{7}{3} \quad \mathbf{c} \quad \int_0^2 x^2 \, dx = \frac{8}{3} \quad \mathbf{d} \quad \int_0^1 3x^2 \, dx = 1$$

$$\mathbf{3} \quad \mathbf{a} \quad \int_0^2 (x^3 - 4x) \, dx = -4 \quad \mathbf{b} \quad \int_2^3 (x^3 - 4x) \, dx = 6\frac{1}{4} \quad \mathbf{c} \quad \int_0^3 (x^3 - 4x) \, dx = 2\frac{1}{4}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \int_a^b -f(x) \, dx &= -\int_a^b f(x) \, dx & \mathbf{b} \quad \int_a^b f(x) \, dx + \int_b^c f(x) \, dx &= \int_a^c f(x) \, dx \\ & & \text{and} \quad \int_a^b kf(x) \, dx &= k \int_a^b f(x) \, dx \end{aligned}$$

$$\mathbf{5} \quad \mathbf{a} \quad \int_0^1 x^2 \, dx = \frac{1}{3}$$

$$\mathbf{b} \quad \int_0^1 \sqrt{x} \, dx = \frac{2}{3}$$

$$\mathbf{c} \quad \int_0^1 (x^2 + \sqrt{x}) \, dx = 1$$

$$\text{this indicates that} \quad \int_0^1 (x^2 + \sqrt{x}) \, dx = \int_0^1 x^2 \, dx + \int_0^1 \sqrt{x} \, dx$$

$$\text{and in general} \quad \int_a^b f(x) \, dx + \int_a^b g(x) \, dx = \int_a^b (f(x) + g(x)) \, dx$$

EXERCISE 25B.3

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int_0^3 f(x) \, dx &= 2 + 3 + 1.5 \\ &= 6.5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_3^7 f(x) \, dx &= -\left(\frac{3}{2} + 3 + \frac{5}{2} + 2\right) \\ &= -9 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int_2^4 f(x) \, dx &= 1.5 - 1.5 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \int_0^7 f(x) \, dx &= 6.5 - 9 \\ &= -2.5 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \int_0^4 f(x) \, dx &= \frac{1}{2}\pi(2)^2 \\ &= 2\pi \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_4^6 f(x) \, dx &= -(2 \times 2) \\ &= -4 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int_6^8 f(x) \, dx &= \frac{1}{2}\pi(1)^2 \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \int_0^8 f(x) \, dx &= 2\pi + (-4) + \frac{\pi}{2} \\ &= \frac{5\pi}{2} - 4 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \int_2^4 f(x) \, dx + \int_4^7 f(x) \, dx \\ &= \int_2^7 f(x) \, dx \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_1^3 g(x) \, dx + \int_3^8 g(x) \, dx + \int_8^9 g(x) \, dx \\ &= \int_1^9 g(x) \, dx \end{aligned}$$

$$\mathbf{4} \quad \mathbf{a} \quad \int_1^3 f(x) \, dx + \int_3^6 f(x) \, dx = \int_1^6 f(x) \, dx$$

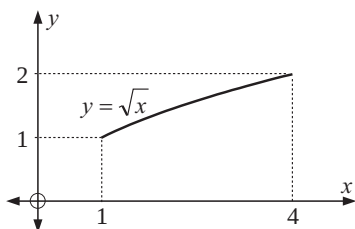
$$\therefore \int_3^6 f(x) \, dx = \int_1^6 f(x) \, dx - \int_1^3 f(x) \, dx = (-3) - (2) = -5$$

$$\mathbf{b} \quad \int_0^2 f(x) \, dx + \int_2^4 f(x) \, dx + \int_4^6 f(x) \, dx = \int_0^6 f(x) \, dx$$

$$\begin{aligned} \therefore \int_2^4 f(x) \, dx &= \int_0^6 f(x) \, dx - \int_4^6 f(x) \, dx - \int_0^2 f(x) \, dx \\ &= (7) - (-2) - (5) \\ &= 7 + 2 - 5 \\ &= 4 \end{aligned}$$

REVIEW SET 25

1 a



x	$\sqrt{x}$
1	1
1.5	1.2247
2	1.4142
2.5	1.5811
3	1.7321
3.5	1.8708
4	2

$$\begin{aligned} \text{b } A_L &= 0.5(1 + 1.2247 + 1.4142 + 1.5811 + 1.7321 + 1.8708) \\ &= 0.5(8.8229) \\ &\doteq 4.411 \end{aligned}$$

2 a A is the *upper half* of a circle centre (2, 0) and radius 2.

$$\therefore (x-2)^2 + (y-0)^2 = 2^2$$

$$(x-2)^2 + y^2 = 4$$

$$y^2 = 4 - (x-2)^2$$

$$y^2 = 4 - x^2 + 4x - 4$$

$$y^2 = 4x - x^2$$

$$\therefore y = \pm\sqrt{4x - x^2}$$

$$\text{So, } y = \sqrt{4x - x^2}$$

$$\text{or } y = -\sqrt{4x - x^2}$$

$$\therefore \text{required equation is } y_A = \sqrt{4x - x^2}$$

b Now B is the *lower half* of a circle centre (5, 0) and radius 1.

$$\therefore (x-5)^2 + (y-0)^2 = 1^2$$

$$(x-5)^2 + y^2 = 1$$

$$y^2 = 1 - (x-5)^2$$

$$y^2 = 1 - x^2 + 10x - 25$$

$$\therefore y = \pm\sqrt{10x - x^2 - 24}$$

$$\therefore y_B = -\sqrt{10x - x^2 - 24}$$

$$\begin{aligned} \text{c } \int_0^4 y_A dx &= \frac{1}{2}\pi r^2 \quad \text{where } r = 2 \\ &= \frac{1}{2}\pi(2)^2 \\ &= 2\pi \end{aligned}$$

$$\begin{aligned} \int_4^6 y_B dx &= -\frac{1}{2}\pi r^2 \quad \text{where } r = 1 \\ &= -\frac{1}{2}\pi(1)^2 \\ &= -\frac{\pi}{2} \end{aligned}$$

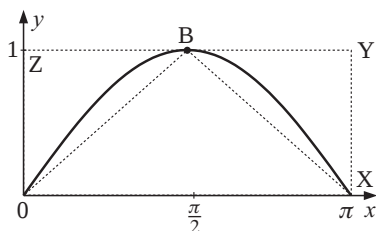
$$\begin{aligned} \text{d } \int_0^6 f(x) dx &= \int_0^4 y_A dx + \int_4^6 y_B dx \\ &= 2\pi + \left(-\frac{\pi}{2}\right) \\ &= \frac{3\pi}{2} \end{aligned}$$

3 a From the graph,

area $\triangle OBX$ < area under the curve < area OXYZ

$$\therefore \frac{1}{2}\pi(1) < \int_0^\pi \sin x dx < \pi(1)$$

$$\text{i.e., } \frac{\pi}{2} < \int_0^\pi \sin x dx < \pi$$



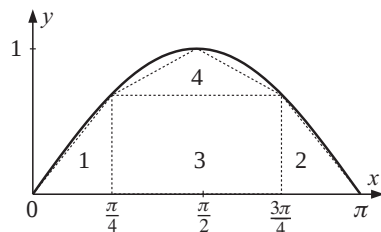
- b** If we partition the diagram into triangles as shown:

$$\text{Area 1} = \frac{1}{2} \left(\frac{\pi}{4} \right) \left(\frac{\sqrt{2}}{2} \right) = \frac{\pi\sqrt{2}}{16}$$

$$\text{Area 2} = \frac{\pi\sqrt{2}}{16}$$

$$\text{Area 3} = \frac{\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}$$

$$\text{Area 4} = \frac{1}{2} \left(\frac{\pi}{2} \right) \left(1 - \frac{\sqrt{2}}{2} \right) = \frac{\pi}{4} \left(1 - \frac{\sqrt{2}}{2} \right)$$



This total area is just less than the area under the arch

$$\begin{aligned} \text{and total area} &= \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{4} + \frac{\pi}{4} \left(1 - \frac{\sqrt{2}}{2} \right) \\ &= \frac{\pi\sqrt{2} + \pi\sqrt{2} + 4\pi\sqrt{2} + 4\pi - 2\pi\sqrt{2}}{16} \\ &= \frac{4\pi + 4\pi\sqrt{2}}{16} \\ &= \frac{\pi + \pi\sqrt{2}}{4} \\ &= \frac{\pi}{4} (1 + \sqrt{2}) \text{ units}^2 \end{aligned}$$

4 a $\int_0^4 f(x)dx = \text{area of triangle} + \text{area of } \frac{1}{4} \text{ circle}$

$$\begin{aligned} &= \frac{1}{2}(2 \times 2) + \frac{1}{4}\pi(2)^2 \\ &= 2 + \pi \end{aligned}$$

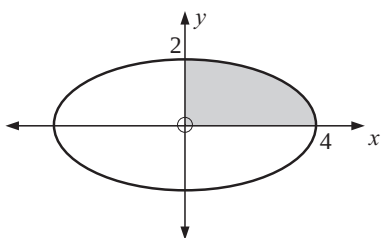
b $\int_4^6 f(x)dx = - \text{area of triangle below } x\text{-axis}$

$$\begin{aligned} &= -\frac{1}{2}(2 \times 2) \\ &= -2 \end{aligned}$$

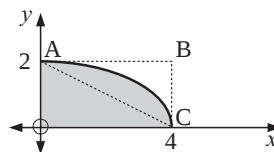
c $\int_0^6 f(x)dx = \int_0^4 f(x)dx + \int_4^6 f(x)dx$

$$\begin{aligned} &= (2 + \pi) + (-2) \\ &= \pi \end{aligned}$$

5 a



b



Now area $\triangle AOC < \text{shaded area} < \text{area } ABCD$

$$\therefore \frac{1}{2}(2 \times 4) < \int_0^4 \frac{1}{2}\sqrt{16-x^2} dx < 2 \times 4$$

$$\therefore 4 < \int_0^4 \frac{1}{2}\sqrt{16-x^2} dx < 8$$

$$\therefore 8 < \int_0^4 \sqrt{16-x^2} dx < 16 \quad \{(\times 2)\}$$

6 Using technology

a $\int_0^2 x^3 dx = 4$

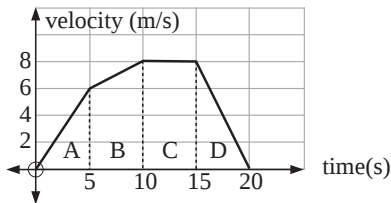
b $\int_1^3 x^3 dx = 20$

Chapter 26

INTEGRATION

EXERCISE 26A

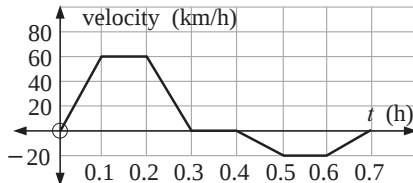
1



Total distance travelled

$$\begin{aligned}
 &= \text{area A} + \text{area B} + \text{area C} + \text{area D} \\
 &= \frac{1}{2}(5 \times 6) + \left(\frac{6+8}{2}\right) 5 + 5 \times 8 + \frac{1}{2}(5 \times 8) \\
 &= 15 + 35 + 40 + 20 \\
 &= 110 \text{ m}
 \end{aligned}$$

2



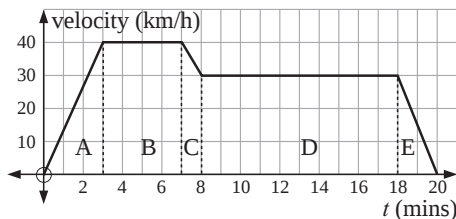
a i The graph above the t -axis indicates that the velocity is positive and the car is travelling forwards.

ii The graph below the t -axis indicates that the velocity is negative and the car is travelling backwards.

b Final displacement = area above the t -axis – area below the t -axis

$$\begin{aligned}
 &= \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right) 60 - \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right) 20 \\
 &= 12 - 4 \\
 &= 8 \text{ km from the starting point in the positive direction}
 \end{aligned}$$

3

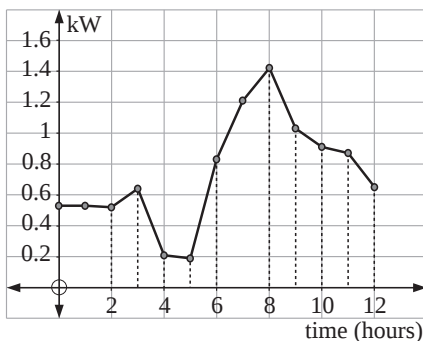


Total distance travelled

$$\begin{aligned}
 &= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E} \\
 &= \frac{1}{60} \left[\frac{1}{2}(3 \times 40) + (40 \times 4) + \left(\frac{40+30}{2}\right) 4 + (10 \times 30) + \frac{1}{2}(2 \times 30) \right] \\
 &= \frac{1}{60} [60 + 160 + 35 + 300 + 30] \\
 &= 9.75 \text{ km}
 \end{aligned}$$

{the factor $\frac{1}{60}$ accounts for the fact that the times are in minutes while the speeds are in km/h}

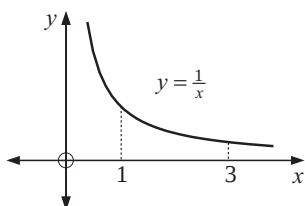
4



Total consumption of electricity

$$\begin{aligned}
 &= \text{area under graph} \\
 &= (2 \times 0.55) + \left(\frac{0.55+0.65}{2}\right) 1 + \left(\frac{0.65+0.2}{2}\right) 1 \\
 &\quad + 1 \times 0.2 + \left(\frac{0.2+0.8}{2}\right) 1 + \left(\frac{0.8+1.2}{2}\right) 1 \\
 &\quad + \left(\frac{1.2+1.4}{2}\right) 1 + \left(\frac{1.4+1}{2}\right) 1 + \left(\frac{1+0.9}{2}\right) 1 \\
 &\quad + \left(\frac{0.9+0.85}{2}\right) 1 + \left(\frac{0.85+0.65}{2}\right) 1 \\
 &\div 8.9 \text{ kWh}
 \end{aligned}$$

5


 Divide the region into 10 strips of width, $\Delta x = \frac{2}{10} = 0.2$

The endpoints of the sub-intervals are: 1, 1.2, 1.4,

, 3.0. Since the function $y = \frac{1}{x}$ is decreasing on the interval, the upper rectangle sums use the y -coordinates at the left-hand end of each sub-interval, while the lower rectangle sums use the y -coordinates at the right-hand ends.

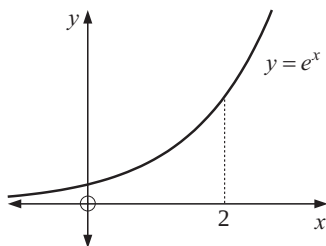
$$\begin{aligned} \text{a Upper rectangular sums: } \therefore A_U &= \frac{1}{1}(0.2) + \frac{1}{1.2}(0.2) + \dots + \frac{1}{2.8}(0.2) \\ &= 0.2 \left[1 + \frac{1}{1.2} + \frac{1}{1.4} + \dots + \frac{1}{2.8} \right] \\ &\doteq 1.17 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{b Lower rectangular sums: } \therefore A_L &= \frac{1}{1.2}(0.2) + \frac{1}{1.4}(0.2) + \frac{1}{1.6}(0.2) + \dots + \frac{1}{3.0}(0.2) \\ &= 0.2 \left[\frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \dots + \frac{1}{3.0} \right] \\ &\doteq 1.03 \text{ units}^2 \end{aligned}$$

 c Midpoints: The midpoints are at $x = 1.1, 1.3, 1.5, 1.7, \dots, 2.9$

$$\begin{aligned} \therefore A_M &= \frac{1}{1.1}(0.2) + \frac{1}{1.3}(0.2) + \dots + \frac{1}{2.9}(0.2) \\ &= 0.2 \left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \dots + \frac{1}{2.9} \right] \\ &\doteq 1.10 \text{ units}^2 \end{aligned}$$

6

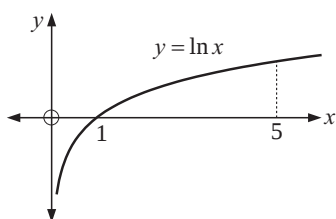

 10 strips, $\therefore$ width, $\Delta x = \frac{2}{10} = 0.2$

The endpoints of the interval are 0, 0.2, 0.4, 0.6,, 2.0

 $\therefore$ the midpoints are 0.1, 0.3, 0.5, 0.7,, 1.9

$$\begin{aligned} \therefore A_M &= 0.2e^{0.1} + 0.2e^{0.3} + \dots + 0.2e^{1.9} \\ &= 0.2[e^{0.1} + e^{0.3} + e^{0.5} + \dots + e^{1.9}] \\ &\doteq 0.2 \times 31.892 \\ &\doteq 6.38 \text{ units}^2 \end{aligned}$$

7


 20 strips, $\therefore$ width, $\Delta x = \frac{4}{20} = 0.2$

Endpoints of the interval are 1.00, 1.2, 1.4,, 5.00

 $\therefore$ the midpoints are 1.1, 1.3, 1.5,, 4.9

$$\begin{aligned} \therefore A_M &= 0.2[\ln 1.1 + \ln 1.3 + \dots + \ln 4.9] \\ &\doteq 0.2 \times 20.24 \\ &\doteq 4.05 \text{ units}^2 \end{aligned}$$

8

a Using technology

$$\begin{aligned} \text{Area} &= 6.389\,06 \\ &\doteq 6.389 \text{ units}^2 \end{aligned}$$

b Using technology

$$\begin{aligned} \text{Area} &= 4.047\,19 \\ &\doteq 4.047 \text{ units}^2 \end{aligned}$$

EXERCISE 26C

1

$$\text{a i } \frac{d}{dx}(x^2) = 2x$$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$$

 $\therefore$ the antiderivative of x is $\frac{1}{2}x^2$

ii

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

 $\therefore$ the antiderivative of x^2 is $\frac{1}{3}x^3$

$$\text{iii} \quad \frac{d}{dx}(x^6) = 6x^5$$

$$\therefore \frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$$

$$\therefore \text{the antiderivative of } x^5 \text{ is } \frac{1}{6}x^6$$

$$\text{iv} \quad \frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$\therefore \frac{d}{dx}(-x^{-1}) = x^{-2}$$

$$\therefore \text{the antiderivative of } x^{-2} \text{ is } -x^{-1} \text{ or } -\frac{1}{x}$$

$$\text{v} \quad \frac{d}{dx}(x^{-3}) = -3x^{-4}$$

$$\therefore \frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$$

$$\therefore \text{the antiderivative of } x^{-4} \text{ is } -\frac{1}{3}x^{-3}$$

$$\text{vi} \quad \frac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}}$$

$$\therefore \frac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$$

$$\therefore \text{the antiderivative of } x^{\frac{1}{3}} \text{ is } \frac{3}{4}x^{\frac{4}{3}}$$

$$\text{viii} \quad \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} \quad \therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = x^{-\frac{1}{2}}$$

$$\therefore \text{the antiderivative of } x^{-\frac{1}{2}} \text{ is } 2x^{\frac{1}{2}} = 2\sqrt{x}$$

$$\text{b} \quad \text{the antiderivative of } x^n \text{ is } \frac{x^{n+1}}{n+1}$$

$$\text{2 a i} \quad \frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$$

$$\therefore \text{the antiderivative of } e^{2x} \text{ is } \frac{1}{2}e^{2x}$$

$$\text{ii} \quad \frac{d}{dx}(e^{5x}) = 5e^{5x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$$

$$\therefore \text{the antiderivative of } e^{5x} \text{ is } \frac{1}{5}e^{5x}$$

$$\text{iii} \quad \frac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$$

$$\therefore \frac{d}{dx}\left(2e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$$

$$\therefore \text{the antiderivative of } e^{\frac{1}{2}x} \text{ is } 2e^{\frac{1}{2}x}$$

$$\text{iv} \quad \frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$$

$$\therefore \frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$$

$$\therefore \text{the antiderivative of } e^{0.01x} \text{ is } 100e^{0.01x}$$

$$\text{v} \quad \frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$$

$$\therefore \text{the antiderivative of } e^{\pi x} \text{ is } \frac{1}{\pi}e^{\pi x}$$

$$\text{vi} \quad \frac{d}{dx}\left(e^{\frac{\pi}{3}}\right) = \frac{1}{3}e^{\frac{\pi}{3}}$$

$$\therefore \frac{d}{dx}\left(3e^{\frac{\pi}{3}}\right) = e^{\frac{\pi}{3}}$$

$$\therefore \text{the antiderivative of } e^{\frac{\pi}{3}} \text{ is } 3e^{\frac{\pi}{3}}$$

$$\text{b} \quad \text{the antiderivative of } e^{kx} \text{ is } \frac{1}{k}e^{kx}$$

$$\text{3 a} \quad \frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$$

$$\frac{d}{dx}(2x^3 + 2x^2) = 6x^2 + 4x$$

$$\therefore \text{antiderivative of } 6x^2 + 4x \text{ is } 2x^3 + 2x^2$$

$$\text{b} \quad \frac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$$

$$\frac{d}{dx}\left(\frac{1}{3}e^{3x+1}\right) = e^{3x+1}$$

$$\therefore \text{antiderivative of } e^{3x+1} \text{ is } \frac{1}{3}e^{3x+1}$$

$$\text{c} \quad \frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$\therefore \frac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$$

$$\therefore \text{antiderivative of } \sqrt{x} \text{ is } \frac{2}{3}x\sqrt{x}$$

$$\text{d} \quad \frac{d}{dx}((2x+1)^4) = 4(2x+1)^3 \times 2 = 8(2x+1)^3$$

$$\therefore \frac{d}{dx}\left(\frac{1}{8}(2x+1)^4\right) = (2x+1)^3$$

$$\therefore \text{antiderivative of } (2x+1)^3 \text{ is } \frac{1}{8}(2x+1)^4$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \frac{dy}{dx} &= 6 \\ \frac{d}{dx}(6x + c) &= 6 \\ \therefore y &= 6x + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \frac{dy}{dx} &= 4x^2 \\ \frac{d}{dx}(x^3) &= 3x^2 \\ \frac{d}{dx}\left(\frac{4}{3}x^3 + c\right) &= 4x^2 \\ \therefore y &= \frac{4}{3}x^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{dy}{dx} &= 5x - x^2 & \text{Now } \frac{d}{dx}(x^2) &= 2x & \therefore \frac{d}{dx}\left(\frac{5}{2}x^2\right) &= 5x \\ & & \text{and } \frac{d}{dx}(x^3) &= 3x^2 & \therefore \frac{d}{dx}\left(-\frac{1}{3}x^3\right) &= -x^2 \end{aligned}$$

$$\text{Now } \frac{d}{dx}\left(\frac{5}{2}x^2 - \frac{1}{3}x^3 + c\right) = 5x - x^2, \quad \therefore y = \frac{5}{2}x^2 - \frac{1}{3}x^3 + c$$

$$\begin{aligned} \mathbf{d} \quad \frac{dy}{dx} &= \frac{1}{x^2} = x^{-2} \\ \text{Now } \frac{d}{dx}(x^{-1}) &= -x^{-2} \\ \therefore \frac{d}{dx}\left(-\frac{1}{x} + c\right) &= x^{-2} \\ \therefore y &= -\frac{1}{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \frac{dy}{dx} &= e^{-3x} \\ \text{Now } \frac{d}{dx}(e^{-3x}) &= -3e^{-3x} \\ \therefore \frac{d}{dx}\left(-\frac{1}{3}e^{-3x} + c\right) &= e^{-3x} \\ \therefore y &= -\frac{1}{3}e^{-3x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{dy}{dx} &= 4x^3 + 3x^2 & \text{Now } \frac{d}{dx}(x^4 + x^3) &= 4x^3 + 3x^2 \\ & & \therefore \frac{d}{dx}(x^4 + x^3 + c) &= 4x^3 + 3x^2 \\ & & \therefore y &= x^4 + x^3 + c \end{aligned}$$

EXERCISE 26D

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int_a^a f(x) \, dx &= F(a) - F(a) = 0 \\ &\text{graphically: } \int_a^a f(x) \, dx = \text{area of} \\ &\text{the strip between } x = a \text{ and } x = a, \\ &\text{i.e., width} = 0 \quad \therefore \text{area} = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_a^b c \, dx &= F(b) - F(a) \\ &= cb - ca \quad \{\text{antideriv. of } c \text{ is } cx\} \\ &= c(b - a) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int_b^a f(x) \, dx &= F(a) - F(b) \\ &= -[F(b) - F(a)] \\ &= -\int_a^b f(x) \, dx \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \text{If } \frac{d}{dx}F(x) &= f(x) \quad \text{then} \\ \frac{d}{dx}cF(x) &= cf(x) \\ \therefore \int_a^b cf(x) \, dx &= c[F(b) - F(a)] \\ &= c \int_a^b f(x) \, dx \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \int_a^b (f(x) + g(x)) \, dx &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad f(x) &= x^3 \quad \text{has antiderivative} \\ F(x) &= \frac{x^4}{4} \end{aligned}$$

$$\begin{aligned} \text{So, area} &= \int_0^1 x^3 \, dx \\ &= F(1) - F(0) \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= x^3 \quad \text{has antiderivative} \\ F(x) &= \frac{x^4}{4} \end{aligned}$$

$$\begin{aligned} \text{So, area} &= \int_1^2 x^3 \, dx \\ &= F(2) - F(1) \\ &= \frac{16}{4} - \frac{1}{4} \\ &= 3\frac{3}{4} \text{ units}^2 \end{aligned}$$

c $f(x) = x^2 + 3x + 2$ has antiderivative

$$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} + 2x$$

$$\begin{aligned}\text{So, area} &= \int_1^3 (x^2 + 3x + 2) dx \\ &= F(3) - F(1) \\ &= \left(\frac{27}{3} + \frac{27}{2} + 6\right) - \left(\frac{1}{3} + \frac{3}{2} + 2\right) \\ &= 24\frac{2}{3} \text{ units}^2\end{aligned}$$

e $f(x) = e^x$ has antiderivative

$$F(x) = e^x$$

$$\begin{aligned}\text{So, area} &= \int_0^{1.5} e^x dx \\ &= F(1.5) - F(0) \\ &= e^{1.5} - e^0 \\ &= e^{1.5} - 1 \\ &\div 3.482 \text{ units}^2\end{aligned}$$

g $f(x) = x^3 + 2x^2 + 7x + 4$ has antiderivative

$$F(x) = \frac{x^4}{4} + \frac{2x^3}{3} + \frac{7x^2}{2} + 4x$$

$$\begin{aligned}\text{So the area} &= \int_1^{1.25} x^3 + 2x^2 + 7x + 4 dx \\ &= F(1.25) - F(1) \\ &= [12.381\ 18 - 8.416\ 67] \\ &\div 3.965 \text{ units}^2\end{aligned}$$

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$$

$$\begin{aligned}\text{So the area} &= \int_0^2 \sqrt{x} dx \\ &= F(2) - F(0) \\ &= \frac{2}{3} \times 2\sqrt{2} - 0 \\ &= \frac{4\sqrt{2}}{3} \text{ units}^2\end{aligned}$$

f $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$$

$$\begin{aligned}\text{So, area} &= \int_1^4 \frac{1}{\sqrt{x}} dx = F(4) - F(1) \\ &= 2\sqrt{4} - 2\sqrt{1} \\ &= 2 \text{ units}^2\end{aligned}$$

EXERCISE 26E.1

$$\begin{aligned}\mathbf{1} \quad y &= x^7 & \therefore \int 7x^6 dx &= x^7 + c_1 \\ \therefore \frac{dy}{dx} &= 7x^6 & \therefore 7 \int x^6 dx &= x^7 + c_1 \\ & & \therefore \int x^6 dx &= \frac{1}{7}x^7 + c\end{aligned}$$

$$\begin{aligned}\mathbf{2} \quad y &= x^3 + x^2 & \therefore \int 3x^2 + 2x dx &= x^3 + x^2 + c \\ \therefore \frac{dy}{dx} &= 3x^2 + 2x & &\end{aligned}$$

$$\begin{aligned}\mathbf{3} \quad y &= e^{2x+1} & \therefore \int 2e^{2x+1} dx &= e^{2x+1} + c_1 \\ \therefore \frac{dy}{dx} &= 2e^{2x+1} & \therefore 2 \int e^{2x+1} dx &= e^{2x+1} + c_1 \\ & & \therefore \int e^{2x+1} dx &= \frac{1}{2}e^{2x+1} + c\end{aligned}$$

$$\begin{aligned}\mathbf{4} \quad y &= (2x+1)^4 & \therefore \int 8(2x+1)^3 dx &= (2x+1)^4 + c_1 \\ \therefore \frac{dy}{dx} &= 4(2x+1)^3 \times 2 & \therefore 8 \int (2x+1)^3 dx &= (2x+1)^4 + c_1 \\ &= 8(2x+1)^3 & \therefore \int (2x+1)^3 dx &= \frac{1}{8}(2x+1)^4 + c\end{aligned}$$

$$\begin{aligned}\mathbf{5} \quad y &= x\sqrt{x} = x^{\frac{3}{2}} & \therefore \int \frac{3}{2}\sqrt{x} dx &= x\sqrt{x} + c_1 \\ \therefore \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x} & \therefore \frac{3}{2} \int \sqrt{x} dx &= x\sqrt{x} + c_1 \\ & & \therefore \int \sqrt{x} dx &= \frac{2}{3}x\sqrt{x} + c\end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad y &= \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} & \therefore \int -\frac{1}{2} \left(\frac{1}{x\sqrt{x}} \right) dx &= \frac{1}{\sqrt{x}} + c_1 \\
 \therefore \frac{dy}{dx} &= -\frac{1}{2} x^{-\frac{3}{2}} = -\frac{1}{2} \left(\frac{1}{x\sqrt{x}} \right) & \therefore -\frac{1}{2} \int \frac{1}{x\sqrt{x}} dx &= \frac{1}{\sqrt{x}} + c_1 \\
 & & \therefore \int \frac{1}{x\sqrt{x}} dx &= -\frac{2}{\sqrt{x}} + c
 \end{aligned}$$

7 Suppose $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ is the antiderivative of $g(x)$.

$$\begin{aligned}
 \therefore \frac{d}{dx} (F(x) + G(x)) &= f(x) + g(x) \\
 \therefore \int (f(x) + g(x)) dx &= F(x) + G(x) + c \\
 &= (F(x) + c_1) + (G(x) + c_2) \\
 &= \int f(x) dx + \int g(x) dx
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad y &= (2x - 1)^6 & \therefore \int 12(2x - 1)^5 dx &= (2x - 1)^6 + c_1 \\
 \therefore \frac{dy}{dx} &= 6(2x - 1)^5 \times 2 & \therefore 12 \int (2x - 1)^5 dx &= (2x - 1)^6 + c_1 \\
 &= 12(2x - 1)^5 & \therefore \int (2x - 1)^5 dx &= \frac{1}{12}(2x - 1)^6 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \sqrt{1 - 4x} & \therefore \int \frac{-2}{\sqrt{1 - 4x}} dx &= \sqrt{1 - 4x} + c_1 \\
 &= (1 - 4x)^{\frac{1}{2}} & \therefore -2 \int \frac{1}{\sqrt{1 - 4x}} dx &= \sqrt{1 - 4x} + c_1 \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}(1 - 4x)^{-\frac{1}{2}}(-4) & \therefore \int \frac{1}{\sqrt{1 - 4x}} dx &= -\frac{1}{2}\sqrt{1 - 4x} + c \\
 &= -2(1 - 4x)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \frac{1}{\sqrt{3x + 1}} = (3x + 1)^{-\frac{1}{2}} & \therefore \int -\frac{3}{2(3x + 1)^{\frac{3}{2}}} dx &= \frac{1}{\sqrt{3x + 1}} + c_1 \\
 \therefore \frac{dy}{dx} &= -\frac{1}{2}(3x + 1)^{-\frac{3}{2}} \times 3 & \therefore -\frac{3}{2} \int \frac{1}{(3x + 1)^{\frac{3}{2}}} dx &= \frac{1}{\sqrt{3x + 1}} + c_1 \\
 &= -\frac{3}{2}(3x + 1)^{-\frac{3}{2}} & \therefore \int \frac{1}{(3x + 1)^{\frac{3}{2}}} dx &= \frac{-2}{3\sqrt{3x + 1}} + c \\
 &= -\frac{3}{2(3x + 1)^{\frac{3}{2}}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad y &= e^{1-3x} & \therefore \int -3e^{1-3x} dx &= e^{1-3x} + c_1 \\
 \therefore \frac{dy}{dx} &= e^{1-3x}(-3) & \therefore -3 \int e^{1-3x} dx &= e^{1-3x} + c_1 \\
 &= -3e^{1-3x} & \therefore \int e^{1-3x} dx &= -\frac{1}{3}e^{1-3x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \ln(4x + 1) & \therefore \int \frac{4}{4x + 1} dx &= \ln(4x + 1) + c_1 \\
 \therefore \frac{dy}{dx} &= \frac{1}{4x + 1} \times 4 & \therefore 4 \int \frac{1}{4x + 1} dx &= \ln(4x + 1) + c_1 \\
 &= \frac{4}{4x + 1} & \therefore \int \frac{1}{4x + 1} dx &= \frac{1}{4} \ln(4x + 1) + c \quad (4x + 1 > 0)
 \end{aligned}$$

$$10 \quad \mathbf{a} \quad \frac{d}{dx}(e^x - x^2) = e^{x-x^2}(1-2x), \quad \therefore \int e^{x-x^2}(1-2x) dx = e^{x-x^2} + c$$

$$\mathbf{b} \quad \frac{d}{dx}(\ln(5-3x+x^2)) = \frac{2x-3}{5-3x+x^2}$$

$$\therefore \int \frac{2x-3}{5-3x+x^2} dx = \ln |5-3x+x^2| + c_1$$

$$\therefore \int \frac{4x-6}{5-3x+x^2} dx = 2\ln(5-3x+x^2) + c \quad \{5-3x+x^2 > 0 \text{ for all } x \text{ as } a > 0 \text{ and } \Delta = -11 < 0\}$$

$$\begin{aligned} \mathbf{c} \quad \frac{d}{dx}(x^2-5x+1)^{-2} & \therefore \int \frac{-4x+10}{(x^2-5x+1)^3} dx = (x^2-5x+1)^{-2} + c_1 \\ & = -2(x^2-5x+1)^{-3}(2x-5) \therefore \int \frac{-2(2x-5)}{(x^2-5x+1)^3} dx = (x^2-5x+1)^{-2} + c_1 \\ & = \frac{-4x+10}{(x^2-5x+1)^3} \therefore -2 \int \frac{2x-5}{(x^2-5x+1)^3} dx = (x^2-5x+1)^{-2} + c_1 \\ & \therefore \int \frac{2x-5}{(x^2-5x+1)^3} dx = -\frac{1}{2}(x^2-5x+1)^{-2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \frac{d}{dx}(xe^x) & = 1 \times e^x + xe^x \therefore \int (e^x + xe^x) dx = xe^x + c_1 \\ & \therefore \int e^x dx + \int xe^x dx = xe^x + c_1 \\ & \therefore e^x + \int xe^x dx = xe^x + c_1 \\ & \therefore \int xe^x dx = xe^x - e^x + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \frac{d}{dx}(2^x) & = \frac{d}{dx}(e^{\ln 2})^x \quad \{\text{since } 2^x = (e^{\ln 2})^x\} \therefore \int 2^x \ln 2 dx = 2^x + c_1 \\ & = e^{(\ln 2)x} \times \ln 2 \therefore \ln 2 \int 2^x dx = 2^x + c_1 \\ & = 2^x \ln 2 \therefore \int 2^x dx = \frac{2^x}{\ln 2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{d}{dx}(x \ln x) & = 1 \times \ln x + x \times \frac{1}{x} \therefore \int (\ln x + 1) dx = x \ln x + c_1 \\ & = \ln x + 1 \therefore \int \ln x dx + x = x \ln x + c_1 \\ & \therefore \int \ln x dx = x \ln x - x + c \end{aligned}$$

EXERCISE 26E.2

$$\mathbf{1} \quad \mathbf{a} \quad \int (x^4 - x^2 - x + 2) dx \\ = \frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c$$

$$\begin{aligned} \mathbf{b} \quad \int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx & = \int x^{\frac{1}{2}} - x^{-\frac{1}{2}} dx \\ & = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ & = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int \left(2e^x - \frac{1}{x^2} \right) dx & \\ & = \int (2e^x - x^{-2}) dx \\ & = 2 \int e^x dx - \frac{x^{-1}}{-1} + c \\ & = 2e^x + \frac{1}{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \int \left(x\sqrt{x} - \frac{1}{x} \right) dx & \\ & = \int \left(x^{\frac{3}{2}} - x^{-1} \right) dx \\ & = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \ln|x| + c \\ & = \frac{2}{5}x^{\frac{5}{2}} - \ln|x| + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int (2x+1)^2 dx \\
 &= \int (4x^2 + 4x + 1) dx \\
 &= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c \\
 &= \frac{4}{3}x^3 + 2x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int \frac{2x-1}{\sqrt{x}} dx \\
 &= \int \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \right) dx \\
 &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int (x+1)^3 dx \\
 &= \int (x^3 + 3x^2 + 3x + 1) dx \\
 &= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \frac{dy}{dx} = (1-2x)^2 \\
 \therefore y &= \int (1-2x)^2 dx \\
 &= \int (1-4x+4x^2) dx \\
 &= x - \frac{4x^2}{2} + \frac{4x^3}{3} + c \\
 &= x - 2x^2 + \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{dy}{dx} = \frac{x^2 + 2x - 5}{x^2} \\
 &= 1 + 2x^{-1} - 5x^{-2} \\
 \therefore y &= \int (1 + 2x^{-1} - 5x^{-2}) dx \\
 &= x + 2 \ln |x| - \frac{5x^{-1}}{-1} + c \\
 &= x + 2 \ln |x| + \frac{5}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & f'(x) = x^3 - 5x + 3 \\
 \therefore f(x) &= \int (x^3 - 5x + 3) dx \\
 &= \frac{x^4}{4} - \frac{5x^2}{2} + 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & f'(x) = 3e^x - \frac{4}{x} \\
 \therefore f(x) &= \int \left(3e^x - \frac{4}{x} \right) dx \\
 &= 3e^x - 4 \ln |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \frac{x^2 + x - 3}{x} dx \\
 &= \int \left(x + 1 - \frac{3}{x} \right) dx \\
 &= \frac{x^2}{2} + x - 3 \ln |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int \left(\frac{1}{x\sqrt{x}} - \frac{4}{x} \right) dx \\
 &= \int \left(x^{-\frac{3}{2}} - \frac{4}{x} \right) dx \\
 &= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - 4 \ln |x| + c \\
 &= -2x^{-\frac{1}{2}} - 4 \ln |x| + c \\
 &= -\frac{2}{\sqrt{x}} - 4 \ln |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{dy}{dx} = \sqrt{x} - \frac{2}{\sqrt{x}} \\
 &= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\
 \therefore y &= \int \left(x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \right) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & f'(x) = 2\sqrt{x}(1-3x) \\
 &= 2x^{\frac{1}{2}} - 6x^{\frac{3}{2}} \\
 \therefore f(x) &= \int \left(2x^{\frac{1}{2}} - 6x^{\frac{3}{2}} \right) dx \\
 &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{6x^{\frac{5}{2}}}{\frac{5}{2}} + c \\
 &= \frac{4}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{2}} + c
 \end{aligned}$$

4 a $f'(x) = 2x - 1$ and $f(0) = 3$

$$\begin{aligned} f(x) &= \int (2x - 1) dx \\ &= \frac{2x^2}{2} - x + c \\ &= x^2 - x + c \end{aligned}$$

But $f(0) = 3$

$$\therefore 0 - 0 + c = 3$$

i.e., $c = 3$

$$\therefore f(x) = x^2 - x + 3$$

b $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

$$\begin{aligned} f(x) &= \int (3x^2 + 2x) dx \\ &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c \end{aligned}$$

But $f(2) = 5$

$$\therefore 8 + 4 + c = 5$$

i.e., $c = -7$

$$\therefore f(x) = x^3 + x^2 - 7$$

c $f'(x) = e^x + \frac{1}{\sqrt{x}}$ and $f(1) = 1$

$$\begin{aligned} f'(x) &= e^x + x^{-\frac{1}{2}} \\ f(x) &= \int \left(e^x + x^{-\frac{1}{2}} \right) dx \\ &= e^x + 2x^{\frac{1}{2}} + c \end{aligned}$$

But $f(1) = 1$

$$\therefore e^1 + 2 + c = 1$$

i.e., $c = -1 - e$

$$\therefore f(x) = e^x + 2\sqrt{x} - 1 - e$$

d $f'(x) = x - \frac{2}{\sqrt{x}}$ and $f(1) = 2$

$$\begin{aligned} f'(x) &= x - 2x^{-\frac{1}{2}} \\ f(x) &= \int \left(x - 2x^{-\frac{1}{2}} \right) dx \\ &= \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{1}{2}x^2 - 4\sqrt{x} + c \end{aligned}$$

But $f(1) = 2$

$$\therefore \frac{1}{2} - 4 + c = 2$$

i.e., $c = \frac{11}{2}$

$$\therefore f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$$

5 a $f''(x) = 2x + 1$ $f'(1) = 3$ $f(2) = 7$

Now $f'(x) = \frac{2x^2}{2} + x + c = x^2 + x + c$

But $f'(1) = 3$

$$\therefore 1 + 1 + c = 3$$

$\therefore c = 1$

$$\therefore f'(x) = x^2 + x + 1$$

$$\therefore f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + k$$

and $f(2) = 7$

$$\therefore \frac{8}{3} + 2 + 2 + k = 7$$

$$k = 7 - 4 - \frac{8}{3}$$

$$\therefore k = \frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$$

b $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}}$ $f'(1) = 12$ $f(0) = 5$

Now $f''(x) = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$$\therefore f'(x) = \frac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$$

But $f'(1) = 12 \therefore 10 + 6 + c = 12$ and so $c = -4$

$$\therefore f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$$

$$\therefore f(x) = \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + k = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + k$$

But $f(0) = 5 \therefore k = 5 \therefore f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c $f''(x) = 2x$ (1, 0) and (0, 5) lie on the curve

Now $f'(x) = \frac{2x^2}{2} + c = x^2 + c$ and $f(x) = \frac{x^3}{3} + cx + k$

But $f(0) = 5 \therefore 0 + 0 + k = 5$ and so $k = 5$

and $f(1) = 0 \therefore \frac{1}{3} + c + 5 = 0$ and so $c = -5\frac{1}{3}$

$\therefore f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 26F

1 a $\int (2x + 5)^3 dx$

$$= \frac{1}{2} \times \frac{(2x + 5)^4}{4} + c$$

$$= \frac{1}{8}(2x + 5)^4 + c$$

b $\int \frac{1}{(3 - 2x)^2} dx$

$$= \int (3 - 2x)^{-2} dx$$

$$= \frac{1}{-2} \times \frac{(3 - 2x)^{-1}}{-1} + c$$

$$= \frac{1}{2(3 - 2x)} + c$$

c $\int \frac{4}{(2x - 1)^4} dx$

$$= \int 4(2x - 1)^{-4} dx$$

$$= 4\left(\frac{1}{2}\right) \times \frac{(2x - 1)^{-3}}{-3} + c$$

$$= -\frac{2}{3}(2x - 1)^{-3} + c$$

d $\int (4x - 3)^7 dx$

$$= \frac{1}{4} \times \frac{(4x - 3)^8}{8} + c$$

$$= \frac{1}{32}(4x - 3)^8 + c$$

e $\int \sqrt{3x - 4} dx$

$$= \int (3x - 4)^{\frac{1}{2}} dx$$

$$= \frac{1}{3} \times \frac{(3x - 4)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{9}(3x - 4)^{\frac{3}{2}} + c$$

f $\int \frac{10}{\sqrt{1 - 5x}} dx$

$$= \int 10(1 - 5x)^{-\frac{1}{2}} dx$$

$$= 10\left(\frac{-1}{5}\right) \times \frac{(1 - 5x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -4(1 - 5x)^{\frac{1}{2}} + c$$

g $\int 3(1 - x)^4 dx$

$$= 3 \int (1 - x)^4 dx$$

$$= 3\left(\frac{-1}{-1}\right) \times \frac{(1 - x)^5}{5} + c$$

$$= -\frac{3}{5}(1 - x)^5 + c$$

h $\int \frac{4}{\sqrt{3 - 4x}} dx$

$$= \int 4(3 - 4x)^{-\frac{1}{2}} dx$$

$$= 4\left(\frac{-1}{-4}\right) \times \frac{(3 - 4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -2\sqrt{3 - 4x} + c$$

i $\int \sqrt[3]{2x - 1} dx$

$$= \int (2x - 1)^{\frac{1}{3}} dx$$

$$= \frac{1}{2} \times \frac{(2x - 1)^{\frac{4}{3}}}{\frac{4}{3}} + c$$

$$= \frac{3}{8}(2x - 1)^{\frac{4}{3}} + c$$

2 a $\frac{dy}{dx} = \sqrt{2x - 7}$

$$= (2x - 7)^{\frac{1}{2}}$$

$$\therefore y = \frac{1}{2} \times \frac{(2x - 7)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}(2x - 7)^{\frac{3}{2}} + c$$

But $y = 11$ when $x = 8$

$$\therefore \frac{1}{3}(9)^{\frac{3}{2}} + c = 11$$

$$\frac{1}{3}(27) + c = 11$$

$$9 + c = 11 \text{ and so } c = 2$$

$$\therefore y = \frac{1}{3}(2x - 7)^{\frac{3}{2}} + 2$$

b $f(x)$ has tangent-slope function $\frac{4}{\sqrt{1 - x}}$

i.e., $f'(x) = \frac{4}{\sqrt{1 - x}} = 4(1 - x)^{-\frac{1}{2}}$

$$f(x) = 4\left(\frac{-1}{-1}\right) \times \frac{(1 - x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -8\sqrt{1 - x} + c$$

But $y = 11$ when $x = -3$

$$\therefore -8\sqrt{1 - (-3)} + c = 11$$

$$\therefore -8\sqrt{4} + c = 11$$

$$\therefore -16 + c = 11 \text{ and } c = 5$$

$$\therefore f(x) = 5 - 8\sqrt{1 - x}$$

When $x = -8$, $y = 5 - 8\sqrt{1 - (-8)} = 5 - 8(3) = -19$, $\therefore$ point is $(-8, -19)$.

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \int 3(2x-1)^2 dx \\
 &= 3 \int (2x-1)^2 dx \\
 &= 3 \left(\frac{1}{2} \right) \frac{(2x-1)^3}{3} + c \\
 &= \frac{1}{2}(2x-1)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (x^2 - x)^2 dx \\
 &= \int (x^4 - 2x^3 + x^2) dx \\
 &= \frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + c \\
 &= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (1-3x)^3 dx \\
 &= \left(\frac{1}{-3} \right) \frac{(1-3x)^4}{4} + c \\
 &= -\frac{1}{12}(1-3x)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int (1-x^2)^2 dx \\
 &= \int (1-2x^2+x^4) dx \\
 &= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int 4\sqrt{5-x} dx \\
 &= 4 \int (5-x)^{\frac{1}{2}} dx \\
 &= 4 \left(\frac{1}{-\frac{1}{2}} \right) \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= -4 \left(\frac{2}{3} \right) (5-x)^{\frac{3}{2}} + c \\
 &= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int (x^2+1)^3 dx \\
 &= \int (x^6+3x^4+3x^2+1) dx \\
 &= \frac{x^7}{7} + \frac{3x^5}{5} + \frac{3x^3}{3} + x + c \\
 &= \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \int (2e^x + 5e^{2x}) dx \\
 &= 2e^x + 5\left(\frac{1}{2}\right)e^{2x} + c \\
 &= 2e^x + \frac{5}{2}e^{2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (x^2 - 2e^{-3x}) dx \\
 &= \frac{x^3}{3} - 2\left(\frac{1}{-3}\right)e^{-3x} + c \\
 &= \frac{1}{3}x^3 + \frac{2}{3}e^{-3x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (\sqrt{x} + 4e^{2x} - e^{-x}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 4\left(\frac{1}{2}\right)e^{2x} - (-1)e^{-x} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + 2e^{2x} + e^{-x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{1}{2x-1} dx \\
 &= \frac{1}{2} \ln |2x-1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \frac{5}{1-3x} dx \\
 &= 5 \int \frac{1}{1-3x} dx \\
 &= 5\left(\frac{1}{-\frac{3}{1}}\right) \ln |1-3x| + c \\
 &= -\frac{5}{3} \ln |1-3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \left(e^{-x} - \frac{4}{2x+1} \right) dx \\
 &= \frac{1}{-1}e^{-x} - 4\left(\frac{1}{2}\right) \ln |2x+1| + c \\
 &= -e^{-x} - 2 \ln |2x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int (e^x + e^{-x})^2 dx \\
 &= \int (e^{2x} + 2 + e^{-2x}) dx \\
 &= \frac{1}{2}e^{2x} + 2x + \left(\frac{1}{-2}\right)e^{-2x} + c \\
 &= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int (e^{-x} + 2)^2 dx \\
 &= \int (e^{-2x} + 4e^{-x} + 4) dx \\
 &= \frac{1}{-2}e^{-2x} + 4\left(\frac{1}{-1}\right)e^{-x} + 4x + c \\
 &= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int \left(x - \frac{5}{1-x} \right) dx = \frac{x^2}{2} - 5\left(\frac{1}{-1}\right) \ln |1-x| + c \\
 &= \frac{1}{2}x^2 + 5 \ln |1-x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & \frac{dy}{dx} = (1-e^x)^2 \\
 &= 1 - 2e^x + e^{2x} \\
 \therefore y &= x - 2e^x + \frac{1}{2}e^{2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{dy}{dx} = 1 - 2x + \frac{3}{x+2} \\
 \therefore y &= x - \frac{2x^2}{2} + 3 \ln |x+2| + c \\
 &= x - x^2 + 3 \ln |x+2| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{dy}{dx} = e^{-2x} + \frac{4}{2x-1} \quad \therefore y = \frac{1}{-2}e^{-2x} + 4\left(\frac{1}{2}\right) \ln |2x-1| + c \\
 &= -\frac{1}{2}e^{-2x} + 2 \ln |2x-1| + c
 \end{aligned}$$

6 Differentiating Tracy's answer gives

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{4} \ln |4x| + c \right) \\ = \frac{1}{4} \left(\frac{1}{4x} \right) \times 4 + 0 \\ = \frac{1}{4x} \end{aligned}$$

$\therefore$ both answers give the correct derivative and both are correct.

Note that this result occurs because $\log 4x = \log 4 + \log x$ $\therefore$ their answers differ by a constant ($\log 4$) which is just part of the constant c .

The derivative of Nadine's answer is

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{4} \ln |x| + c \right) \\ = \frac{1}{4} \left(\frac{1}{x} \right) + 0 \\ = \frac{1}{4x} \end{aligned}$$

7 a $f'(x) = 2e^{-2x}$ $f(0) = 3$

$$\begin{aligned} \therefore f(x) &= 2\left(\frac{1}{-2}\right)e^{-2x} + c \\ &= -e^{-2x} + c \end{aligned}$$

But $f(0) = 3$

$$\begin{aligned} \therefore -e^0 + c &= 3 \\ \therefore c &= 4 \\ \therefore f(x) &= -e^{-2x} + 4 \end{aligned}$$

b $f'(x) = 2x - \frac{2}{1-x}$ $f(-1) = 3$

$$\begin{aligned} \therefore f(x) &= \frac{2x^2}{2} - \frac{2}{-1} \ln |1-x| + c \\ &= x^2 + 2 \ln |1-x| + c \end{aligned}$$

But $f(-1) = 3$

$$\begin{aligned} \therefore 1 + 2 \ln |2| + c &= 3 \\ \therefore c &= 2 - 2 \ln 2 \\ \therefore f(x) &= x^2 + 2 \ln |1-x| + 2 - 2 \ln 2 \end{aligned}$$

c $f'(x) = \sqrt{x} + \frac{1}{2}e^{-4x}$
 $\therefore f'(x) = x^{\frac{1}{2}} + \frac{1}{2}e^{-4x}$
 and $f(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}\left(\frac{1}{-4}\right)e^{-4x} + c$
 $= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + c$

But $f(1) = 0$

$$\begin{aligned} \therefore \frac{2}{3} - \frac{1}{8}e^{-4} + c &= 0 \\ \therefore c &= \frac{1}{8}e^{-4} - \frac{2}{3} \\ \therefore f(x) &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3} \end{aligned}$$

8

$$\begin{aligned} \frac{3}{x+2} - \frac{1}{x-2} \\ = \frac{3(x-2) - 1(x+2)}{(x+2)(x-2)} \\ = \frac{3x - 6 - x - 2}{x^2 - 4} \\ = \frac{2x - 8}{x^2 - 4} \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{2x-8}{x^2-4} dx \\ = \int \left(\frac{3}{x+2} - \frac{1}{x-2} \right) dx \\ = 3 \ln |x+2| - \ln |x-2| + c \end{aligned}$$

9

$$\begin{aligned} \frac{1}{2x-1} - \frac{1}{2x+1} \\ = \frac{1(2x+1) - 1(2x-1)}{(2x-1)(2x+1)} \\ = \frac{2x+1-2x+1}{(2x-1)(2x+1)} \\ = \frac{2}{4x^2-1} \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{2}{4x^2-1} dx \\ = \int \left(\frac{1}{2x-1} - \frac{1}{2x+1} \right) dx \\ = \frac{1}{2} \ln |2x-1| - \frac{1}{2} \ln |2x+1| + c \end{aligned}$$

EXERCISE 26G

1 a Let $u = x^3 + 1$, $\therefore \frac{du}{dx} = 3x^2$

$$\begin{aligned}\therefore \int 3x^2(x^3 + 1)^4 dx &= \int u^4 \frac{du}{dx} dx \\ &= \int u^4 du \\ &= \frac{1}{5}u^5 + c \\ &= \frac{1}{5}(x^3 + 1)^5 + c\end{aligned}$$

b Let $u = x^2 + 3$ $\therefore \frac{du}{dx} = 2x$

$$\begin{aligned}\therefore \int \frac{2x}{\sqrt{x^2 + 3}} dx &= \int \left((x^2 + 3)^{-\frac{1}{2}} \times 2x \right) dx \\ &= \int u^{-\frac{1}{2}} \frac{du}{dx} dx \\ &= \int u^{-\frac{1}{2}} du \\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{u} + c \\ &= 2\sqrt{x^2 + 3} + c\end{aligned}$$

c Let $u = x^3 + x$ $\therefore \frac{du}{dx} = 3x^2 + 1$

$$\begin{aligned}\therefore \int \sqrt{x^3 + x} (3x^2 + 1) dx &= \int \sqrt{u} \frac{du}{dx} dx \\ &= \int u^{\frac{1}{2}} du \\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{3}(x^3 + x)^{\frac{3}{2}} + c\end{aligned}$$

d Let $u = 2 + x^4$ $\therefore \frac{du}{dx} = 4x^3$

$$\begin{aligned}\therefore \int 4x^3(2 + x^4)^3 dx &= \int u^3 \frac{du}{dx} dx \\ &= \int u^3 du \\ &= \frac{u^4}{4} + c \\ &= \frac{1}{4}(2 + x^4)^4 + c\end{aligned}$$

e Let $u = x^3 + 2x + 1$ $\therefore \frac{du}{dx} = 3x^2 + 2$

$$\begin{aligned}\therefore \int (x^3 + 2x + 1)^4 (3x^2 + 2) dx &= \int u^4 \frac{du}{dx} dx \\ &= \int u^4 du \\ &= \frac{u^5}{5} + c \\ &= \frac{1}{5}(x^3 + 2x + 1)^5 + c\end{aligned}$$

f Let $u = 3x^3 - 1$ $\therefore \frac{du}{dx} = 9x^2$

$$\begin{aligned}\therefore \int \frac{x^2}{(3x^3 - 1)^4} dx &= \int (3x^3 - 1)^{-4} \times x^2 dx \\ &= \int u^{-4} \left(\frac{1}{9} \frac{du}{dx} \right) dx \\ &= \frac{1}{9} \int u^{-4} du \\ &= \frac{1}{9} \frac{u^{-3}}{-3} + c \\ &= -\frac{1}{27(3x^3 - 1)^3} + c\end{aligned}$$

$$\mathbf{g} \quad \text{Let } u = 1 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$$

$$\begin{aligned} \therefore \quad & \int \frac{x}{(1-x^2)^5} dx \\ &= \int ((1-x^2)^{-5} \times x) dx \\ &= \int u^{-5} \left(-\frac{1}{2} \frac{du}{dx}\right) dx \\ &= -\frac{1}{2} \int u^{-5} du \\ &= -\frac{1}{2} \frac{u^{-4}}{-4} + c \\ &= \frac{1}{8(1-x^2)^4} + c \end{aligned}$$

$$\mathbf{h} \quad \text{Let } u = x^2 + 4x - 3 \quad \therefore \quad \frac{du}{dx} = 2x + 4$$

$$\begin{aligned} \therefore \quad & \int \frac{x+2}{(x^2+4x-3)^2} dx \\ &= \int (x^2+4x-3)^{-2} (x+2) dx \\ &= \int u^{-2} \left(\frac{1}{2} \frac{du}{dx}\right) dx \\ &= \frac{1}{2} \int u^{-2} du \\ &= \frac{1}{2} \frac{u^{-1}}{-1} + c \\ &= \frac{-1}{2(x^2+4x-3)} + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad \text{Let } u = x^2 + x \quad \therefore \quad \frac{du}{dx} = 2x + 1 \quad \therefore \quad & \int x^4(x+1)^4(2x+1) dx \\ &= \int (x^2+x)^4(2x+1) dx \quad \{\text{as } a^4b^4 = (ab)^4\} \\ &= \int u^4 \frac{du}{dx} dx = \int u^4 du \\ &= \frac{1}{5} u^5 + c \\ &= \frac{1}{5} (x^2+x)^5 + c \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \text{Let } u = 1 - 2x \quad \therefore \quad \frac{du}{dx} = -2$$

$$\begin{aligned} \therefore \quad & \int -2e^{1-2x} dx \\ &= \int e^u \frac{du}{dx} dx \\ &= \int e^u du \\ &= e^u + c \\ &= e^{1-2x} + c \end{aligned}$$

$$\mathbf{b} \quad \text{Let } u = x^2 \quad \therefore \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \therefore \quad & \int 2xe^{x^2} dx \\ &= \int e^u \frac{du}{dx} dx \\ &= \int e^u du \\ &= e^u + c \\ &= e^{x^2} + c \end{aligned}$$

$$\mathbf{c} \quad \text{Let } u = x^3 + 1 \quad \therefore \quad \frac{du}{dx} = 3x^2$$

$$\begin{aligned} \therefore \quad & \int x^2 e^{x^3+1} dx \\ &= \int e^u \left(\frac{1}{3} \frac{du}{dx}\right) dx \\ &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + c \\ &= \frac{1}{3} e^{x^3+1} + c \end{aligned}$$

$$\mathbf{d} \quad \text{Let } u = \sqrt{x} \quad \therefore \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} \therefore \quad & \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \\ &= \int e^u \left(2 \frac{du}{dx}\right) dx \\ &= 2 \int e^u du \\ &= 2e^u + c \\ &= 2e^{\sqrt{x}} + c \end{aligned}$$

$$\mathbf{e} \quad \text{Let } u = x - x^2 \quad \therefore \quad \frac{du}{dx} = 1 - 2x$$

$$\begin{aligned} \therefore \quad & \int (2x-1)e^{x-x^2} dx \\ &= \int e^u \left(-\frac{du}{dx}\right) dx \\ &= \int -e^u du \\ &= -e^u + c \\ &= -e^{x-x^2} + c \end{aligned}$$

$$\mathbf{f} \quad \text{Let } u = \frac{x-1}{x} \quad \therefore \quad \frac{du}{dx} = 0 - (-1)x^{-2} = \frac{1}{x^2}$$

$$\begin{aligned} \therefore \quad & \int \frac{e^{\frac{x-1}{x}}}{x^2} dx = \int e^u \frac{du}{dx} dx \\ &= \int e^u du \\ &= e^u + c \\ &= e^{\frac{x-1}{x}} + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \text{Let } u = x^2 + 1 \quad \therefore \quad \frac{du}{dx} = 2x \\
 & \therefore \quad \int \frac{2x}{x^2 + 1} dx \\
 & = \int \frac{1}{x^2 + 1} (2x) dx \\
 & = \int \frac{1}{u} \frac{du}{dx} dx \\
 & = \int \frac{1}{u} du \\
 & = \ln |u| + c \\
 & = \ln |x^2 + 1| + c \\
 & = \ln(x^2 + 1) + c \quad \text{as } x^2 + 1 > 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{Let } u = 2 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x \\
 & \therefore \quad \int \frac{x}{2 - x^2} dx \\
 & = \int \left(\frac{1}{2 - x^2} \times x \right) dx \\
 & = \int \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\
 & = -\frac{1}{2} \int \frac{1}{u} du \\
 & = -\frac{1}{2} \ln |u| + c \\
 & = -\frac{1}{2} \ln |2 - x^2| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \text{Let } u = x^2 - 3x \quad \therefore \quad \frac{du}{dx} = 2x - 3 \\
 & \therefore \quad \int \frac{2x - 3}{x^2 - 3x} dx \\
 & = \int \frac{1}{x^2 - 3x} (2x - 3) dx \\
 & = \int \frac{1}{u} \frac{du}{dx} dx \\
 & = \int \frac{1}{u} du \\
 & = \ln |u| + c \\
 & = \ln |x^2 - 3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \text{Let } u = x^3 - x \quad \therefore \quad \frac{du}{dx} = 3x^2 - 1 \\
 & \therefore \quad \int \frac{6x^2 - 2}{x^3 - x} dx \\
 & = \int \frac{1}{x^3 - x} (6x^2 - 2) dx \\
 & = \int \frac{1}{u} \left(2 \frac{du}{dx} \right) dx \\
 & = 2 \int \frac{1}{u} du \\
 & = 2 \ln |u| + c \\
 & = 2 \ln |x^3 - x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \text{Let } u = 5x - x^2 \quad \therefore \quad \frac{du}{dx} = 5 - 2x \\
 & \therefore \quad \int \frac{4x - 10}{5x - x^2} dx \\
 & = \int \frac{1}{5x - x^2} (4x - 10) dx \\
 & = \int \frac{1}{u} \left(-2 \frac{du}{dx} \right) dx \\
 & = -2 \int \frac{1}{u} dx \\
 & = -2 \ln |u| + c \\
 & = -2 \ln |5x - x^2| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \text{Let } u = x^3 - 3x \quad \therefore \quad \frac{du}{dx} = 3x^2 - 3 \\
 & \therefore \quad \int \frac{1 - x^2}{x^3 - 3x} dx \\
 & = \int \frac{1}{x^3 - 3x} (1 - x^2) dx \\
 & = \int \frac{1}{u} \left(-\frac{1}{3} \frac{du}{dx} \right) dx \\
 & = -\frac{1}{3} \int \frac{1}{u} du \\
 & = -\frac{1}{3} \ln |u| + c \\
 & = -\frac{1}{3} \ln |x^3 - 3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & f(x) = \int x^2(3 - x^3)^2 dx = \int u^2 \left(-\frac{1}{3} \frac{du}{dx} \right) dx \quad \{ \text{letting } u = 3 - x^3 \therefore \frac{du}{dx} = -3x^2 \} \\
 & = -\frac{1}{3} \int u^2 du \\
 & = -\frac{1}{3} \times \frac{u^3}{3} + c \\
 & = -\frac{1}{9} (3 - x^3)^3 + c
 \end{aligned}$$

$$\mathbf{b} \quad \text{Let } u = x^2 - 2 \quad \therefore \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \therefore f(x) &= \int \frac{3x}{x^2 - 2} dx \\ &= \int \frac{1}{x^2 - 2} (3x) dx \\ &= \int \frac{1}{u} \left(\frac{3}{2} \frac{du}{dx} \right) dx \\ &= \frac{3}{2} \int \frac{1}{u} du \\ &= \frac{3}{2} \ln |u| + c \\ &= \frac{3}{2} \ln |x^2 - 2| + c \end{aligned}$$

$$\mathbf{d} \quad \text{Let } u = 1 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$$

$$\begin{aligned} \therefore f(x) &= \int x e^{1-x^2} dx \\ &= \int e^u \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + c \\ &= -\frac{1}{2} e^{1-x^2} + c \end{aligned}$$

$$\mathbf{f} \quad \text{Let } u = \ln x \quad \therefore \quad \frac{du}{dx} = \frac{1}{x}$$

$$\begin{aligned} \therefore f(x) &= \int \frac{(\ln x)^3}{x} dx \\ &= \int u^3 \frac{du}{dx} dx \\ &= \int u^3 du \\ &= \frac{u^4}{4} + c \\ &= \frac{1}{4} (\ln x)^4 + c \end{aligned}$$

$$\mathbf{h} \quad \text{Let } u = \ln x \quad \therefore \quad \frac{du}{dx} = \frac{1}{x}$$

$$\begin{aligned} \therefore f(x) &= \int \frac{4}{x \ln x} dx \\ &= \int u^{-1} \left(4 \frac{du}{dx} \right) dx \\ &= 4 \int \frac{1}{u} du \\ &= 4 \ln |u| + c \\ &= 4 \ln |\ln x| + c \end{aligned}$$

$$\mathbf{c} \quad \text{Let } u = 1 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$$

$$\begin{aligned} \therefore f(x) &= \int x \sqrt{1 - x^2} dx \\ &= \int \sqrt{u} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int u^{\frac{1}{2}} du \\ &= -\frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= -\frac{1}{3} u^{\frac{3}{2}} + c \\ &= -\frac{1}{3} (1 - x^2)^{\frac{3}{2}} + c \end{aligned}$$

$$\mathbf{e} \quad \text{Let } u = x^3 - x \quad \therefore \quad \frac{du}{dx} = 3x^2 - 1$$

$$\begin{aligned} \therefore f(x) &= \int \frac{1 - 3x^2}{x^3 - x} dx \\ &= \int \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\ &= - \int \frac{1}{u} du \\ &= - \ln |u| + c \\ &= - \ln |x^3 - x| + c \end{aligned}$$

$$\mathbf{g} \quad \text{Let } u = x^3 + 2x^2 - 1 \quad \therefore \quad \frac{du}{dx} = 3x^2 + 4x$$

$$\begin{aligned} \therefore f(x) &= \int \frac{4x + 3x^2}{x^3 + 2x^2 - 1} dx \\ &= \int \frac{1}{u} \frac{du}{dx} dx \\ &= \int \frac{1}{u} du \\ &= \ln |u| + c \\ &= \ln |x^3 + 2x^2 - 1| + c \end{aligned}$$

$$\mathbf{i} \quad \text{Let } u = \ln x \quad \therefore \quad \frac{du}{dx} = \frac{1}{x}$$

$$\begin{aligned} \therefore f(x) &= \int \frac{1}{x(\ln x)^2} dx \\ &= \int u^{-2} \frac{du}{dx} dx \\ &= \int u^{-2} du \\ &= \frac{u^{-1}}{-1} + c \\ &= -\frac{1}{\ln x} + c \end{aligned}$$

EXERCISE 26H

1 $v(t) = 1 - 2t \text{ cms}^{-1}, \quad t \geq 0$

$v(t) = s'(t) = 1 - 2t$ which has sign diagram:

$\therefore$ a direction reversal occurs at $t = \frac{1}{2}$.

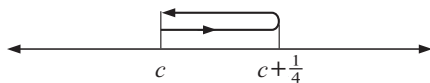


Now $s(t) = \int (1 - 2t) dt = t - \frac{2t^2}{2} + c = t - t^2 + c$

$\therefore s(0) = c$

and $s(\frac{1}{2}) = \frac{1}{4} + c \quad \therefore$ motion diagram is:

and $s(1) = c$



$\therefore$ total distance travelled $= (c + \frac{1}{4} - c) + (c + \frac{1}{4} - c)$
 $= \frac{1}{2} \text{ cm}$

2 $v(t) = t^2 - t - 2 \text{ cms}^{-1}, \quad t \geq 0$

$v(t) = s'(t) = t^2 - t - 2$

$= (t - 2)(t + 1)$ which has sign diagram:

$\therefore$ a direction reversal occurs at $t = 2$.

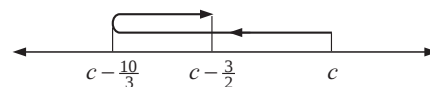


Now $s(t) = \int (t^2 - t - 2) dt = \frac{t^3}{3} - \frac{t^2}{2} - 2t + c$

$s(0) = c$

$s(2) = c - \frac{10}{3} \quad \therefore$ motion diagram is:

$s(3) = c - \frac{3}{2}$



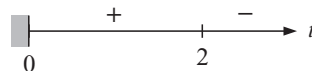
$\therefore$ total distance travelled $= (c - [c - \frac{10}{3}]) + (c - \frac{3}{2} - [c - \frac{10}{3}])$
 $= \frac{10}{3} - \frac{3}{2} + \frac{10}{3}$
 $= \frac{31}{6}$
 $= 5\frac{1}{6} \text{ cm}$

3 $x'(t) = 16t - 4t^3 \text{ units s}^{-1}, \quad t \geq 0$

$= 4t(4 - t^2)$

$= 4t(2 + t)(2 - t)$ which has sign diagram:

$\therefore$ a direction reversal occurs at $t = 2$.



Now $x(t) = \int (16t - 4t^3) dt = 8t^2 - t^4 + c$

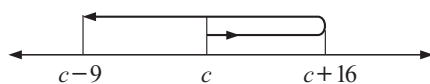
a $0 \leq t \leq 3$

$\therefore$ motion diagram is:

$x(0) = c$

$x(2) = 32 - 16 + c = c + 16$

$x(3) = 72 - 81 + c = c - 9$



$\therefore$ total distance travelled $= (c + 16 - c) + (c + 16 - [c - 9])$
 $= 41 \text{ units}$

b $1 \leq t \leq 3$

$x(1) = 7 + c = c + 7 \quad \therefore$ motion diagram is:



$\therefore$ total distance travelled $= (c + 16 - [c + 7]) + (c + 16 - [c - 9])$
 $= 34 \text{ units}$

4 $v(t) = 50 - 10e^{-0.5t} \text{ ms}^{-1} \quad t \geq 0$

a $v(0) = 50 - \frac{10}{e^0} = 50 - 10 = 40 \text{ ms}^{-1}$

b $v(3) = 50 - \frac{10}{e^{1.5}} \div 47.77 \text{ ms}^{-1}$

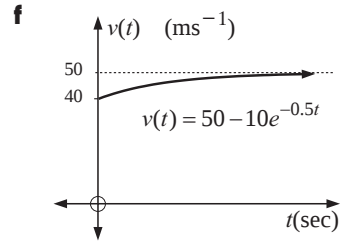
c The velocity reaches 45 ms^{-1} when

$$\begin{aligned} 45 &= 50 - 10e^{-0.5t} \\ \therefore 10e^{-\frac{t}{2}} &= 5 \\ \therefore e^{-\frac{t}{2}} &= \frac{1}{2} \\ \therefore e^{\frac{t}{2}} &= 2 \\ \therefore \frac{t}{2} &= \ln 2 \\ \therefore t &= 2 \ln 2 \div 1.386 \text{ sec} \end{aligned}$$

d $v(t) = 50 - \frac{10}{e^{\frac{t}{2}}}$, $\therefore$ as $t \rightarrow \infty$, $\frac{10}{e^{\frac{t}{2}}} \rightarrow +0$ and so $v(t) \rightarrow 50 \text{ ms}^{-1}$ (below)

e $a(t) = v'(t)$
 $= -10e^{-0.5t}(-0.5)$
 $= 5e^{-0.5t} \text{ ms}^{-2}$
 $= \frac{5}{e^{0.5t}} \text{ ms}^{-2}$

$\therefore a(t) > 0$ for all t $\{e^x > 0 \text{ for all } x\}$
 i.e., the acceleration is always positive



g $s(t) = \int (50 - 10e^{-0.5t}) dt = 50t - 10 \left(\frac{1}{-0.5} \right) e^{-0.5t} + c = 50t + 20e^{-0.5t} + c$

$s(0) = 0 + 20 + c = 20 + c$ and $s(3) = 150 + 20e^{-1.5} + c$

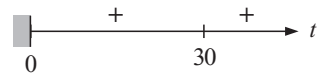
and from the graph in **f** it can be seen that the particle does not change direction,

$\therefore$ distance travelled $= (150 + 20e^{-\frac{3}{2}} + c) - (20 + c)$
 $= 130 + 20e^{-\frac{3}{2}}$ or $\div 134.5 \text{ m}$

5 $a(t) = \frac{t}{10} - 3 \text{ ms}^{-2} \therefore v(t) = \int \left(\frac{t}{10} - 3 \right) dt = \frac{t^2}{20} - 3t + c$

But $v(0) = 45 \therefore c = 45$

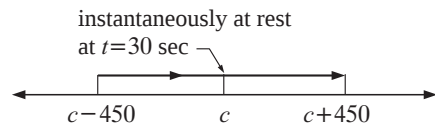
Now $v(t) = \frac{t^2}{20} - 3t + 45$
 $= \frac{t^2 - 60t + 900}{20}$
 $= \frac{(t - 30)^2}{20}$ which has sign diagram:



$\therefore$ the train only stops and does not change direction at $t = 30$ seconds

Now $s(t) = \int \frac{(t - 30)^2}{20} dt$
 $= \frac{1}{20} \frac{(t - 30)^3}{3} + c$
 $= \frac{1}{60} (t - 30)^3 + c$

the motion diagram is:



$\therefore s(0) = -\frac{27,000}{60} + c = c - 450$

and $s(60) = \frac{27,000}{60} + c = c + 450$,

$\therefore$ the total distance travelled $= c + 450 - (c - 450) = 900 \text{ m}$

$$6 \quad a(t) = 4e^{-\frac{t}{20}} \text{ ms}^{-2}$$

$$\therefore v(t) = \int 4e^{-\frac{t}{20}} dt = 4 \frac{1}{-\frac{1}{20}} e^{-\frac{t}{20}} + c = -80e^{-\frac{t}{20}} + c$$

$$\text{Now } v(0) = 20 \text{ ms}^{-1} \quad \therefore c = 100$$

$$\therefore v(t) = 100 - 80e^{-\frac{t}{20}}$$

a as $t \rightarrow \infty$, $e^{-\frac{t}{20}} \rightarrow 0$ (above) $\therefore v(t) \rightarrow 100$ (below)

i.e., the body approaches a limiting velocity of 100 ms^{-1}

b Now for all $t > 0$, $v(t) > 0$

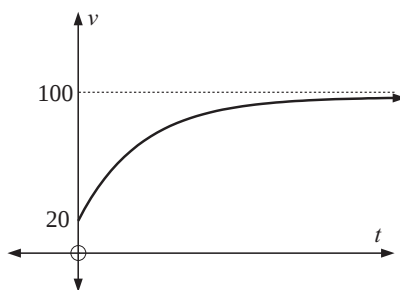
$\therefore$ no change in direction occurs.

$$\begin{aligned} s(t) &= \int \left(100 - 80e^{-\frac{t}{20}} \right) dt \\ &= 100t - 80 \frac{1}{-\frac{1}{20}} e^{-\frac{t}{20}} + c \\ &= 100t + 1600e^{-\frac{t}{20}} + c \end{aligned}$$

$$s(0) = 0 + 1600 + c = 1600 + c$$

$$s(10) = 1000 + 1600e^{-\frac{1}{2}} + c$$

$$\therefore \text{distance travelled} = (1000 + 1600e^{-\frac{1}{2}} + c) - (1600 + c) \div 370.4 \text{ m}$$



EXERCISE 26I

$$\begin{aligned} 1 \quad \mathbf{a} \quad & \int_0^1 x^3 dx \\ &= \left[\frac{x^4}{4} \right]_0^1 \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int_0^2 (x^2 - x) dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^2 \\ &= \left(\frac{8}{3} - 2 \right) - (0 - 0) \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int_0^1 e^x dx \\ &= [e^x]_0^1 \\ &= e^1 - e^0 \\ &= e - 1 \\ &\div 1.718 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int_1^4 \left(x - \frac{3}{\sqrt{x}} \right) dx \\ &= \int_1^4 \left(x - 3x^{-\frac{1}{2}} \right) dx \\ &= \left[\frac{x^2}{2} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 \\ &= \left[\frac{x^2}{2} - 6\sqrt{x} \right]_1^4 \\ &= \left[\frac{16}{2} - 12 \right] - \left(\frac{1}{2} - 6 \right) \\ &= 1\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int_4^9 \frac{x-3}{\sqrt{x}} dx \\ &= \int_4^9 \left(x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} \right) dx \\ &= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_4^9 \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} \right]_4^9 \\ &= \left[\frac{2}{3}(27) - 6(3) \right] - \left[\frac{2}{3}(8) - 6(2) \right] \\ &= (18 - 18) - \left(\frac{16}{3} - 12 \right) \\ &= 6\frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int_1^3 \frac{1}{x} dx \\ &= [\ln|x|]_1^3 \\ &= \ln 3 - \ln 1 \\ &= \ln 3 - 0 \\ &= \ln 3 \\ &\div 1.099 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int_1^2 (e^{-x} + 1)^2 dx \\
 &= \int_1^2 (e^{-2x} + 2e^{-x} + 1) dx \\
 &= \left[\left(-\frac{1}{2}\right)e^{-2x} + 2\left(-\frac{1}{1}\right)e^{-x} + x \right]_1^2 \\
 &= \left[-\frac{e^{-2x}}{2} - 2e^{-x} + x \right]_1^2 \\
 &= \left(-\frac{e^{-4}}{2} - 2e^{-2} + 2 \right) - \left(-\frac{e^{-2}}{2} - 2e^{-1} + 1 \right) \\
 &\doteq 1.524
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int_2^6 \frac{1}{\sqrt{2x-3}} dx \\
 &= \int_2^6 (2x-3)^{-\frac{1}{2}} dx \\
 &= \left[\frac{1}{\frac{1}{2}} \frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^6 \\
 &= [\sqrt{2x-3}]_2^6 \\
 &= \sqrt{9} - \sqrt{1} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int_0^1 e^{1-x} dx \\
 &= \left[\left(-\frac{1}{1}\right)e^{1-x} \right]_0^1 \\
 &= \left(\frac{e^0}{-1} \right) - \left(\frac{e^1}{-1} \right) \\
 &= -1 + e \\
 &\doteq 1.718
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \ln \int_1^2 \frac{x}{(x^2+2)^2} dx \qquad \therefore \int_1^2 \frac{x}{(x^2+2)^2} dx = \int_1^2 u^{-2} \left(\frac{1}{2} \frac{du}{dx} \right) dx \\
 & \text{we let } u = x^2 + 2 \\
 & \therefore \frac{du}{dx} = 2x \\
 & \text{and when } x = 1, \quad u = 3 \\
 & \quad \text{when } x = 2, \quad u = 6 \\
 & \qquad \qquad \qquad = \frac{1}{2} \int_3^6 u^{-2} du \\
 & \qquad \qquad \qquad = \frac{1}{2} \left[\frac{u^{-1}}{-1} \right]_3^6 \\
 & \qquad \qquad \qquad = \frac{1}{2} \left[-\frac{1}{6} - \left(-\frac{1}{3}\right) \right] \\
 & \qquad \qquad \qquad = \frac{1}{2} \left(\frac{1}{6} \right) \\
 & \qquad \qquad \qquad = \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \ln \int_0^1 x^2 e^{x^3+1} dx \qquad \therefore \int_0^1 x^2 e^{x^3+1} dx = \int_0^1 e^u \left(\frac{1}{3} \frac{du}{dx} \right) dx \\
 & \text{we let } u = x^3 + 1 \\
 & \therefore \frac{du}{dx} = 3x^2 \\
 & \text{and when } x = 0, \quad u = 1 \\
 & \quad \text{when } x = 1, \quad u = 2 \\
 & \qquad \qquad \qquad = \frac{1}{3} \int_1^2 e^u du \\
 & \qquad \qquad \qquad = \frac{1}{3} [e^u]_1^2 \\
 & \qquad \qquad \qquad = \frac{1}{3} (e^2 - e) \quad (\doteq 1.557)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \ln \int_0^3 x\sqrt{x^2+16} dx \qquad \therefore \int_0^3 x\sqrt{x^2+16} dx = \int_0^3 u^{\frac{1}{2}} \left(\frac{1}{2} \frac{du}{dx} \right) dx \\
 & \text{we let } u = x^2 + 16 \\
 & \therefore \frac{du}{dx} = 2x \\
 & \text{and when } x = 0, \quad u = 16 \\
 & \quad \text{when } x = 3, \quad u = 25 \\
 & \qquad \qquad \qquad = \frac{1}{2} \int_{16}^{25} u^{\frac{1}{2}} du \\
 & \qquad \qquad \qquad = \frac{1}{2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{16}^{25} \\
 & \qquad \qquad \qquad = \frac{1}{2} \times \frac{2}{3} \left[u^{\frac{3}{2}} \right]_{16}^{25} \\
 & \qquad \qquad \qquad = \frac{1}{3} (125 - 64) = 20\frac{1}{3}
 \end{aligned}$$

- d** In $\int_1^2 x e^{-2x^2} dx$ $\therefore \int_1^2 x e^{-2x^2} dx = \int_1^2 e^u \left(-\frac{1}{4} \frac{du}{dx}\right) dx$
 we let $u = -2x^2$ $= -\frac{1}{4} \int_{-2}^{-8} e^u du$
 $\therefore \frac{du}{dx} = -4x$ $= -\frac{1}{4} [e^u]_{-2}^{-8}$
 and when $x = 1$, $u = -2$ $= -\frac{1}{4}(e^{-8} - e^{-2})$
 when $x = 2$, $u = -8$ $\div 0.0337$
- e** In $\int_2^3 \frac{x}{2-x^2} dx$ $\therefore \int_2^3 \frac{x}{2-x^2} = \int_2^3 \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx}\right) dx$
 we let $u = 2-x^2$ $= -\frac{1}{2} \int_{-2}^{-7} \frac{1}{u} du$
 $\therefore \frac{du}{dx} = -2x$ $= -\frac{1}{2} [\ln |u|]_{-2}^{-7}$
 and when $x = 2$, $u = -2$ $= -\frac{1}{2}(\ln 7 - \ln 2)$
 when $x = 3$, $u = -7$ $= -\frac{1}{2} \ln\left(\frac{7}{2}\right) \quad (\div -0.6264)$
- f** In $\int_1^2 \frac{\ln x}{x} dx$ $\therefore \int_1^2 \frac{\ln x}{x} dx = \int_1^2 u \frac{du}{dx} dx$
 we let $u = \ln x$ $= \int_0^{\ln 2} u du$
 $\therefore \frac{du}{dx} = \frac{1}{x}$ $= \left[\frac{u^2}{2}\right]_0^{\ln 2}$
 and when $x = 1$, $u = 0$ $= \frac{(\ln 2)^2}{2} - 0$
 when $x = 2$, $u = \ln 2$ $\div 0.2402$
- g** In $\int_0^1 \frac{1-3x^2}{1-x^3+x} dx$ we let $u = 1-x^3+x$
 $\therefore \frac{du}{dx} = -3x^2+1$ and when $x = 0$, $u = 1$
 when $x = 1$, $u = 1$
 $\therefore \int_0^1 \frac{1-3x^2}{1-x^3+x} dx = \int_0^1 \frac{1}{u} \frac{du}{dx} dx$
 $= \int_1^1 \frac{1}{u} du$
 $= 0 \quad \{\text{since limits of integration are the same}\}$
- h** In $\int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x} dx$ we let $u = x^3-x^2+2x$
 $\therefore \frac{du}{dx} = 3x^2-2x+2$ and when $x = 2$, $u = 8$
 when $x = 4$, $u = 56$
 $\therefore \int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x} dx = \int_2^4 \frac{1}{u} \left(2 \frac{du}{dx}\right) dx$
 $= 2 \int_8^{56} \frac{1}{u} du$
 $= 2 [\ln |u|]_8^{56}$
 $= 2(\ln 56 - \ln 8)$
 $= 2 \ln 7 \quad (\div 3.892)$

i In $\int_0^1 (x^2 + 2x)^n (x + 1) dx$ we let $u = x^2 + 2x$

$$\therefore \frac{du}{dx} = 2x + 2 \quad \text{and} \quad \begin{array}{ll} \text{when } x = 0, & u = 0 \\ \text{when } x = 1, & u = 3 \end{array}$$

$$\therefore \int_0^1 (x^2 + 2x)^n (x + 1) dx = \int_0^1 u^n \left(\frac{1}{2} \frac{du}{dx}\right) dx = \frac{1}{2} \int_0^3 u^n du$$

$$\text{Now if } n \neq -1, \text{ the integral} = \frac{1}{2} \left[\frac{u^{n+1}}{n+1} \right]_0^3 = \frac{1}{2} \left(\frac{3^{n+1}}{n+1} \right)$$

$$\text{but if } n = -1, \text{ the integral} = \frac{1}{2} \int_0^3 \frac{1}{u} du = \frac{1}{2} [\ln |u|]_0^3 \quad \text{which is undefined as } \ln 0 \text{ is not defined}$$

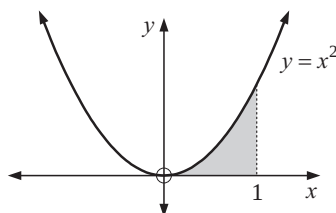
3

$$\begin{aligned} \frac{3}{x+4} - \frac{2}{x-1} &= \frac{3(x-1) - 2(x+4)}{(x+4)(x-1)} \\ &= \frac{3x - 3 - 2x - 8}{x^2 + 3x - 4} \\ &= \frac{x - 11}{x^2 + 3x - 4} \end{aligned}$$

$$\begin{aligned} \therefore \int_{-2}^{-1} \frac{x - 11}{x^2 + 3x - 4} dx &= \int_{-2}^{-1} \left(\frac{3}{x+4} - \frac{2}{x-1} \right) dx \\ &= [3 \ln |x+4| - 2 \ln |x-1|]_{-2}^{-1} \\ &= (3 \ln 3 - 2 \ln 2) - (3 \ln 2 - 2 \ln 3) \\ &= 3 \ln 3 - 2 \ln 2 - 3 \ln 2 + 2 \ln 3 \\ &= 5 \ln 3 - 5 \ln 2 \\ &= 5 \ln \left(\frac{3}{2} \right) \end{aligned}$$

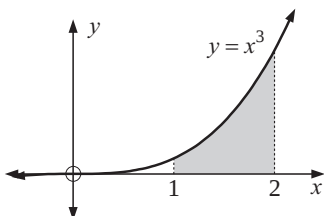
EXERCISE 26J

1 a



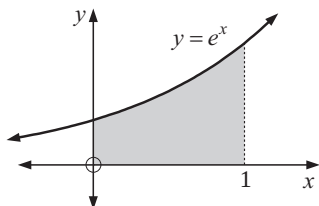
$$\begin{aligned} \text{Area} &= \int_0^1 x^2 dx \\ &= \left[\frac{x^3}{3} \right]_0^1 \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3} \text{ units}^2 \end{aligned}$$

b

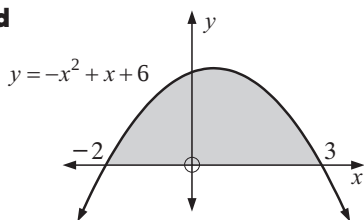


$$\begin{aligned} \text{Area} &= \int_1^2 x^3 dx \\ &= \left[\frac{x^4}{4} \right]_1^2 \\ &= \frac{16}{4} - \frac{1}{4} \\ &= 3\frac{3}{4} \text{ units}^2 \end{aligned}$$

c



$$\begin{aligned} \text{Area} &= \int_0^1 e^x dx \\ &= [e^x]_0^1 \\ &= e - 1 \\ &\doteq 1.718 \text{ units}^2 \end{aligned}$$

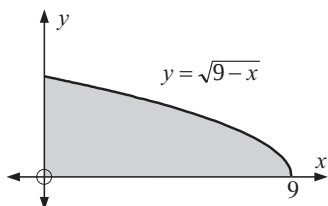
dCuts the x -axis at $y = 0 \quad \therefore 6 + x - x^2 = 0$

$$\therefore (3 - x)(2 + x) = 0$$

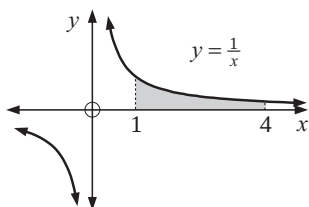
$$\therefore x = 3 \text{ or } -2$$

i.e., x -intercepts are 3 and -2

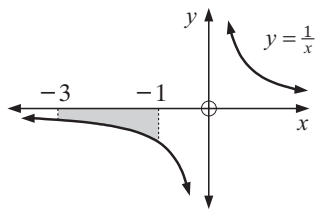
$$\begin{aligned} \text{Area} &= \int_{-2}^3 (6 + x - x^2) dx \\ &= \left[6x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^3 \\ &= \left(18 + \frac{9}{2} - 9 \right) - \left(-12 + 2 + \frac{8}{3} \right) \\ &= 20\frac{5}{6} \text{ units}^2 \end{aligned}$$

e

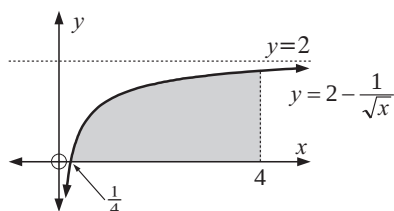
$$\begin{aligned} \text{Area} &= \int_0^9 (9 - x)^{\frac{1}{2}} dx \\ &= \left[\left(\frac{1}{-\frac{1}{2}} \right) \frac{(9 - x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^9 \\ &= -\frac{2}{3} \left[(9 - x)^{\frac{3}{2}} \right]_0^9 \\ &= -\frac{2}{3} [0 - 27] \\ &= 18 \text{ units}^2 \end{aligned}$$

f

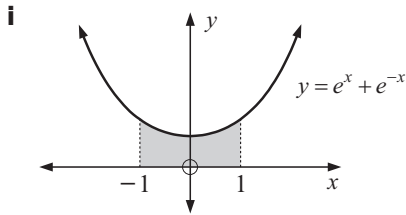
$$\begin{aligned} \text{Area} &= \int_1^4 \frac{1}{x} dx \\ &= [\ln |x|]_1^4 \\ &= \ln 4 - \ln 1 \\ &= \ln 4 - 0 \\ &\div 1.386 \text{ units}^2 \end{aligned}$$

g As the curve lies below the x -axis, the integral will be negative, but we want to find the area, which is positive.

$$\begin{aligned} \text{Area} &= - \int_{-3}^{-1} \frac{1}{x} dx \\ &= - [\ln |x|]_{-3}^{-1} \\ &= -(\ln 1 - \ln 3) \\ &= 0 + \ln 3 \\ &\div 1.099 \text{ units}^2 \end{aligned}$$

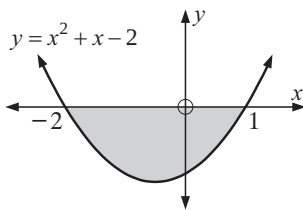
h

$$\begin{aligned} \text{Area} &= \int_{\frac{1}{4}}^4 (2 - x^{-\frac{1}{2}}) dx \\ &= \left[2x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_{\frac{1}{4}}^4 \\ &= [2x - 2\sqrt{x}]_{\frac{1}{4}}^4 \\ &= (8 - 4) - \left(\frac{1}{2} - 1 \right) \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$



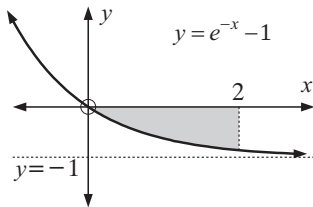
$$\begin{aligned}
 \text{Area} &= \int_{-1}^1 (e^x + e^{-x}) \, dx \\
 &= [e^x - e^{-x}]_{-1}^1 \\
 &= (e - e^{-1}) - (e^{-1} - e) \\
 &= 2e - \frac{2}{e} \\
 &\doteq 4.701 \text{ units}^2
 \end{aligned}$$

- 2 a** The curve cuts the x -axis when $y = 0$ $\therefore x^2 + x - 2 = 0$
 $\therefore (x + 2)(x - 1) = 0$
 $\therefore x = -2 \text{ or } 1$
 i.e., x -intercepts are -2 and 1



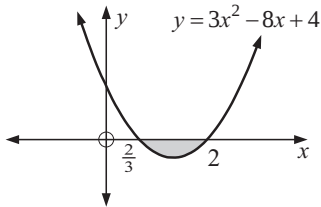
$$\begin{aligned}
 \text{Area} &= \int_{-2}^1 [0 - (x^2 + x - 2)] \, dx \\
 &= \int_{-2}^1 (-x^2 - x + 2) \, dx \\
 &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\
 &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\
 &= 4\frac{1}{2} \text{ units}^2
 \end{aligned}$$

- b** The curve cuts the x -axis at $(0, 0)$.



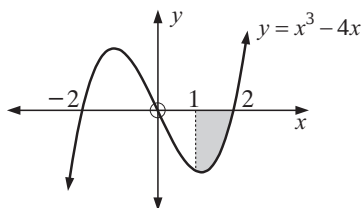
$$\begin{aligned}
 \text{Area} &= \int_0^2 [0 - (e^{-x} - 1)] \, dx \\
 &= \int_0^2 (1 - e^{-x}) \, dx \\
 &= [x + e^{-x}]_0^2 \\
 &= \left(2 + \frac{1}{e^2} \right) - (0 + e^0) \\
 &= 1 + \frac{1}{e^2} \quad (\doteq 1.135 \text{ units}^2)
 \end{aligned}$$

- c** The curve cuts the x -axis when $y = 0$ $\therefore 3x^2 - 8x + 4 = 0$
 $\therefore (3x - 2)(x - 2) = 0$
 $\therefore x = \frac{2}{3} \text{ or } 2$
 i.e., x -intercepts are $\frac{2}{3}$ and 2



$$\begin{aligned}
 \text{Area} &= \int_{\frac{2}{3}}^2 [0 - (3x^2 - 8x + 4)] \, dx \\
 &= \int_{\frac{2}{3}}^2 (-3x^2 + 8x - 4) \, dx \\
 &= \left[-x^3 + 4x^2 - 4x \right]_{\frac{2}{3}}^2 \\
 &= (-8 + 16 - 8) - \left(-\frac{8}{27} + \frac{16}{3} - \frac{8}{3} \right) \\
 &= 1\frac{5}{27} \text{ units}^2
 \end{aligned}$$

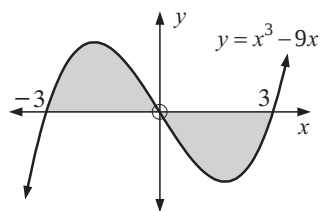
- d** The curve cuts the x -axis when $y = 0$ $\therefore x^3 - 4x = 0$
 $\therefore x(x^2 - 4) = 0$
 $\therefore x(x + 2)(x - 2) = 0$
 i.e., x -intercepts are 0 and ± 2



$$\begin{aligned}
 \text{Area} &= \int_1^2 [0 - f(x)] \, dx \\
 &= \int_1^2 (-x^3 + 4x) \, dx \\
 &= \left[-\frac{x^4}{4} + 2x^2 \right]_1^2 \\
 &= (-4 + 8) - \left(-\frac{1}{4} + 2\right) \\
 &= 2\frac{1}{4} \text{ units}^2
 \end{aligned}$$

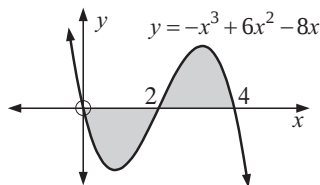
- 3 a** $f(x) = x^3 - 9x$
 $= x(x^2 - 9)$
 $= x(x+3)(x-3)$
 $\therefore y = f(x)$ cuts the x -axis at $0, \pm 3$

$$\begin{aligned}
 \text{Area} &= \int_{-3}^0 (x^3 - 9x) \, dx + \int_0^3 [0 - (x^3 - 9x)] \, dx \\
 &= \left[\frac{x^4}{4} - \frac{9x^2}{2} \right]_{-3}^0 + \left[-\frac{x^4}{4} + \frac{9x^2}{2} \right]_0^3 \\
 &= (0 - [\frac{81}{4} - \frac{81}{2}]) + ([-\frac{81}{4} + \frac{81}{2}] - 0) \\
 &= 40\frac{1}{2} \text{ units}^2
 \end{aligned}$$



- b** $f(x) = -x(x-2)(x-4)$
 $= -x^3 + 6x^2 - 8x$
 $\therefore y = f(x)$ cuts the x -axis at $0, 2$ and 4

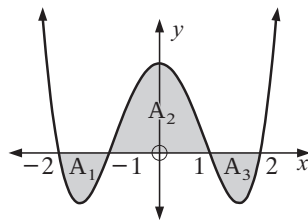
$$\begin{aligned}
 \text{Area} &= \int_0^2 [0 - (-x^3 + 6x^2 - 8x)] \, dx + \int_2^4 [(-x^3 + 6x^2 - 8x) - 0] \, dx \\
 &= \int_0^2 (x^3 - 6x^2 + 8x) \, dx + \int_2^4 (-x^3 + 6x^2 - 8x) \, dx \\
 &= \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 + \left[-\frac{x^4}{4} + 2x^3 - 4x^2 \right]_2^4 \\
 &= ([4 - 16 + 16] - 0) + ([-64 + 128 - 64] - [-4 + 16 - 16]) \\
 &= 8 \text{ units}^2
 \end{aligned}$$



- c** $f(x) = x^4 - 5x^2 + 4$
 $= (x^2 - 1)(x^2 - 4)$
 $= (x+1)(x-1)(x+2)(x-2)$
 $\therefore y = f(x)$ cuts the x -axis at $\pm 1, \pm 2$

$$\begin{aligned}
 A_1 &= \int_{-2}^{-1} [0 - (x^4 - 5x^2 + 4)] \, dx \\
 &= \int_{-2}^{-1} (-x^4 + 5x^2 - 4) \, dx \\
 &= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_{-2}^{-1} \\
 &= \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(\frac{32}{5} - \frac{40}{3} + 8\right) \\
 &= \frac{22}{15} \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 A_2 &= \int_{-1}^1 (x^4 - 5x^2 + 4) \, dx \\
 &= \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_{-1}^1 \\
 &= \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right) \\
 &= \frac{76}{15} \text{ units}^2
 \end{aligned}$$

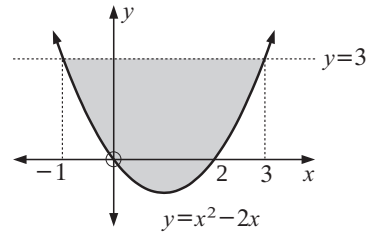


Now by symmetry, $A_3 = A_1$ $\therefore A = \frac{22}{15} + \frac{76}{15} + \frac{22}{15} = \frac{120}{15} = 8 \text{ units}^2$

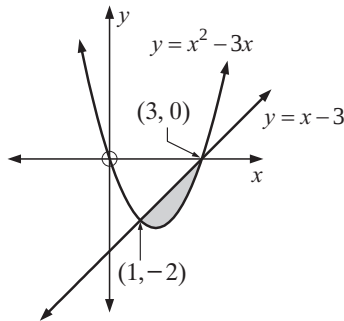
- 4 a** $y = x^2 - 2x$ meets $y = 3$ where

$$\begin{aligned} x^2 - 2x &= 3 \\ \therefore x^2 - 2x - 3 &= 0 \\ \therefore (x-3)(x+1) &= 0 \\ \therefore x &= 3 \text{ and } x = -1 \end{aligned}$$

$$\begin{aligned} A &= \int_{-1}^3 [3 - (x^2 - 2x)] dx \\ &= \int_{-1}^3 (3 + 2x - x^2) dx \\ &= \left[3x + x^2 - \frac{x^3}{3} \right]_{-1}^3 \\ &= (9 + 9 - 9) - (-3 + 1 + \frac{1}{3}) \\ &= 10\frac{2}{3} \text{ units}^2 \end{aligned}$$



- b i**



- ii** $y = x - 3$ meets $y = x^2 - 3x$

where $x = 3$ i.e., $(3, 0)$
and where $x = 1$ i.e., $(1, -2)$

Checking algebraically:

the graphs meet where $x - 3 = x^2 - 3x$

$$\therefore x^2 - 3x - x + 3 = 0$$

$$\therefore x^2 - 4x + 3 = 0$$

$$\therefore (x-1)(x-3) = 0$$

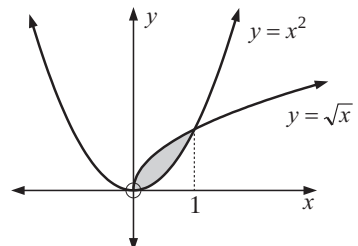
$$\therefore x = 1 \text{ or } 3 \quad \checkmark$$

$$\begin{aligned} \text{iii Area} &= \int_1^3 [(x-3) - (x^2 - 3x)] dx \\ &= \int_1^3 (-3 + 4x - x^2) dx \\ &= \left[-3x + 2x^2 - \frac{x^3}{3} \right]_1^3 \\ &= (-9 + 18 - 9) - (-3 + 2 - \frac{1}{3}) \\ &= 1\frac{1}{3} \text{ units}^2 \end{aligned}$$

- c** $y = \sqrt{x}$ meets $y = x^2$ where $\sqrt{x} = x^2$
 $\therefore x = x^4$
 $\therefore x^4 - x = 0$
 $\therefore x(x^3 - 1) = 0$
 $\therefore x(x-1)(x^2 + x + 1) = 0$
 $\therefore x = 0 \text{ or } 1$

The factor $(x^2 + x + 1)$ has no real root since $\Delta = -3$ which is < 0 .

$$\begin{aligned} \text{Area} &= \int_0^1 (\sqrt{x} - x^2) dx \\ &= \int_0^1 \left(x^{\frac{1}{2}} - x^2 \right) dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \text{ unit}^2 \end{aligned}$$



d $y = e^x - 1$ meets $y = 2 - 2e^{-x}$ where

$$e^x - 1 = 2 - 2e^{-x}$$

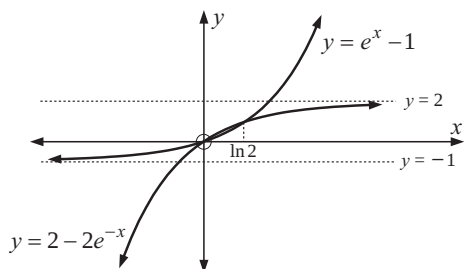
$$\therefore e^{2x} - e^x = 2e^x - 2 \quad \{\times e^x\}$$

$$\therefore e^{2x} - 3e^x + 2 = 0$$

$$\therefore (e^x - 1)(e^x - 2) = 0$$

$$\therefore e^x = 1 \text{ or } 2$$

$$\therefore x = 0 \text{ or } \ln 2$$



$$A = \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)] dx$$

$$= \int_0^{\ln 2} (3 - e^x - 2e^{-x}) dx$$

$$= [3x - e^x + 2e^{-x}]_0^{\ln 2}$$

$$= (3 \ln 2 - 2 + 1) - (0 - 1 + 2)$$

$$= 3 \ln 2 - 2$$

$$\doteq 0.0794 \text{ units}^2$$

e $y = 2e^x$ meets $y = e^{2x}$ where

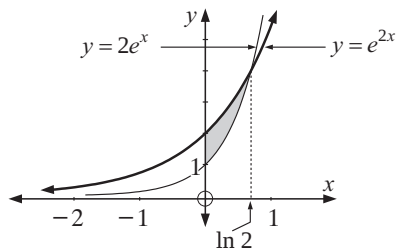
$$2e^x = e^{2x}$$

$$\therefore e^{2x} - 2e^x = 0$$

$$\therefore e^x(e^x - 2) = 0$$

$$\therefore e^x = 2 \quad \{\text{as } e^x > 0 \text{ for all } x\}$$

$$\therefore x = \ln 2$$



$$A = \int_0^{\ln 2} (2e^x - e^{2x}) dx$$

$$= [2e^x - \frac{1}{2}e^{2x}]_0^{\ln 2}$$

$$= (4 - 2) - (2 - \frac{1}{2})$$

$$= \frac{1}{2} \text{ unit}^2$$

5 $y = 2x$ meets $y^2 = 4x$ where

$$(2x)^2 = 4x$$

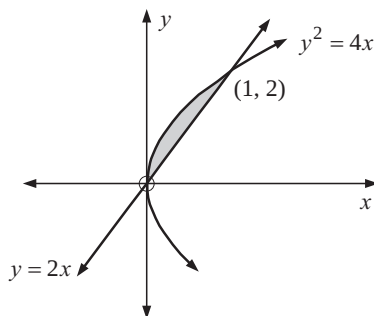
$$\therefore 4x^2 = 4x$$

$$\therefore 4x^2 - 4x = 0$$

$$\therefore 4x(x - 1) = 0$$

$$\therefore x = 0 \text{ or } 1$$

The upper part of $y^2 = 4x$
is $y = \sqrt{4x}$
i.e., $y = 2\sqrt{x}$



$$\text{Area} = \int_0^1 (2\sqrt{x} - 2x) dx$$

$$= \int_0^1 (2x^{\frac{1}{2}} - 2x) dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - x^2 \right]_0^1$$

$$= \frac{4}{3} - 1$$

$$= \frac{1}{3} \text{ unit}^2$$

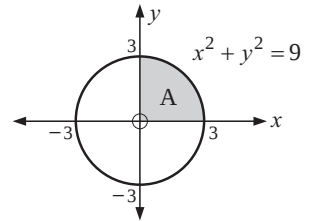
6 a Now $x^2 + y^2 = 9$

$$\therefore y^2 = 9 - x^2$$

$$y = \pm\sqrt{9 - x^2}$$

In the upper half of the circle all y -values are ≥ 0

$\therefore y = +\sqrt{9 - x^2}$ is the required equation.



b Now the shaded area above is A where $A = \int_0^3 \sqrt{9 - x^2} dx$

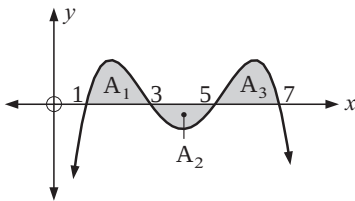
This is a quarter of the area of a circle radius 3 units $\therefore A = \frac{1}{4}(\pi \times 3^2)$

$$= \frac{9}{4}\pi$$

$$\div 7.069 \text{ units}^2$$

c The answer checks using technology.

7



a $\int_1^7 f(x) dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leq x \leq 7$.

But $f(x)$ is not positive for $3 \leq x \leq 5$,

so $\int_1^7 f(x) dx$ will give us

(Area enclosed in $1 \leq x \leq 3$)

– (Area enclosed in $3 \leq x \leq 5$)

+ (Area enclosed in $5 \leq x \leq 7$)

which is NOT the shaded area.

b shaded area $= \int_1^3 f(x) dx + \int_3^5 [0 - f(x)] dx + \int_5^7 f(x) dx$

$$= \int_1^3 f(x) dx - \int_3^5 f(x) dx + \int_5^7 f(x) dx$$

8

$$\text{Area} = \int_1^k \frac{1}{1+2x} dx = 0.2 \text{ units}^2$$

$$\therefore \left[\frac{1}{2} \ln |1+2x| \right]_1^k = 0.2$$

$$\therefore [\ln |1+2x|]_1^k = 0.4$$

$$\therefore \ln |1+2k| - \ln 3 = 0.4$$

$$\therefore \ln(1+2k) - \ln 3 = 0.4 \quad \{\text{as } k > 0, \text{ then } 1+2k > 0 \text{ also}\}$$

$$\therefore \ln \left(\frac{1+2k}{3} \right) = 0.4$$

$$\therefore \frac{1+2k}{3} = e^{0.4}$$

$$\therefore 1+2k = 3e^{0.4}$$

$$\therefore k = \frac{3e^{0.4} - 1}{2} \div 1.7377$$

9

$$\text{Area} = \int_0^b \sqrt{x} dx$$

$$\therefore \int_0^b x^{\frac{1}{2}} dx = 1$$

$$\therefore \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^b = 1$$

$$\therefore \frac{2}{3} b\sqrt{b} - 0 = 1$$

$$\therefore b\sqrt{b} = \frac{3}{2}$$

$$\therefore b^{\frac{3}{2}} = 1.5$$

$$\therefore b = (1.5)^{\frac{2}{3}} \div 1.3104$$

10 $y = x^2$ meets $y = k$ where $x^2 = k \quad \therefore \quad x = \pm\sqrt{k}$

$$\text{Now area} = \int_0^{\sqrt{k}} (k - x^2) dx \quad \therefore \quad \int_0^{\sqrt{k}} (k - x^2) dx = 2.4$$

$$\therefore \left[kx - \frac{x^3}{3} \right]_0^{\sqrt{k}} = 2.4$$

$$\therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} - 0 = 2.4$$

$$\therefore \frac{2k\sqrt{k}}{3} = 2.4$$

$$\therefore k^{\frac{3}{2}} = 3.6$$

$$\therefore k = (3.6)^{\frac{2}{3}} \div 2.3489$$

11 By symmetry, the area bounded by $x = 0$ and $x = a$ is $\frac{1}{2}(3a)$ units².

$$\therefore \int_0^a (x^2 + 2) dx = \frac{3}{2}a$$

$$\therefore \left[\frac{x^3}{3} + 2x \right]_0^a = \frac{3}{2}a$$

$$\therefore \frac{a^3}{3} + 2a - 0 = \frac{3}{2}a$$

$$\therefore 2a^3 + 12a = 9a \quad \{\times \text{ by } 6\}$$

$$\therefore 2a^3 + 3a = 0$$

$$\therefore a(2a^2 + 3) = 0$$

$$\therefore a = 0 \quad \{a^2 \geq 0 \text{ for all } a, \therefore 2a^2 + 3 > 0 \text{ for all } a\}$$

EXERCISE 26K

1 The marginal cost is $C'(x)$ and $C'(x) = 3.15 + 0.004x$ \$/gadget

$$\begin{aligned} \therefore C(x) &= \int (3.15 + 0.004x) dx \\ &= 3.15x + 0.002x^2 + c \end{aligned}$$

$$\text{But } C(0) = 450 \quad \therefore \quad c = 450$$

$$\therefore C(x) = 3.15x + 0.002x^2 + 450 \text{ dollars}$$

$$\begin{aligned} C(800) &= 3.15(800) + 0.002(800)^2 + 450 \\ &= \$4250 \end{aligned}$$

$$\therefore \text{total cost is } \$4250$$

2 The marginal cost is $C'(x)$ and $C'(x) = 10 - \frac{4}{\sqrt{x+1}}$ \$/item

$$\begin{aligned} \therefore C(x) &= \int \left(10 - 4(x+1)^{-\frac{1}{2}} \right) dx \\ &= 10x - 8(x+1)^{\frac{1}{2}} + c \\ &= 10x - 8\sqrt{x+1} + c \end{aligned}$$

$$\begin{aligned} \text{But } C(0) &= 200 \quad \therefore \quad 0 - 8 + c = 200 \\ &\quad \therefore \quad c = 208 \end{aligned}$$

$$\therefore C(x) = 10x - 8\sqrt{x+1} + 208 \text{ dollars}$$

$$\begin{aligned} C(100) &= 10(100) - 8\sqrt{101} + 208 \\ &\div \$1127.60 \end{aligned}$$

$$\therefore \text{total cost is } \$1127.60$$

- 3** The marginal cost is $\frac{dC}{dx}$ and $\frac{dC}{dx} = 25x - 4x^{0.8} + 0.0024x^2$ million dollars/aeroplane

$$\begin{aligned}\therefore C(x) &= \int (25x - 4x^{0.8} + 0.0024x^2) dx \\ &= \frac{25x^2}{2} - \frac{4}{1.8}x^{1.8} + 0.0008x^3 + c \\ &= 12.5x^2 - \frac{20}{9}x^{1.8} + 0.0008x^3 + c\end{aligned}$$

But $C(0) = 275 \quad \therefore c = 275$

$$\therefore C(x) = 12.5x^2 - \frac{20}{9}x^{1.8} + 0.0008x^3 + 275 \text{ million dollars}$$

$$\begin{aligned}C(20) &= 12.5(400) - \frac{20}{9}(20^{1.8}) + 0.0008(8000) + 275 \\ &= \$4793.2 \text{ million} \quad \text{i.e., total cost is } \$4793.2 \text{ million}\end{aligned}$$

- 4** $\frac{dS}{dx} = 10 + \frac{3}{x} + x^2$ for $2 \leq x \leq 10$

$$\begin{aligned}\therefore S(x) &= 10x + 3 \ln|x| + \frac{x^3}{3} + c \quad \therefore S(3) \doteq 42.30 + c \\ &\quad \text{and } S(6) \doteq 137.38 + c\end{aligned}$$

$$\begin{aligned}\therefore \text{total change in sales from 3 to 6} &= S(6) - S(3) \\ &\doteq 95.08 \text{ units}\end{aligned}$$

- 5 a** The marginal profit is $P'(x)$ and $P'(x) = 15 - 0.03x$ \$/plate

$$\begin{aligned}\therefore P(x) &= \int (15 - 0.03x) dx \\ &= 15x - 0.015x^2 + c\end{aligned}$$

But $P(0) = -650 \quad \therefore c = -650$

$$\therefore P(x) = 15x - 0.015x^2 - 650 \text{ dollars}$$

- b** The maximum profit occurs when $P'(x) = 0$ i.e., when $15 - 0.03x = 0$

$$\begin{aligned}\therefore 0.03x &= 15 \\ \therefore x &= \frac{15}{0.03} \\ \text{i.e., } x &= 500\end{aligned}$$

and $P''(x) = -0.03 < 0 \quad \therefore$ profit is at a maximum when $x = 500$

$$\begin{aligned}\therefore \text{maximum profit} &= P(500) \\ &= 15(500) - 0.015(500)^2 - 650 \\ &= \$3100\end{aligned}$$

- c** In order for a profit to be made $P(x)$ must be greater than 0

$$\text{i.e., } 15x - 0.015x^2 - 650 > 0$$

Using technology to graph $P(x)$ and find x intercepts we get $x_1 = 45.39$ and $x_2 = 954.6$

$$\therefore 46 \leq x \leq 954 \quad \{\text{We cannot produce part plates.}\}$$

- 6** $E(t) = 350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)$ calories/day

$$\begin{aligned}\text{Total energy needs over the first week} &= \int_0^7 E(t) dt \\ &= \int_0^7 [350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)] dt \\ &= \left[\frac{1}{0.15} \times \frac{350(80 + 0.15t)^{1.8}}{1.8} - 9600t - 9t^2 \right]_0^7 \\ &\doteq 14\,400 \text{ calories}\end{aligned}$$

7

$$\begin{aligned}\frac{dT}{dx} &= \frac{-20}{x^{0.63}} = -20x^{-0.63} \quad \therefore T = \int -20x^{-0.63} dx \\ &= \frac{-20x^{0.37}}{0.37} + c\end{aligned}$$

$$\begin{aligned}\text{But when } x = 3, T = 100, \therefore \frac{-20(3^{0.37})}{0.37} + c &= 100 \\ \therefore c &= 100 + \frac{20(3^{0.37})}{0.37} \doteq 181.1639 \\ \text{i.e., } T &= \frac{-20x^{0.37}}{0.37} + 181.1639 \\ \text{and when } x = 6, T &= -104.8925 + 181.1639 \\ &\doteq 76.27\end{aligned}$$

$\therefore$ the outer surface temperature is 76.27°C

8

$$\begin{aligned}\text{a } \frac{d^2y}{dx^2} &= -\frac{1}{10}(1-x)^2 \quad \text{and } \frac{dy}{dx} = \int \frac{d^2y}{dx^2} dx \\ &= \int -\frac{1}{10}(1-x)^2 dx \\ &= -\frac{1}{10}\left(\frac{1}{-1}\right) \times \frac{(1-x)^3}{3} + c \\ &= \frac{1}{30}(1-x)^3 + c\end{aligned}$$

$$\text{But when } x = 0, \frac{dy}{dx} = 0 \quad \{\text{horizontal tangent}\}$$

$$\therefore \frac{1}{30}(1-0)^3 + c = 0 \quad \text{and so } c = -\frac{1}{30}$$

$$\therefore \frac{dy}{dx} = \frac{1}{30}(1-x)^3 - \frac{1}{30}$$

$$\begin{aligned}\therefore y &= \int \left[\frac{1}{30}(1-x)^3 - \frac{1}{30} \right] dx \\ &= \frac{1}{30} \left(\frac{1}{-1} \right) \frac{(1-x)^4}{4} - \frac{1}{30}x + d \\ &= -\frac{(1-x)^4}{120} - \frac{x}{30} + d\end{aligned}$$

$$\text{Also, when } x = 0, y = 0$$

$$\therefore -\frac{1}{120} - 0 + d = 0 \quad \therefore d = \frac{1}{120}$$

$$\text{and so } y = \frac{1}{120} - \frac{(1-x)^4}{120} - \frac{x}{30}$$

b Maximum deflection occurs at the right hand end where $x \doteq 1$

$$\text{and at } x \doteq 1, y \doteq \frac{1}{120} - 0 - \frac{1}{30} \doteq -0.025 \text{ m}$$

i.e., maximum deflection is about 2.5 cm.

9

$$\begin{aligned}\text{a } \frac{d^2y}{dx^2} &= \frac{1}{100} \left(2x - \frac{x^2}{2} \right) = \frac{1}{50}x - \frac{1}{200}x^2 \\ \therefore \frac{dy}{dx} &= \int \left(\frac{1}{50}x - \frac{1}{200}x^2 \right) dx = \frac{1}{100}x^2 - \frac{1}{600}x^3 + c\end{aligned}$$

$$\text{The sag, } y = \int \left(\frac{1}{100}x^2 - \frac{1}{600}x^3 + c \right) dx$$

$$\text{i.e., } y = \frac{1}{300}x^3 - \frac{1}{2400}x^4 + cx + d$$

Now when $x = 0$, $y = 0 \quad \therefore \quad 0 - 0 + 0 + d = 0$

$$\therefore d = 0$$

$$\therefore y = \frac{1}{300}x^3 - \frac{1}{2400}x^4 + cx$$

Also when $x = 4$, $y = 0 \quad \therefore \quad \frac{1}{300}(4^3) - \frac{1}{2400}(4^4) + 4c = 0$

$$\therefore 4c = \frac{1}{2400}(4^4) - \frac{1}{300}(4^3)$$

$$\therefore c = \frac{1}{2400}(4^3) - \frac{1}{300}(4^2)$$

$$\therefore c = -\frac{2}{75}$$

$$\therefore y = \left(\frac{1}{300}x^3 - \frac{1}{2400}x^4 - \frac{2}{75}x\right) \text{ m}$$

- b** The maximum sag occurs when $\frac{dy}{dx} = 0$ i.e., $\frac{1}{100}x^2 - \frac{1}{600}x^3 - \frac{2}{75} = 0$
 $6x^2 - x^3 - 16 = 0$

Using technology the three solutions are: $x = -1.464$, 2 or 5.4641

But the maximum lies between 0 and 4 , $\therefore$ maximum sag occurs when $x = 2$.

$$\text{When } x = 2, \quad y = \frac{1}{300}(2^3) - \frac{1}{2400}(2^4) - \frac{2}{75}(2)$$

$$\doteq -0.03333 \text{ m}$$

$$\doteq -3.333 \text{ cm} \quad \therefore \text{ the maximum sag is } \doteq 3.33 \text{ cm}$$

- c** When 1 m from P, i.e., $x = 3 \text{ m}$,

$$y = \frac{1}{300}(3^3) - \frac{1}{2400}(3^4) - \frac{2}{75}(3)$$

$$= -0.02375 \text{ m}$$

$$= -2.375 \text{ cm} \quad \therefore \text{ the sag is } 2.375 \text{ cm}$$

- d** When 1 m from P, i.e., $x = 3 \text{ m}$, $\frac{dy}{dx} = \frac{1}{100}(3^2) - \frac{1}{600}(3^3) - \frac{2}{75} \doteq 0.0183$

i.e., the angle θ , that the plank makes with the horizontal is such that $\tan \theta \doteq 0.0183$

$$\text{i.e., } \theta = \tan^{-1}(0.0183) \doteq 1.05^\circ$$

- 10** The cost per unit volume V is $\frac{1}{2}x^2 + 4 \text{ \$/m}^3$

$$\therefore \frac{dC}{dV} = \left(\frac{1}{2}x^2 + 4\right) \text{ \$/m}^3 \quad \text{and as } V = \pi r^2 x, \quad \frac{dV}{dx} = \pi r^2$$

$$\text{Now } \frac{dC}{dx} = \frac{dC}{dV} \frac{dV}{dx} \quad \{\text{Chain rule}\}$$

$$\therefore \frac{dC}{dx} = \left(\frac{1}{2}x^2 + 4\right) \times \pi r^2$$

$$\begin{aligned} \therefore \frac{dC}{dx} &= \pi r^2 \left(\frac{1}{2}x^2 + 4\right) \quad \text{and the cost } C, \text{ of digging a well} = \int \frac{dC}{dx} dx \\ &= \int [\pi r^2 (\frac{1}{2}x^2 + 4)] dx \\ &= \pi r^2 \left(\frac{x^3}{6} + 4x\right) + c \end{aligned}$$

$$\therefore \text{ cost of digging a well } h \text{ metres deep} = \pi r^2 \left(\frac{h^3}{6} + 4h\right) + c$$

$$\text{Now if the initial cost} = C_0, \quad \pi r^2 \left(\frac{0}{6} + 0\right) + c = C_0$$

$$\therefore c = C_0$$

$$\therefore C(h) = \pi r^2 \left(\frac{h^3 + 24h}{6}\right) + C_0$$

- 11 a** The yield Y per unit area A is proportional to $\frac{1}{\sqrt{x+4}}$.

$$\text{i.e., } \frac{dY}{dA} \propto \frac{1}{\sqrt{x+4}}$$

$$\therefore \frac{dY}{dA} = k \left(\frac{1}{\sqrt{x+4}} \right) = \frac{k}{\sqrt{x+4}} \quad \text{where } k \text{ is a constant}$$

- b** Now the shaded area $A = \text{length} \times \text{width}$

$$\therefore A = (4 - 2p)x$$

$$\therefore \frac{dA}{dx} = 4 - 2p$$

$$\text{Now } \frac{dY}{dx} = \frac{dY}{dA} \frac{dA}{dx} \quad \{\text{Chain rule}\}$$

$$\therefore \frac{dY}{dx} = \frac{k}{\sqrt{x+4}} \times (4 - 2p)$$

$$\text{i.e., } \frac{dY}{dx} = \frac{k(4 - 2p)}{\sqrt{x+4}}$$

- c** $\frac{dY}{dx}$ is the instantaneous rate of change of the yield with respect to the distance x from the canal.

$$\begin{aligned} \therefore \text{total yield} = Y &= \int_0^p \frac{dY}{dx} dx \\ &= \int_0^p \frac{k(4 - 2p)}{\sqrt{x+4}} dx \end{aligned}$$

- d** $Y = k(4 - 2p) \int_0^p (x + 4)^{-\frac{1}{2}} dx$ {using **c**}

$$= k(4 - 2p) \times \left[\frac{(x + 4)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^p$$

$$= 2k(4 - 2p) \left[\sqrt{x+4} \right]_0^p$$

$$= 4k(2 - p) \left[\sqrt{p+4} - \sqrt{4} \right]$$

$$\therefore Y = 4k(2 - p) (\sqrt{p+4} - 2)$$

- e** For yield to be a maximum we need to maximise Y . Using technology to graph Y and find the maximum, we find that the maximum occurs when $p \doteq 0.9735$ km

i.e., orchard is $0.9735 \text{ km} \times 2.053 \text{ km}$

REVIEW SET 26A

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int \frac{4}{\sqrt{x}} dx &= 4 \int x^{-\frac{1}{2}} dx \\ &= 4 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 8\sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int \frac{3}{1-2x} dx &= 3 \int \frac{1}{1-2x} dx \\ &= 3 \left(\frac{1}{-2} \right) \ln |1-2x| + c \\ &= -\frac{3}{2} \ln |1-2x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int x e^{1-x^2} dx &= \int e^u \left(-\frac{1}{2} \frac{du}{dx} \right) du \quad \{\text{letting } u = 1 - x^2 \therefore \frac{du}{dx} = -2x\} \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + c \\ &= -\frac{1}{2} e^{1-x^2} + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int_{-5}^{-1} \sqrt{1-3x} \, dx \\
 &= \int_{-5}^{-1} (1-3x)^{\frac{1}{2}} \, dx \\
 &= \left[\frac{1}{-\frac{3}{2}} \times \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-5}^{-1} \\
 &= -\frac{2}{9} \left[(1-3x)^{\frac{3}{2}} \right]_{-5}^{-1} \\
 &= -\frac{2}{9} \left[4^{\frac{3}{2}} - 16^{\frac{3}{2}} \right] \\
 &= -\frac{2}{9} [8 - 64] \\
 &= 12\frac{4}{9}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad & y = \sqrt{x^2 - 4} = (x^2 - 4)^{\frac{1}{2}} \\
 \therefore \quad & \frac{dy}{dx} = \frac{1}{2} (x^2 - 4)^{-\frac{1}{2}} \times 2x \\
 &= \frac{x}{(x^2 - 4)^{\frac{1}{2}}} \\
 &= \frac{x}{\sqrt{x^2 - 4}}
 \end{aligned}$$

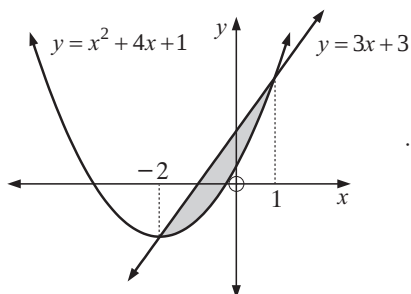
$$\begin{aligned}
 \mathbf{4} \quad v(t) &= 2t - 3t^2 \\
 &= t(2 - 3t) \quad \text{which has sign diagram:}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } s(t) &= \int (2t - 3t^2) \, dt \\
 s(t) &= t^2 - t^3 + c \text{ metres}
 \end{aligned}$$

$$\begin{aligned}
 \text{and so } s(0) &= c \\
 s\left(\frac{2}{3}\right) &= \frac{4}{9} - \frac{8}{27} + c = c + \frac{4}{27} \\
 s(1) &= 1 - 1 + c = c
 \end{aligned}$$

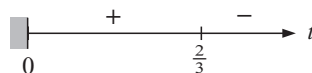
$$\therefore \text{ total distance travelled} = (c + \frac{4}{27} - c) + (c + \frac{4}{27} - c) = \frac{8}{27} \text{ m} \div 29.6 \text{ cm}$$

$$\mathbf{5} \quad y = x^2 + 4x + 1 \text{ meets } y = 3x + 3 \text{ where } x^2 + 4x + 1 = 3x + 3$$



$$\begin{aligned}
 \mathbf{b} \quad & \ln \int_0^1 \frac{4x^2}{(x^3 + 2)^3} \, dx \\
 & \text{we let } u = x^3 + 2 \\
 \therefore \quad & \frac{du}{dx} = 3x^2 \quad \text{and when } x = 0, u = 2 \\
 & \quad \quad \quad \text{when } x = 1, u = 3 \\
 \therefore \quad & \int_0^1 \frac{4x^2}{(x^3 + 2)^3} \, dx = \int_2^3 \frac{1}{u^3} \left(\frac{4}{3} \frac{du}{dx} \right) dx \\
 &= \frac{4}{3} \int_2^3 u^{-3} \, du \\
 &= \frac{4}{3} \left[\frac{u^{-2}}{-2} \right]_2^3 \\
 &= \frac{4}{3} \left[-\frac{1}{2u^2} \right]_2^3 \\
 &= \frac{4}{3} \left[\left(-\frac{1}{18}\right) - \left(-\frac{1}{8}\right) \right] \\
 &= \frac{5}{54}
 \end{aligned}$$

$$\therefore \int \frac{x}{\sqrt{x^2 - 4}} \, dx = \sqrt{x^2 - 4} + c$$



with motion diagram:



$$\begin{aligned}
 \therefore \quad & \text{area} = \int_{-2}^1 [(3x + 3) - (x^2 + 4x + 1)] \, dx \\
 &= \int_{-2}^1 (-x^2 - x + 2) \, dx \\
 &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\
 &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\
 &= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4 \\
 &= 4\frac{1}{2} \text{ units}^2
 \end{aligned}$$

6 $\frac{dI}{dt} = -\frac{100}{t^2} \quad t \geq 0.2 \text{ seconds}$

$$\therefore I(t) = \int \frac{dI}{dt} dt = \int -100t^{-2} dt = 100t^{-1} + c$$

Now $I(2) = 150$ amps

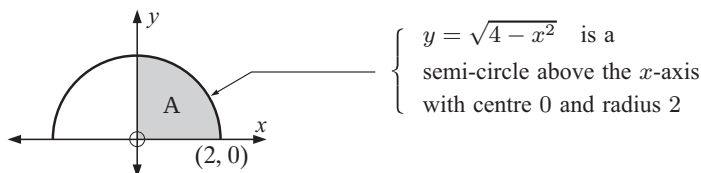
$$\therefore \frac{100}{2} + c = 150 \quad \text{and so } c = 100$$

$$\therefore I(t) = \left(\frac{100}{t} + 100 \right) \text{ amps}$$

a $I(20) = \frac{100}{20} + 100$
 $= 105$ amps

b as $t \rightarrow +\infty$
 $I(t) \rightarrow 100$ amps (above)

7



Now $\int_0^2 \sqrt{4-x^2} dx = \text{shaded area} = \frac{1}{4}$ of the area of a circle of radius 2 units
 $= \frac{1}{4}\pi(2^2)$
 $= \pi \text{ units}^2$

8 $\int_{-1}^3 f(x) dx$ gives us the correct area only if $f(x)$ is positive on the interval $-1 \leq x \leq 3$.

But $f(x)$ is not positive for $1 \leq x \leq 3$ so $\int_{-1}^3 f(x) dx$ does not provide the correct answer.

The shaded area below the x -axis is given by $\int_1^3 [0 - f(x)] dx = -\int_1^3 f(x) dx$

9 a $f(x) = \frac{x}{1+x^2}$

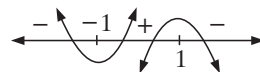
$$\therefore f'(x) = \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

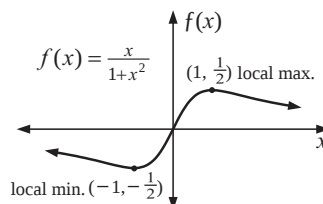
$$= \frac{(1+x)(1-x)}{(1+x^2)^2}$$

which has sign diagram:



$\therefore$ there is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(1, \frac{1}{2})$

b as $x \rightarrow \infty$ $f(x) \rightarrow 0$ (above) and **c**
as $x \rightarrow -\infty$ $f(x) \rightarrow 0$ (below)



$$\begin{aligned}
 \mathbf{d} \quad \text{Area} &= \int_{-2}^0 \left[0 - \frac{x}{1+x^2} \right] dx \quad \therefore \int_{-2}^0 \frac{-x}{1+x^2} dx = \int_{-2}^0 \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\
 &= \int_{-2}^0 \frac{-x}{1+x^2} dx &= -\frac{1}{2} \int_5^1 \frac{1}{u} du \\
 \text{Let } u &= 1+x^2 \quad \therefore \frac{du}{dx} = 2x &= -\frac{1}{2} [\ln |u|]_5^1 \\
 \text{and when } x &= 0, \quad u = 1 &= -\frac{1}{2} (0 - \ln 5) \\
 \text{when } x &= -2, \quad u = 5 &= \frac{1}{2} \ln 5 \text{ units}^2 \quad (\div 0.8047 \text{ units}^2)
 \end{aligned}$$

$$\mathbf{10} \quad \text{Shaded area} = \int_{-4}^{-1} [g(x) - f(x)] dx + \int_{-1}^5 [f(x) - g(x)] dx$$

$$\begin{aligned}
 \mathbf{11} \quad \text{The coordinates of B are } (2, 4+k) &\quad \therefore \int_0^2 (x^2 + k) dx = 4 + k \\
 \therefore \text{area rectangle OABC} &= 2 \times (4+k) \\
 &= 8 + 2k \\
 \therefore \text{since the two shaded regions are equal in area,} &\quad \therefore \left[\frac{x^3}{3} + kx \right]_0^2 = 4 + k \\
 \text{each area is } 4 + k \text{ units}^2. &\quad \therefore \frac{8}{3} + 2k = 4 + k \\
 &\quad \therefore k = 4 - \frac{8}{3} \\
 &\quad \therefore k = \frac{4}{3}
 \end{aligned}$$

REVIEW SET 26B

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad &\int (2e^{-x} - \frac{1}{x} + 3) dx \\
 &= (\frac{1}{-1})2e^{-x} - \ln |x| + 3x + c \\
 &= -2e^{-x} - \ln |x| + 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad &\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx \\
 &= \int \left(x - 2 + \frac{1}{x} \right) dx \\
 &= \frac{x^2}{2} - 2x + \ln |x| + c
 \end{aligned}$$

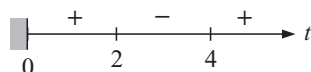
$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad &\int_1^2 (x^2 - 1)^2 dx \\
 &= \int_1^2 (x^4 - 2x^2 + 1) dx \\
 &= \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_1^2 \\
 &= \left[\frac{32}{5} - \frac{16}{3} + 2 \right] - \left[\frac{1}{5} - \frac{2}{3} + 1 \right] \\
 &= \frac{31}{5} - \frac{14}{3} + 1 \\
 &= 2\frac{8}{15}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{In } &\int_1^2 x(x^2 - 1)^2 dx \\
 \text{We let } u &= x^2 - 1 \quad \therefore \frac{du}{dx} = 2x \\
 \text{and when } x &= 1, \quad u = 0 \\
 \text{when } x &= 2, \quad u = 3 \\
 \therefore \int_1^2 x(x^2 - 1)^2 dx &= \int_0^3 u^2 \left(\frac{1}{2} \frac{du}{dx} dx \right) \\
 &= \frac{1}{2} \int_0^3 u^2 du \\
 &= \frac{1}{2} \left[\frac{u^3}{3} \right]_0^3 \\
 &= \frac{1}{2} \left(\frac{27}{3} - 0 \right) \\
 &= 4\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad &f(x) = (3x^2 + x)^3 \\
 \therefore f'(x) &= 3(3x^2 + x)^2(6x + 1) \\
 \therefore \int 3(3x^2 + x)^2(6x + 1) dx &= (3x^2 + x)^3 + c_1 \\
 \therefore \int 3(3x^2 + x)^2(6x + 1) dx &= (3x^2 + x)^3 + c_1 \\
 \therefore \int (3x^2 + x)^2(6x + 1) dx &= \frac{1}{3}(3x^2 + x)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad v(t) &= t^2 - 6t + 8 \text{ ms}^{-1}, \quad t \geq 0 \\
 &= (t-4)(t-2)
 \end{aligned}$$

which has sign diagram:



$$\begin{aligned}\text{b} \quad \text{Now } s(t) &= \int (t^2 - 6t + 8) dt \\ &= \frac{t^3}{3} - 3t^2 + 8t + c\end{aligned}$$

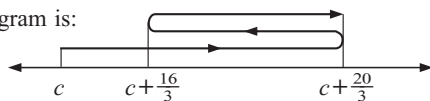
$$\text{and so } s(0) = c$$

$$s(2) = c + 6\frac{2}{3}$$

$$s(4) = c + 5\frac{1}{3}$$

$$s(5) = c + 6\frac{2}{3}$$

the motion diagram is:



The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$ it is $6\frac{2}{3}$ m from its starting point.

c After 5 seconds, the particle is $6\frac{2}{3}$ m to the right of its starting point.

$$\begin{aligned}\text{d} \quad \text{total distance travelled} &= (c + \frac{20}{3} - c) + [(c + \frac{20}{3}) - (c + \frac{16}{3})] + [(c + \frac{20}{3}) - (c + \frac{16}{3})] \\ &= 9\frac{1}{3} \text{ m}\end{aligned}$$

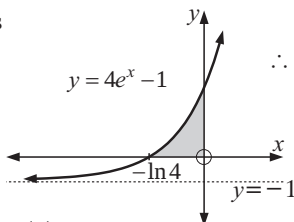
5 $y = 4e^x - 1$ cuts the x -axis

$$\text{when } 4e^x - 1 = 0$$

$$\therefore e^x = \frac{1}{4}$$

$$\therefore x = \ln \frac{1}{4}$$

$$= -\ln 4$$



$$\begin{aligned}\therefore \text{area} &= \int_{-\ln 4}^0 (4e^x - 1) dx \\ &= [4e^x - x]_{-\ln 4}^0 \\ &= [4 - 0] - [1 + \ln 4] \\ &= 3 - \ln 4 \quad (\div 1.614 \text{ units}^2)\end{aligned}$$

6 $y = f(x)$ $f(0) = -1$ $f(1) = 13$

$$f''(x) = 18x + 10$$

$$\therefore f'(x) = 9x^2 + 10x + c$$

$$\text{and } f(x) = 3x^3 + 5x^2 + cx + d$$

$$\text{But } f(0) = -1 \therefore d = -1$$

$$\therefore f(x) = 3x^3 + 5x^2 + ax - 1$$

$$\text{But } f(1) = 13$$

$$\therefore 3 + 5 + a - 1 = 13$$

$$\therefore a + 7 = 13$$

$$\therefore a = 6$$

$$\therefore f(x) = 3x^3 + 5x^2 + 6x - 1$$

7 The area between $x = 0$ and $x = a$ is 2 units².

$$\therefore \int_0^a e^x dx = 2$$

$$\therefore [e^x]_0^a = 2$$

$$\therefore e^a - e^0 = 2$$

$$\therefore e^a = 3$$

$$\therefore a = \ln 3$$

The area between $x = a = \ln 3$ and $x = b$ is 2 units².

$$\therefore \int_{\ln 3}^b e^x dx = 2$$

$$\therefore [e^x]_{\ln 3}^b = 2$$

$$\therefore e^b - e^{\ln 3} = 2$$

$$\therefore e^b - 3 = 2$$

$$\therefore e^b = 5$$

$$\therefore b = \ln 5$$

$$8 \quad \frac{4x-3}{2x+1} = \frac{2(2x+1)-5}{2x+1} = 2 + \frac{-5}{2x+1} \therefore A = 2 \quad B = -5$$

$$\begin{aligned}\therefore \int_0^2 \frac{4x-3}{2x+1} dx &= \int_0^2 \left(2 - 5 \left(\frac{1}{2x+1} \right) \right) dx \\ &= \left[2x - 5 \left(\frac{1}{2} \right) \ln |2x+1| \right]_0^2 \\ &= \left[4 - \frac{5}{2} \ln 5 \right] - \left[0 - \frac{5}{2} \ln 1 \right] \\ &= 4 - \frac{5}{2} \ln 5 \\ &\div -0.0236\end{aligned}$$

9 The shaded area $= \int_0^2 ax(x-2) dx = 4 \text{ units}^2$

$$\therefore \int_0^2 (ax^2 - 2ax) dx = 4$$

$$\therefore \left[\frac{ax^3}{3} - ax^2 \right]_0^2 = 4$$

$$\therefore \left[\frac{8a}{3} - 4a \right] - 0 = 4$$

$$\therefore \frac{8a}{3} - \frac{12a}{3} = 4$$

$$\therefore -\frac{4a}{3} = 4$$

$$\therefore a = -3$$

$$\therefore y = -3x(x-2)$$

Suppose A has coordinates $(k, -3k(k-2))$

$$\therefore \text{slope OA} = \frac{-3k(k-2) - 0}{k - 0} = -3(k-2) \quad \therefore \text{equation of OA is } y = -3(k-2)x$$

Now if OA divides the shaded area into equal areas,

$$\int_0^k [-3x(x-2) - (-3(k-2)x)] dx = 2 \quad \therefore -k^3 + \frac{3k^3}{2} = 2$$

$$\therefore \int_0^k (-3x^2 + 6x + 3kx - 6x) dx = 2 \quad \therefore \frac{k^3}{2} = 2$$

$$\therefore \int_0^k (-3x^2 + 3kx) dx = 2 \quad \therefore k^3 = 4$$

$$\therefore \left[-x^3 + \frac{3kx^2}{2} \right]_0^k = 2 \quad \therefore k = \sqrt[3]{4}$$

$$\therefore \text{the } x\text{-coordinate of A is } \sqrt[3]{4}$$

10 $f(x) = \frac{3x-5}{(x-2)^2}$

a $f(x)$ cuts the y -axis when $x = 0$ i.e., $y = -\frac{5}{4}$

$f(x)$ cuts the x -axis when $f(x) = 0 \quad \therefore 3x-5 = 0$ and so $x = \frac{5}{3}$

$\therefore$ the x -intercept is $\frac{5}{3}$ and the y -intercept is $-\frac{5}{4}$

b There is one vertical asymptote, $x = 2$

c $\frac{3x-5}{(x-2)^2}$ is a quotient with $u = 3x-5$ and $v = (x-2)^2$
 $\therefore u' = 3$ and $v' = 2(x-2)^1$

$$\therefore f'(x) = \frac{3(x-2)^2 - 2(3x-5)(x-2)}{(x-2)^4} \quad \{\text{quotient rule}\}$$

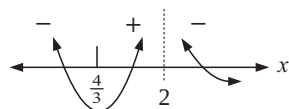
$$= \frac{(x-2)[3(x-2) - 2(3x-5)]}{(x-2)^4}$$

$$= \frac{3x-6-6x+10}{(x-2)^3}$$

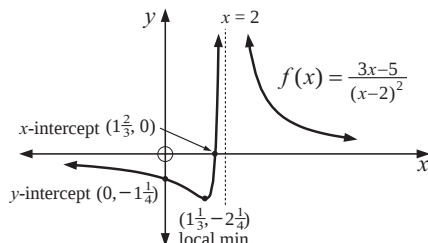
$$= \frac{4-3x}{(x-2)^3}$$

$\therefore$ it has a local minimum at $(\frac{4}{3}, -\frac{9}{4})$

which has sign diagram:



d



e

$$\frac{3x-5}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

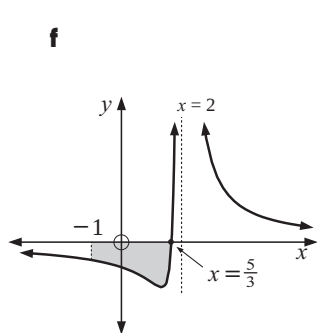
$$\therefore \frac{3x-5}{(x-2)^2} = \frac{A(x-2) + B}{(x-2)^2}$$

$$\therefore \frac{3x-5}{(x-2)^2} = \frac{Ax + (B-2A)}{(x-2)^2}$$

Equating coefficients, $A = 3$

and $B - 2A = -5$

$$\therefore B = 2A - 5 = 6 - 5 = 1$$



$$\begin{aligned}
 \text{Area} &= -\left[3 \ln |x-2|\right]_{-1}^{\frac{5}{3}} - \left[-\frac{1}{x-2}\right]_{-1}^{\frac{5}{3}} \\
 &= -3 \ln \frac{1}{3} + 3 \ln 3 + \frac{1}{(-\frac{1}{3})} - \frac{1}{(-3)} \\
 &= 3 \ln 3 + 3 \ln 3 - 3 + \frac{1}{3} \\
 &= 6 \ln 3 - \frac{8}{3} \div 3.925 \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 &= -\int_{-1}^{\frac{5}{3}} f(x) \, dx \\
 &= -\int_{-1}^{\frac{5}{3}} \frac{3x-5}{(x-2)^2} \, dx \\
 &= -\int_{-1}^{\frac{5}{3}} \left(\frac{3}{x-2} + \frac{1}{(x-2)^2} \right) \, dx \\
 &= -\int_{-1}^{\frac{5}{3}} \frac{3}{x-2} \, dx - \int_{-1}^{\frac{5}{3}} (x-2)^{-2} \, dx
 \end{aligned}$$

- 11** The line $y = mx + c$ passes through $(-1, 0)$. $\therefore 0 = -m + c$ and so $c = m$
 $\therefore$ the line is $y = cx + c$

The curve and the line meet when $cx + c = -x^2 + 2x + 3$

$$\therefore x^2 + (c-2)x + (c-3) = 0$$

$$\therefore (x+1)(x+[c-3]) = 0 \quad \{\text{as we know } x = -1 \text{ is a solution}\}$$

$$\therefore x = -1 \text{ or } 3 - c$$

If we let $a = 3 - c$, then the enclosed area $= \int_{-1}^a [(-x^2 + 2x + 3) - (3-a)(x+1)] \, dx$

$$\begin{aligned}
 &= \int_{-1}^a [-x^2 + (a-1)x + a] \, dx \\
 &= \left[-\frac{x^3}{3} + \frac{(a-1)x^2}{2} + ax \right]_{-1}^a \\
 &= \left(-\frac{a^3}{3} + \frac{(a-1)a^2}{2} + a^2 \right) - \left(\frac{1}{3} + \frac{(a-1)}{2} - a \right) \\
 &= -\frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{2}a^2 + a^2 - \frac{1}{3} - \frac{1}{2}a + \frac{1}{2} + a \\
 &= \frac{1}{6}a^3 + \frac{1}{2}a^2 + \frac{1}{2}a + \frac{1}{6}
 \end{aligned}$$

But this area is 4.5 units^2 , $\therefore \frac{1}{6}a^3 + \frac{1}{2}a^2 + \frac{1}{2}a + \frac{1}{6} = \frac{9}{2}$

$$\therefore a^3 + 3a^2 + 3a + 1 = 27$$

$$\therefore a^3 + 3a^2 + 3a - 26 = 0$$

$$\therefore (a-2)(a^2 + 5a + 13) = 0 \quad \text{where the quadratic has } \Delta < 0, \\ \therefore \text{no real solutions}$$

$$\therefore a = 2 \quad \text{and so } c = 1 \quad (=m)$$

From technology, the only real solution to this equation is $c = 1$.

So $c = 1$, $m = 1$, and the line has equation $y = x + 1$.

REVIEW SET 26C

1 a $\frac{dy}{dx} = (x^2 - 1)^2$ **b** $\frac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$ $\therefore y = \int (400 - 20e^{-\frac{x}{2}}) \, dx$

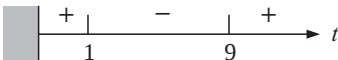
$$\begin{aligned}
 \therefore y &= \int (x^2 - 1)^2 \, dx & &= 400x - \frac{20e^{-\frac{x}{2}}}{-\frac{1}{2}} + c \\
 &= \int (x^4 - 2x^2 + 1) \, dx & &= 400x + 40e^{-\frac{x}{2}} + c \\
 &= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int_{-2}^0 \frac{4}{2x-1} dx \\
 &= 4 \int_{-2}^0 \frac{1}{2x-1} dx \\
 &= 4 \left[\left(\frac{1}{2} \right) \ln |2x-1| \right]_{-2}^0 \\
 &= 2 [\ln |2x-1|]_{-2}^0 \\
 &= 2 [\ln |-1| - \ln |-5|] \\
 &= 2[0 - \ln 5] \\
 &= -2 \ln 5 \\
 &\doteq -3.219
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \ln \int_0^1 \frac{10x}{\sqrt{3x^2+1}} dx \\
 & \text{we let } u = 3x^2 + 1, \quad \therefore \frac{du}{dx} = 6x \\
 & \text{When } x = 0, u = 1. \text{ When } x = 1, u = 4. \\
 \therefore & \int_0^1 \frac{10x}{\sqrt{3x^2+1}} dx = \int_1^4 u^{-\frac{1}{2}} \left(\frac{5}{3} \frac{du}{dx} \right) dx \\
 &= \frac{5}{3} \int_1^4 u^{-\frac{1}{2}} du \\
 &= \frac{5}{3} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 \\
 &= \frac{10}{3} (\sqrt{4} - \sqrt{1}) \\
 &= \frac{10}{3}
 \end{aligned}$$

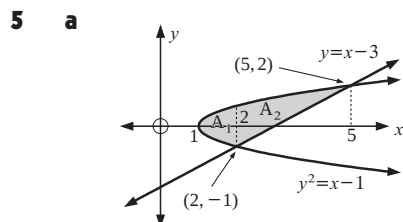
$$\begin{aligned}
 \mathbf{3} \quad & y = (3x^2 + 1)^{\frac{1}{2}} \quad \therefore \int \frac{3x}{\sqrt{3x^2+1}} dx = \sqrt{3x^2+1} + c_1 \\
 \therefore \frac{dy}{dx} &= \frac{1}{2} (3x^2 + 1)^{-\frac{1}{2}} \cdot 6x \\
 &= \frac{3x}{\sqrt{3x^2+1}} \quad 3 \int \frac{x}{\sqrt{3x^2+1}} dx = \sqrt{3x^2+1} + c_1 \\
 \therefore \int \frac{x}{\sqrt{3x^2+1}} dx &= \frac{1}{3} \sqrt{3x^2+1} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad & a(t) = 6t - 30 \text{ cms}^{-2} \\
 \therefore v(t) &= \int (6t - 30) dt = 3t^2 - 30t + c \\
 \text{But } v(0) &= 27 \quad \therefore c = 27 \\
 \therefore v(t) &= 3t^2 - 30t + 27 \text{ cms}^{-1} \\
 \therefore s(t) &= \int (3t^2 - 30t + 27) dt = t^3 - 15t^2 + 27t + d \\
 \text{But } s(0) &= 0 \quad \therefore d = 0 \\
 \therefore s(t) &= t^3 - 15t^2 + 27t \\
 \text{Now } v(t) &= 3t^2 - 30t + 27 \\
 &= 3(t^2 - 10t + 9) \\
 &= 3(t-1)(t-9)
 \end{aligned}$$

which has sign diagram: 

The particle comes to rest for the second time at $t = 9$ seconds.

$$\begin{aligned}
 \text{and } s(0) &= 0 \text{ cm} \\
 s(1) &= 13 \text{ cm} \\
 s(9) &= -243 \text{ cm} \quad \therefore D = 13 + 13 + 243 = 269 \text{ cm}
 \end{aligned}$$



$$\begin{aligned}
 y^2 &= x - 1 \text{ meets } y = x - 3 \text{ where} \\
 x - 1 &= (x - 3)^2 \\
 \therefore x - 1 &= x^2 - 6x + 9 \\
 \therefore x^2 - 7x + 10 &= 0 \\
 \therefore (x - 5)(x - 2) &= 0 \\
 \therefore x &= 2 \text{ or } x = 5 \\
 \therefore \text{at } (5, 2) \text{ and } (2, -1)
 \end{aligned}$$

$$\mathbf{b} \quad \text{Area} = A_1 + A_2$$

$$\begin{aligned} &= 2 \int_1^2 (x-1)^{\frac{1}{2}} dx + \int_2^5 [(x-1)^{\frac{1}{2}} - (x-3)] dx \\ &= 2 \left[\frac{2}{3}(x-1)^{\frac{3}{2}} \right]_1^2 + \left[\frac{2}{3}(x-1)^{\frac{3}{2}} - \frac{x^2}{2} + 3x \right]_2^5 \\ &= 2 \left[\frac{2}{3} - 0 \right] + \left[\left(\frac{2}{3}(8) - \frac{25}{2} + 15 \right) - \left(\frac{2}{3} - 2 + 6 \right) \right] \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

$$\mathbf{6} \quad y = k \text{ meets } y = x^2 \text{ where } x^2 = k \quad \therefore x = \pm\sqrt{k}$$

$$\begin{aligned} \text{By symmetry,} \quad \int_0^{\sqrt{k}} (k - x^2) dx &= \frac{1}{2} \times 5\frac{1}{3} = \frac{1}{2} \times \frac{16}{3} & \therefore \frac{2}{3}k\sqrt{k} &= \frac{8}{3} \\ \therefore \left[kx - \frac{x^3}{3} \right]_0^{\sqrt{k}} &= \frac{8}{3} & \therefore k\sqrt{k} &= 4 \\ \therefore k^{\frac{3}{2}} &= 4 & \therefore k^{\frac{3}{2}} &= 4 \\ \therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} &= \frac{8}{3} & \therefore k &= 4^{\frac{2}{3}} = \sqrt[3]{16} \end{aligned}$$

$$\mathbf{7} \quad f'(x) = 2\sqrt{x} + \frac{a}{\sqrt{x}} = 2x^{\frac{1}{2}} + ax^{-\frac{1}{2}}$$

$$\therefore f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}} + c = \frac{4x\sqrt{x}}{3} + 2a\sqrt{x} + c$$

$$\text{Now } f(0) = 2 \quad \therefore c = 2 \quad \therefore f(x) = \frac{4x\sqrt{x}}{3} + 2a\sqrt{x} + 2$$

$$\text{Also } f(1) = 4 \quad \therefore \frac{4}{3} + 2a + 2 = 4 \quad \therefore 2a = \frac{2}{3} \quad \therefore a = \frac{1}{3}$$

$$\text{Now } f'(x) = 2\sqrt{x} + \frac{1}{3\sqrt{x}} = \frac{6x+1}{3\sqrt{x}} > 0 \quad \text{since } f(x) \text{ is only defined for } x > 0$$

$$\text{and for } x > 0, \quad 6x+1 \quad \text{and} \quad 3\sqrt{x} \quad \text{are} > 0$$

$$\therefore \text{the function has no stationary point as } f'(x) \neq 0 \quad \text{for any value of } x.$$

$$\mathbf{8} \quad \frac{-2x}{4-x^2} = \frac{A}{x+2} + \frac{B}{2-x} \quad \text{for all } x$$

$$\therefore \frac{-2x}{4-x^2} = \frac{A(2-x) + B(x+2)}{(x+2)(2-x)}$$

$$\therefore \frac{-2x}{4-x^2} = \frac{2A - Ax + Bx + 2B}{4-x^2}$$

$$\therefore \frac{0-2x}{4-x^2} = \frac{(2A+2B) + x(-A+B)}{4-x^2}$$

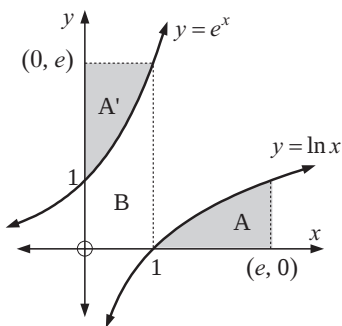
$$\text{Equating coefficients,} \quad -A + B = -2 \quad \text{and} \quad 2A + 2B = 0, \quad \text{i.e., } B = -A$$

$$\therefore -A + -A = -2$$

$$\therefore -2A = -2 \quad \therefore A = 1 \quad \text{and} \quad B = -1$$

$$\begin{aligned} \therefore \int_3^4 -\frac{2x}{4-x^2} dx &= \int_3^4 \left(\frac{1}{x+2} + \frac{1}{x-2} \right) dx \quad \left\{ \text{as } \frac{-1}{2-x} = \frac{1}{x-2} \right\} \\ &= [\ln|x+2| + \ln|x-2|]_3^4 \\ &= (\ln 6 + \ln 2) - (\ln 5 + \ln 1) \\ &= \ln 12 - \ln 5 \\ &= \ln\left(\frac{12}{5}\right) \end{aligned}$$

9



$y = e^x$ and $y = \ln x$ are inverse functions,
i.e., they are symmetrical about $y = x$

$$\therefore \text{area } A = \text{area } A'$$

but $\text{area } A' + \text{area } B = \text{area of rectangle}$

$$\therefore \text{area } A + \text{area } B = e \times 1 = e$$

$$\text{and as } \text{area } A = \int_1^e \ln x \, dx$$

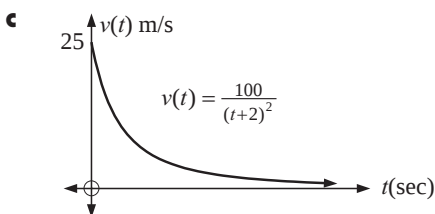
$$\text{and } \text{area } B = \int_0^1 e^x \, dx$$

$$\text{then } \int_1^e \ln x \, dx + \int_0^1 e^x \, dx = e$$

$$10 \quad v(t) = \frac{100}{(t+2)^2} = 100(t+2)^{-2} \text{ ms}^{-1}$$

$$\mathbf{a} \quad \text{At } t = 0, \quad v(0) = \frac{100}{2^2} = 25 \text{ ms}^{-1} \quad \text{At } t = 3, \quad v(3) = \frac{100}{5^2} = 4 \text{ ms}^{-1}$$

$$\mathbf{b} \quad \text{as } t \rightarrow +\infty, \quad v(t) \rightarrow 0 \text{ ms}^{-1} \quad (\text{above})$$



$$\begin{aligned} \mathbf{e} \quad a(t) &= v'(t) \\ &= -200(t+2)^{-3} \text{ ms}^{-2} \\ &= \frac{-200}{(t+2)^3} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{dv}{dt} &= \frac{-200}{(t+2)^3} \\ &= -\frac{1}{5} \frac{1000}{(t+2)^3} \\ &= -\frac{1}{5} \left(\frac{100}{(t+2)^2} \right)^{\frac{3}{2}} \\ &= -\frac{1}{5} v^{\frac{3}{2}} \\ \therefore \frac{dv}{dt} &= -kv^{\frac{3}{2}} \quad \text{where } k = \frac{1}{5} \end{aligned}$$

$$\begin{aligned} 11 \quad \frac{d}{dx} (\ln x)^2 &= 2(\ln x)^1 \left(\frac{1}{x} \right) \\ &= \frac{2 \ln x}{x} \end{aligned}$$

$$\therefore \int \frac{2 \ln x}{x} dx = (\ln x)^2 + c_1$$

$$\therefore \int \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 + c$$

$\mathbf{d}$ As $v(t)$ is always positive, the boat is always travelling forwards.

$$\begin{aligned} s(t) &= \int v(t) \, dt \\ &= \int 100(t+2)^{-2} \, dt \\ &= -100(t+2)^{-1} + c \\ &= \frac{-100}{t+2} + c \end{aligned}$$

$$\therefore s(0) = c - 50 \text{ m}$$

$\therefore$ when the boat has travelled 30 m,

$$s(t) = c - 20$$

$$\therefore c - 20 = \frac{-100}{t+2} + c$$

$$\therefore \frac{-100}{t+2} = -20$$

$$\therefore t+2 = 5$$

$$\therefore t = 3 \text{ seconds}$$

Chapter 27

TRIGONOMETRIC INTEGRATION

EXERCISE 27A

- 1** **a** $\int (3 \sin x - 2) \, dx$
 $= -3 \cos x - 2x + c$
- b** $\int (4x - 2 \cos x) \, dx$
 $= 2x^2 - 2 \sin x + c$
- c** $\int (2\sqrt{x} + \frac{4}{\cos^2 x}) \, dx$
 $= \int (2x^{\frac{1}{2}} + 4 \frac{1}{\cos^2 x}) \, dx$
 $= \frac{4}{3} x^{\frac{3}{2}} + 4 \tan x + c$
- d** $\int (\frac{1}{\cos^2 x} + 2 \sin x) \, dx$
 $= \tan x - 2 \cos x + c$
- e** $\int (\frac{x}{2} - \frac{1}{\cos^2 x}) \, dx$
 $= \frac{x^2}{4} - \tan x + c$
- f** $\int (\sin x - 2 \cos x + e^x) \, dx$
 $= -\cos x - 2 \sin x + e^x + c$
- 2** **a** $\int (\sqrt{x} + \frac{1}{2} \cos x) \, dx$
 $= \int (x^{\frac{1}{2}} + \frac{1}{2} \cos x) \, dx$
 $= \frac{2}{3} x^{\frac{3}{2}} + \frac{1}{2} \sin x + c$
- b** $\int (\theta - \sin \theta) \, d\theta$
 $= \frac{\theta^2}{2} + \cos \theta + c$
- c** $\int (t\sqrt{t} + \frac{2}{\cos^2 t}) \, dt$
 $= \int (t^{\frac{3}{2}} + 2 \sec^2 t) \, dt$
 $= \frac{2}{5} t^{\frac{5}{2}} + 2 \tan t + c$
- d** $\int (2e^t - 4 \sin t) \, dt$
 $= 2e^t + 4 \cos t + c$
- e** $\int (3 \cos t - \frac{1}{t}) \, dt$
 $= 3 \sin t - \ln |t| + c$
- f** $\int (3 - \frac{2}{\theta} + \frac{1}{\cos^2 \theta}) \, d\theta$
 $= 3\theta - 2 \ln |\theta| + \tan \theta + c$
- 3** **a** $\frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x \quad \therefore \int e^x (\sin x + \cos x) \, dx = \int (e^x \sin x + e^x \cos x) \, dx$
 $= e^x \sin x + c$
- b** $\frac{d}{dx}(e^{-x} \sin x) = -e^{-x} \sin x + e^{-x} \cos x$
 $= \frac{\cos x - \sin x}{e^x}$
 $\therefore \int \frac{\cos x - \sin x}{e^x} \, dx = e^{-x} \sin x + c$
- c** $\frac{d}{dx}(x \cos x) = \cos x + x(-\sin x)$
 $= \cos x - x \sin x$
 $\therefore \int (\cos x - x \sin x) \, dx = x \cos x + c_1$
 $\therefore \int \cos x \, dx - \int x \sin x \, dx = x \cos x + c_1$
 $\therefore \sin x - \int x \sin x \, dx = x \cos x + c_1$
 $\therefore \int x \sin x \, dx = -x \cos x + \sin x + c$
- d** $\frac{1}{\cos x} = (\cos x)^{-1} \quad \therefore \frac{d}{dx}(\frac{1}{\cos x}) = -(\cos x)^{-2}(-\sin x)$
 $= \frac{\sin x}{\cos^2 x}$
 $= \frac{\sin x}{\cos x}$
 $= \frac{\tan x}{\cos x} \quad \therefore \int \frac{\tan x}{\sin x} \, dx = \frac{1}{\cos x} + c$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad f'(x) &= x^2 - 4 \cos x \\ \therefore f(x) &= \int (x^2 - 4 \cos x) \, dx \\ &= \frac{x^3}{3} - 4 \sin x + c \end{aligned}$$

$$\text{But } f(0) = 3$$

$$\therefore 0 - 4 \sin(0) + c = 3$$

$$\therefore c = 3$$

$$\therefore f(x) = \frac{x^3}{3} - 4 \sin x + 3$$

$$\begin{aligned} \mathbf{c} \quad f'(x) &= \sqrt{x} - \frac{2}{\cos^2 x} \\ \therefore f(x) &= \int (x^{\frac{1}{2}} - \frac{2}{\cos^2 x}) \, dx \\ &= \frac{2}{3} x^{\frac{3}{2}} - 2 \tan x + c \end{aligned}$$

$$\text{But } f(\pi) = 0$$

$$\therefore \frac{2}{3} \pi^{\frac{3}{2}} - 2 \tan \pi + c = 0$$

$$\therefore c = -\frac{2}{3} \pi^{\frac{3}{2}}$$

$$\therefore f(x) = \frac{2}{3} x^{\frac{3}{2}} - 2 \tan x - \frac{2}{3} \pi^{\frac{3}{2}}$$

$$\begin{aligned} \mathbf{b} \quad f'(x) &= 2 \cos x - 3 \sin x \\ \therefore f(x) &= \int (2 \cos x - 3 \sin x) \, dx \\ &= 2 \sin x + 3 \cos x + c \end{aligned}$$

$$\text{But } f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore 2 \sin \frac{\pi}{4} + 3 \cos \frac{\pi}{4} + c = \frac{1}{\sqrt{2}}$$

$$\therefore 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) + c = \frac{1}{\sqrt{2}}$$

$$\therefore c = -\frac{4}{\sqrt{2}}$$

$$\therefore c = -2\sqrt{2}$$

$$\therefore f(x) = 2 \sin x + 3 \cos x - 2\sqrt{2}$$

EXERCISE 27B

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int \sin(3x) \, dx \\ = -\frac{1}{3} \cos(3x) + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int 2 \cos(4x) \, dx \\ = 2 \times \frac{1}{4} \sin(4x) + c \\ = \frac{1}{2} \sin(4x) + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int \frac{1}{\cos^2(2x)} \, dx \\ = \frac{1}{2} \tan(2x) + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \int 3 \cos\left(\frac{x}{2}\right) \, dx \\ = 6 \sin\left(\frac{x}{2}\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \int (3 \sin(2x) - e^{-x}) \, dx \\ = -\frac{3}{2} \cos(2x) + e^{-x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \int \left[e^{2x} - \frac{2}{\sec^2\left(\frac{x}{2}\right)} \right] \, dx \\ = \frac{1}{2} e^{2x} - 2 \times 2 \tan\left(\frac{x}{2}\right) + c \\ = \frac{1}{2} e^{2x} - 4 \tan\left(\frac{x}{2}\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad \int 2 \sin\left(2x + \frac{\pi}{6}\right) \, dx \\ = -\frac{2}{2} \cos\left(2x + \frac{\pi}{6}\right) + c \\ = -\cos\left(2x + \frac{\pi}{6}\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad \int -3 \cos\left(\frac{\pi}{4} - x\right) \, dx \\ = -3(-\sin\left(\frac{\pi}{4} - x\right)) + c \\ = 3 \sin\left(\frac{\pi}{4} - x\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad \int \frac{4}{\cos^2\left(\frac{\pi}{3} - 2x\right)} \, dx \\ = 4 \times \left(-\frac{1}{2}\right) \tan\left(\frac{\pi}{3} - 2x\right) + c \\ = -2 \tan\left(\frac{\pi}{3} - 2x\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad \int \cos(2x) + \sin(2x) \, dx \quad \mathbf{k} \quad \int 2 \sin(3x) + 5 \cos(4x) \, dx \\ = \frac{1}{2} \sin(2x) - \frac{1}{2} \cos(2x) + c \quad = -\frac{2}{3} \cos(3x) + \frac{5}{4} \sin(4x) + c \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad \int \frac{1}{2} \cos(8x) - 3 \sin x \, dx \\ = \frac{1}{2} \left(\frac{1}{8}\right) \sin(8x) + 3 \cos x + c \\ = \frac{1}{16} \sin(8x) + 3 \cos x + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int \cos^2 x \, dx \\
 &= \int \left(\frac{1}{2} + \frac{1}{2} \cos(2x) \right) dx \\
 &= \frac{1}{2}x + \frac{1}{4} \sin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (1 + \cos^2(2x)) \, dx \\
 &= \int \left(1 + \frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \int \left(\frac{3}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \frac{3}{2}x + \frac{1}{8} \sin(4x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \frac{1}{2} \cos^2(4x) \, dx \\
 &= \int \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \cos(8x) \right) dx \\
 &= \int \left(\frac{1}{4} + \frac{1}{4} \cos(8x) \right) dx \\
 &= \frac{1}{4}x + \frac{1}{32} \sin(8x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \sin^2 x \, dx \\
 &= \int \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx \\
 &= \frac{1}{2}x - \frac{1}{4} \sin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int (3 - \sin^2(3x)) \, dx \\
 &= \int \left(3 - \left(\frac{1}{2} - \frac{1}{2} \cos(6x) \right) \right) dx \\
 &= \int \left(\frac{5}{2} + \frac{1}{2} \cos(6x) \right) dx \\
 &= \frac{5}{2}x + \frac{1}{12} \sin(6x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int (1 + \cos x)^2 \, dx \\
 &= \int (1 + 2 \cos x + \cos^2 x) \, dx \\
 &= \int \left(1 + 2 \cos x + \frac{1}{2} + \frac{1}{2} \cos(2x) \right) dx \\
 &= \frac{3}{2}x + 2 \sin x + \frac{1}{4} \sin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \cos^2 \theta &= \frac{1}{2} + \frac{1}{2} \cos(2\theta) \quad \therefore \quad \cos^4 \theta = \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right)^2 \\
 &= \frac{1}{4} + \frac{1}{4} \cos^2(2\theta) + \frac{1}{2} \cos(2\theta) \\
 &= \frac{1}{4} + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos(4\theta) \right) + \frac{1}{2} \cos(2\theta) \\
 &= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) \\
 &= \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) + \frac{3}{8} \quad \text{as required}
 \end{aligned}$$

$$\therefore \int \cos^4 x \, dx = \int \left(\frac{1}{8} \cos(4x) + \frac{1}{2} \cos(2x) + \frac{3}{8} \right) dx = \frac{1}{32} \sin(4x) + \frac{1}{4} \sin(2x) + \frac{3}{8}x + c$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \text{Consider } \int \sin^4 x \cos x \, dx \\
 & \text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x \\
 & \therefore \int \sin^4 x \cos x \, dx \\
 &= \int u^4 \frac{du}{dx} \, dx \\
 &= \int u^4 \, du \\
 &= \frac{u^5}{5} + c \\
 &= \frac{1}{5} \sin^5 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{Consider } \int \frac{\sin x}{\sqrt{\cos x}} \, dx \\
 & \text{Let } u = \cos x, \quad \frac{du}{dx} = -\sin x \\
 & \therefore \int \frac{\sin x}{\sqrt{\cos x}} \, dx \\
 &= \int u^{-\frac{1}{2}} \left(-\frac{du}{dx} \right) dx \\
 &= \int -u^{-\frac{1}{2}} \, du \\
 &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \quad \text{or} \quad -2\sqrt{\cos x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \text{Consider } \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \\
 & \text{Let } u = \cos x, \quad \frac{du}{dx} = -\sin x \\
 & \therefore \int \frac{\sin x}{\cos x} \, dx \\
 &= \int \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\
 &= \int -\frac{1}{u} \, du \\
 &= -\ln|u| + c \\
 &= -\ln|\cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \text{Consider } \int \sqrt{\sin x} \cos x \, dx \\
 & \text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x \\
 & \therefore \int \sqrt{\sin x} \cos x \, dx \\
 &= \int u^{\frac{1}{2}} \frac{du}{dx} \, dx \\
 &= \int u^{\frac{1}{2}} \, du \\
 &= \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{3} (\sin x)^{\frac{3}{2}} + c
 \end{aligned}$$

e Consider $\int \frac{\cos x}{(2 + \sin x)^2} dx$

Let $u = 2 + \sin x$, $\frac{du}{dx} = \cos x$

$$\therefore \int \frac{\cos x}{(2 + \sin x)^2} dx$$

$$= \int u^{-2} \frac{du}{dx} dx$$

$$= \int u^{-2} du$$

$$= -u^{-1} + c$$

$$= \frac{-1}{2 + \sin x} + c$$

f Consider $\int \frac{\sin x}{\cos^3 x} dx$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$$\therefore \int \frac{\sin x}{\cos^3 x} dx$$

$$= \int u^{-3} \left(-\frac{du}{dx}\right) dx$$

$$= \int -u^{-3} du$$

$$= \frac{-u^{-2}}{-2} + c$$

$$= \frac{1}{2} u^{-2} + c$$

$$= \frac{1}{2} (\cos x)^{-2} + c$$

$$= \frac{1}{2 \cos^2 x} + c$$

g Consider $\int \frac{\sin x}{1 - \cos x} dx$

Let $u = 1 - \cos x$, $\frac{du}{dx} = \sin x$

$$\therefore \int \frac{\sin x}{1 - \cos x} dx$$

$$= \int \frac{1}{u} \frac{du}{dx} dx$$

$$= \int \frac{1}{u} du$$

$$= \ln |u| + c$$

$$= \ln |1 - \cos x| + c$$

h Consider $\int \frac{\cos(2x)}{\sin(2x) - 3} dx$

Let $u = \sin(2x) - 3$, $\frac{du}{dx} = 2 \cos(2x)$

$$\therefore \cos(2x) = \frac{1}{2} \frac{du}{dx}$$

$$\therefore \int \frac{\cos(2x)}{\sin(2x) - 3} dx$$

$$= \int \frac{1}{u} \left(\frac{1}{2} \frac{du}{dx}\right) dx$$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| + c$$

$$= \frac{1}{2} \ln |\sin(2x) - 3| + c$$

i Consider $\int \frac{\cos(3x)}{\sin(3x)} dx$

Let $u = \sin(3x)$, $\frac{du}{dx} = 3 \cos(3x)$

$$\therefore \cos(3x) = \frac{1}{3} \frac{du}{dx}$$

$$\therefore \int \frac{\cos(3x)}{\sin(3x)} dx$$

$$= \int \frac{1}{u} \left(\frac{1}{3} \frac{du}{dx}\right) dx$$

$$= \frac{1}{3} \int \frac{1}{u} du$$

$$= \frac{1}{3} \ln |u| + c$$

$$= \frac{1}{3} \ln |\sin(3x)| + c$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx & \therefore \int \cos^3 x \, dx &= \int (1 - u^2) \frac{du}{dx} dx \\
 &= \int (1 - \sin^2 x) \cos x \, dx & &= \int (1 - u^2) du \\
 \text{Let } u = \sin x, \quad \frac{du}{dx} &= \cos x & &= u - \frac{u^3}{3} + c \\
 & & &= \sin x - \frac{\sin^3 x}{3} + c \\
 & & &= \sin x - \frac{1}{3} \sin^3 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \sin^5 x \, dx & & \therefore \int \sin^5 x \, dx &= \int (1 - 2u^2 + u^4) \left(-\frac{du}{dx}\right) dx \\
 &= \int \sin^4 x \sin x \, dx & &= \int (-1 + 2u^2 - u^4) du \\
 &= \int (1 - \cos^2 x)^2 \sin x \, dx & &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + c \\
 &= \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx & &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c \\
 \text{Let } u = \cos x, \quad \frac{du}{dx} &= -\sin x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int \sin^4 x \cos^3 x \, dx & & \therefore \int \sin^4 x \cos^3 x \, dx &= \int (u^4 - u^6) \frac{du}{dx} dx \\
 &= \int \sin^4 x \cos^2 x \cos x \, dx & &= \int (u^4 - u^6) du \\
 &= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx & &= \frac{1}{5}u^5 - \frac{1}{7}u^7 + c \\
 &= \int (\sin^4 x - \sin^6 x) \cos x \, dx & &= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c \\
 \text{Let } u = \sin x, \quad \frac{du}{dx} &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad \text{Let } u = \cos x, \quad \frac{du}{dx} &= -\sin x & \mathbf{b} \quad \text{Let } u = \sin(2x), \quad \frac{du}{dx} &= 2\cos(2x) \\
 \therefore f(x) &= \int \sin x e^{\cos x} \, dx & \therefore f(x) &= \int \sin^3(2x) \cos(2x) \, dx \\
 &= \int e^{\cos x} \sin x \, dx & &= \int u^3 \left(\frac{1}{2} \frac{du}{dx}\right) dx \\
 &= \int e^u \left(-\frac{du}{dx}\right) dx & &= \frac{1}{2} \int u^3 du \\
 &= -\int e^u du & &= \frac{1}{2} \times \frac{u^4}{4} + c \\
 &= -e^u + c & &= \frac{1}{8} \sin^4(2x) + c \\
 &= -e^{\cos x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{Let } u &= \sin x - \cos x, \\
 \frac{du}{dx} &= \cos x + \sin x \\
 \therefore f(x) &= \int \frac{\sin x + \cos x}{\sin x - \cos x} dx \\
 &= \int \frac{1}{u} \frac{du}{dx} dx \\
 &= \int \frac{1}{u} du \\
 &= \ln |u| + c \\
 &= \ln |\sin x - \cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \text{Let } u &= \tan x \\
 \therefore \frac{du}{dx} &= \frac{1}{\cos^2 x} \\
 \text{So,} \quad \int \frac{e^{\tan x}}{\cos^2 x} dx & \\
 &= \int e^u \left(\frac{du}{dx}\right) dx \\
 &= \int e^u du \\
 &= e^u + c \\
 &= e^{\tan x} + c
 \end{aligned}$$

EXERCISE 27C

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int_0^{\frac{\pi}{6}} \cos x \, dx \\
 &= [\sin x]_0^{\frac{\pi}{6}} \\
 &= \sin \frac{\pi}{6} - \sin 0 \\
 &= \frac{1}{2} - 0 \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \, dx \\
 &= [-\cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= -\cos \frac{\pi}{2} + \cos \frac{\pi}{3} \\
 &= 0 + \frac{1}{2} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \, dx \\
 &= [\tan x]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
 &= \tan \frac{\pi}{3} - \tan \frac{\pi}{4} \\
 &= \sqrt{3} - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_0^{\frac{\pi}{6}} \sin(3x) \, dx \\
 &= \left[-\frac{1}{3} \cos(3x)\right]_0^{\frac{\pi}{6}} \\
 &= -\frac{1}{3} \left[\cos \frac{\pi}{2} - \cos 0\right] \\
 &= -\frac{1}{3} [0 - 1] \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_0^{\frac{\pi}{4}} \cos^2 x \, dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(2x)\right) dx \\
 &= \left[\frac{x}{2} + \frac{1}{4} \sin(2x)\right]_0^{\frac{\pi}{4}} \\
 &= \left[\frac{\pi}{8} + \frac{1}{4} \sin \frac{\pi}{2}\right] - 0 \\
 &= \frac{\pi}{8} + \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \\
 &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos(2x)\right) dx \\
 &= \left[\frac{x}{2} - \frac{1}{4} \sin(2x)\right]_0^{\frac{\pi}{2}} \\
 &= \left[\frac{\pi}{4} - \frac{1}{4} \sin \pi\right] - 0 \\
 &= \frac{\pi}{4}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \text{Let } u = \cos x, \quad \frac{du}{dx} = -\sin x$$

When $x = 0$, $u = \cos 0 = 1$
 When $x = \frac{\pi}{3}$, $u = \cos \frac{\pi}{3} = \frac{1}{2}$

$$\begin{aligned}
 \therefore \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}} \, dx &= \int_1^{\frac{1}{2}} u^{-\frac{1}{2}} \left(-\frac{du}{dx}\right) dx \\
 &= \int_{\frac{1}{2}}^1 u^{-\frac{1}{2}} du \\
 &= \left[2u^{\frac{1}{2}}\right]_{\frac{1}{2}}^1 \\
 &= 2\sqrt{1} - 2\sqrt{\frac{1}{2}} \\
 &= 2 - \sqrt{2}
 \end{aligned}$$

$$\mathbf{b} \quad \text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x$$

When $x = 0$, $u = \sin 0 = 0$
 When $x = \frac{\pi}{6}$, $u = \sin \frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned}
 \therefore \int_0^{\frac{\pi}{6}} \sin^2 x \cos x \, dx &= \int_0^{\frac{1}{2}} u^2 \frac{du}{dx} dx \\
 &= \int_0^{\frac{1}{2}} u^2 du \\
 &= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}} \\
 &= \frac{1}{3} \left(\frac{1}{2}\right)^3 - 0 \quad \text{or} \quad \frac{1}{24}
 \end{aligned}$$

$$\mathbf{c} \quad \text{Let } u = \cos x \quad \therefore \frac{du}{dx} = -\sin x$$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

$$= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \left(-\frac{du}{dx}\right) dx$$

$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} \, du$$

$$= [\ln |u|]_{\frac{1}{\sqrt{2}}}^1$$

$$= \ln 1 - \ln \frac{1}{\sqrt{2}}$$

$$= \ln \sqrt{2}$$

$$= \frac{1}{2} \ln 2$$

d Let $u = \sin x$, $\frac{du}{dx} = \cos x$

When $x = \frac{\pi}{6}$, $u = \sin \frac{\pi}{6} = \frac{1}{2}$

When $x = \frac{\pi}{2}$, $u = \sin \frac{\pi}{2} = 1$

$$\begin{aligned} \therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\tan x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u} \frac{du}{dx} dx \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u} du \\ &= [\ln |u|]_{\frac{1}{2}}^1 \\ &= \ln 1 - \ln \frac{1}{2} \\ &= 0 + \ln 2 \quad \left\{ \ln \left(\frac{1}{a} \right) = -\ln a \right\} \\ &= \ln 2 \end{aligned}$$

e Let $u = 1 - \sin x$, $\frac{du}{dx} = -\cos x$

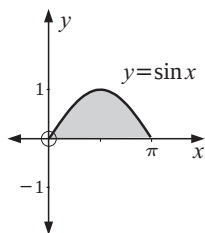
When $x = 0$, $u = 1 - \sin 0 = 1$

When $x = \frac{\pi}{6}$, $u = 1 - \sin \frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{6}} \frac{\cos x}{1 - \sin x} dx &= \int_1^{\frac{1}{2}} \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u} du \\ &= [\ln |u|]_{\frac{1}{2}}^1 \\ &= \ln 1 - \ln \frac{1}{2} \\ &= 0 + \ln 2 \\ &= \ln 2 \end{aligned}$$

EXERCISE 27D

1 a

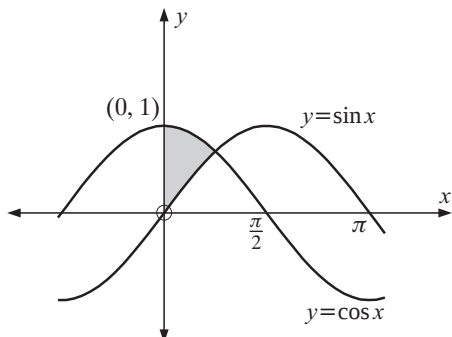


$$\begin{aligned} A &= \int_0^{\pi} \sin x \, dx \\ &= [-\cos x]_0^{\pi} \\ &= [-\cos \pi + \cos 0] \\ &= -(-1) + 1 \\ &= 2 \text{ units}^2 \end{aligned}$$

b We first note that $\sin^2 x \geq 0$ always, so the function never drops below the x -axis.

$$\begin{aligned} \therefore A &= \int_0^{\pi} \sin^2 x \, dx \\ &= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx \\ &= \left[\frac{x}{2} - \frac{1}{4} \sin(2x) \right]_0^{\pi} \\ &= \left[\frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right] - \left[0 - \frac{1}{4} \sin 0 \right] \\ &= \frac{\pi}{2} \text{ units}^2 \end{aligned}$$

2



The curves $y = \cos x$ and $y = \sin x$ meet when $x = \frac{\pi}{4}$.

$$\begin{aligned} \therefore A &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) \, dx \\ &= [\sin x + \cos x]_0^{\frac{\pi}{4}} \\ &= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0 + 1) \\ &= \frac{2}{\sqrt{2}} - 1 \\ &= (\sqrt{2} - 1) \text{ units}^2 \end{aligned}$$

3 a $y = \tan x$

A has y -coordinate 1 and lies on the graph of $y = \tan x$ on the interval $[0, \frac{\pi}{2}]$.

$$\therefore \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

i.e., A is at $(\frac{\pi}{4}, 1)$

b $\tan x = \frac{\sin x}{\cos x}$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\begin{aligned} \therefore \text{area} &= \int_0^{\frac{\pi}{4}} \tan x \, dx \\ &= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \left(-\frac{du}{dx}\right) dx \\ &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} \, du \\ &= [\ln |u|]_{\frac{1}{\sqrt{2}}}^1 \\ &= \ln 1 - \ln \frac{1}{\sqrt{2}} \\ &= \ln \sqrt{2} \quad \text{or} \quad \frac{1}{2} \ln 2 \text{ units}^2 \end{aligned}$$

4 a $y = \sin(2x)$ is the curve C_1
 $y = \sin x$ is the curve C_2

b The curves meet when $\sin(2x) = \sin x$

$$\therefore 2 \sin x \cos x - \sin x = 0$$

$$\therefore \sin x(2 \cos x - 1) = 0$$

i.e., $\sin x = 0$ or $\cos x = \frac{1}{2}$

$$\therefore x \text{ coordinate of A} = \frac{\pi}{3}$$

$$\therefore \text{A is at } \left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

c Area $= \int_0^{\frac{\pi}{3}} (\sin(2x) - \sin x) \, dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin(2x)) \, dx$

$$\begin{aligned} &= \left[-\frac{1}{2} \cos(2x) + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2} \cos(2x)\right]_{\frac{\pi}{3}}^{\pi} \\ &= \left(-\frac{1}{2} \cos \frac{2\pi}{3} + \cos \frac{\pi}{3}\right) - \left(-\frac{1}{2} \cos 0 + \cos 0\right) + \left(-\cos \pi + \frac{1}{2} \cos 2\pi\right) \\ &\quad - \left(-\cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3}\right) \\ &= \left(\frac{1}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2} + 1\right) + \left(1 + \frac{1}{2}\right) - \left(-\frac{1}{2} - \frac{1}{4}\right) \\ &= 2\frac{1}{2} \text{ units}^2 \end{aligned}$$

5 a $y = \cos(2x)$ is the curve C_2 and $y = \cos^2 x$ is the curve C_1

b When $x = 0$, $y = \cos 0 = 1$.

$$\therefore \text{A is at } (0, 1)$$

When $x = \frac{\pi}{4}$, $y = \cos(\frac{\pi}{2}) = 0$.

$$\therefore \text{B is at } \left(\frac{\pi}{4}, 0\right)$$

Also, $\cos^2(\frac{\pi}{2}) = 0$,

$$\therefore \text{C is at } \left(\frac{\pi}{2}, 0\right)$$

When $x = \frac{3\pi}{4}$, $y = \cos(\frac{3\pi}{2}) = 0$,

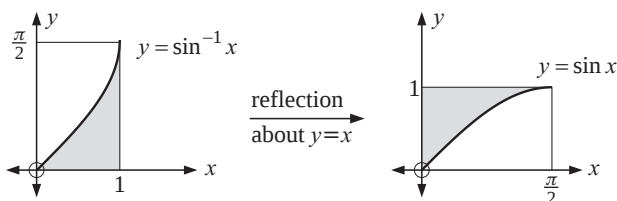
$$\therefore \text{D is at } \left(\frac{3\pi}{4}, 0\right)$$

$\cos(2\pi) = \cos^2 \pi = 1 \therefore y = \cos(2x)$ and $y = \cos^2 x = 1$ when $x = \pi \therefore \text{E is at } (\pi, 1)$

c $A = \int_0^{\pi} (\cos^2 x - \cos(2x)) \, dx$

$$\begin{aligned} &= \int_0^{\pi} \frac{1}{2} + \frac{1}{2} \cos(2x) - \cos(2x) \, dx \\ &= \int_0^{\pi} \frac{1}{2} - \frac{1}{2} \cos(2x) \, dx \\ &= \left[\frac{x}{2} - \frac{1}{4} \sin(2x)\right]_0^{\pi} = \left(\frac{\pi}{2} - 0\right) - (0 - 0) = \frac{\pi}{2} \text{ units}^2 \end{aligned}$$

- 6** $y = \sin^{-1} x$ is the reflection of the function $y = \sin x$ in the line $y = x$.
 $\therefore$ since $\sin^{-1} 1 = \frac{\pi}{2}$, the shaded areas in the figures below are equal.

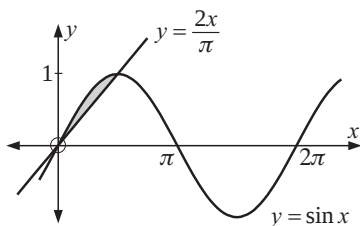


$$\begin{aligned}
 \therefore \int_0^1 \sin^{-1} x \, dx &= \text{area of rectangle} - \int_0^{\frac{\pi}{2}} \sin x \, dx \\
 &= \frac{\pi}{2} \times 1 - [-\cos x]_0^{\frac{\pi}{2}} \\
 &= \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}} \\
 &= \left(\frac{\pi}{2} - 1\right) \text{ units}^2
 \end{aligned}$$

REVIEW SET 27A

- 1 a** $\int \sin^7 x \cos x \, dx$
 $= \int u^7 \left(\frac{du}{dx}\right) dx$ on letting $u = \sin x$
 $= \int u^7 \, du$
 $= \frac{u^8}{8} + c$
 $= \frac{1}{8} \sin^8 x + c$
- b** Consider $\int \tan(2x) \, dx = \int \frac{\sin(2x)}{\cos(2x)} \, dx$
Let $u = \cos(2x)$, $\frac{du}{dx} = -2 \sin(2x)$
 $\therefore \sin(2x) = -\frac{1}{2} \frac{du}{dx}$
 $\therefore \int \tan(2x) \, dx = \int \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx}\right) dx$
 $= -\frac{1}{2} \int \frac{1}{u} \, du$
 $= -\frac{1}{2} \ln |u| + c$
 $= -\frac{1}{2} \ln |\cos(2x)| + c$
- c** Consider $\int e^{\sin x} \cos x \, dx$
Let $u = \sin x$, $\frac{du}{dx} = \cos x$
 $\therefore \int e^{\sin x} \cos x \, dx = \int e^u \frac{du}{dx} \, dx$
 $= \int e^u \, du$
 $= e^u + c$
 $= e^{\sin x} + c$
- 2** If $y = x \tan x$ then
 $\frac{dy}{dx} = 1 \tan x + x \left(\frac{1}{\cos^2 x}\right)$
 $= \frac{\sin x}{\cos x} + \frac{x}{\cos^2 x}$
- $\therefore \int \frac{\sin x}{\cos x} \, dx + \int \frac{x}{\cos^2 x} \, dx = x \tan x + c_1$
 $\therefore \int \frac{1}{u} \left(-\frac{du}{dx}\right) dx + \int \frac{x}{\cos^2 x} \, dx = x \tan x + c_1$
{letting $u = \cos x$, $\therefore \frac{du}{dx} = -\sin x$ }
 $\therefore -\int \frac{1}{u} \, du + \int \frac{x}{\cos^2 x} \, dx = x \tan x + c_1$
 $\therefore \int \frac{x}{\cos^2 x} \, dx = x \tan x + \ln |u| + c$
 $= x \tan x + \ln |\cos x| + c$

- 3** Using technology, the graphs meet when $x = 0$ and $x = \frac{\pi}{2}$. $\therefore$ enclosed area



$$\begin{aligned}
 &= \int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2x}{\pi} \right) dx \\
 &= \left[-\cos x - \frac{x^2}{\pi} \right]_0^{\frac{\pi}{2}} \\
 &= \left(0 - \frac{\pi}{4} \right) - (-1 - 0) \\
 &= \left(1 - \frac{\pi}{4} \right) \text{ units}^2
 \end{aligned}$$

4 a
$$\begin{aligned}
 \int_0^{\frac{\pi}{3}} \cos^2 \left(\frac{x}{2} \right) dx &= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2} \cos x \right) dx \\
 &= \left[\frac{x}{2} + \frac{1}{2} \sin x \right]_0^{\frac{\pi}{3}} \\
 &= \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} - 0 - \frac{1}{2} \sin 0 \\
 &= \frac{\pi}{6} + \frac{\sqrt{3}}{4}
 \end{aligned}$$

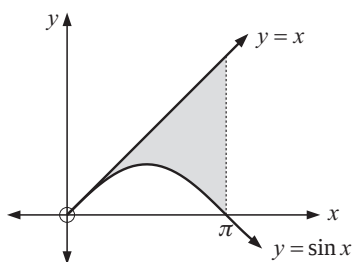
b
$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \tan x \, dx &= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx & \therefore \int_0^{\frac{\pi}{4}} \tan x \, dx &= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\
 & & &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} \, du \\
 & & &= [\ln |u|]_{\frac{1}{\sqrt{2}}}^1 \\
 & & &= \ln 1 - \ln \frac{1}{\sqrt{2}} \\
 & & &= \ln \sqrt{2} \\
 & & &= \frac{1}{2} \ln 2
 \end{aligned}$$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$
 When $x = 0$, $u = \cos 0 = 1$
 When $x = \frac{\pi}{4}$, $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

5
$$\begin{aligned}
 y &= \ln \left(\frac{1}{\cos x} \right) \\
 &= -\ln(\cos x) \\
 \therefore \frac{dy}{dx} &= -\frac{1}{\cos x} \times (-\sin x) \\
 &= \frac{\sin x}{\cos x} \\
 &= \tan x \\
 \therefore \int \tan x \, dx &= \ln \left(\frac{1}{\cos x} \right) + c \\
 &= -\ln(\cos x) + c \\
 &\quad \left\{ \text{since } \frac{1}{\cos x} > 0 \right\}
 \end{aligned}$$

6 Let $u = \sin \theta$ $\therefore \frac{du}{d\theta} = \cos \theta$

$$\begin{aligned}
 u\left(\frac{\pi}{6}\right) &= \sin \frac{\pi}{6} = \frac{1}{2} \\
 u\left(\frac{\pi}{2}\right) &= \sin \frac{\pi}{2} = 1 \\
 &\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta} d\theta \\
 &= \int_{\frac{1}{2}}^1 \frac{1}{u} \frac{du}{d\theta} d\theta \\
 &= \int_{\frac{1}{2}}^1 \frac{1}{u} \, du \\
 &= [\ln |u|]_{\frac{1}{2}}^1 \\
 &= \ln 1 - \ln \left(\frac{1}{2} \right) \\
 &= 0 + \ln 2 \\
 &= \ln 2
 \end{aligned}$$

7

Required area

$$\begin{aligned}
 &= \text{area of } \triangle - \text{area under sine curve} \\
 &= \frac{1}{2} \pi \times \pi - \int_0^{\pi} \sin x \, dx \\
 &= \frac{\pi^2}{2} - [-\cos x]_0^{\pi} \\
 &= \frac{\pi^2}{2} - [-\cos \pi + \cos 0] \\
 &= \frac{\pi^2}{2} - 2 \text{ units}^2
 \end{aligned}$$

REVIEW SET 27B

1 a $\int 4 \sin^2 \left(\frac{x}{2} \right) dx$

$$= \int 2 \left(2 \sin^2 \left(\frac{x}{2} \right) \right) dx$$

Now $\cos(2\theta) = 1 - 2 \sin^2 \theta$

$$\therefore 2 \sin^2 \theta = 1 - \cos(2\theta)$$

$$\therefore \int 4 \sin^2 \left(\frac{x}{2} \right) dx = 2 \int (1 - \cos x) dx$$

$$= 2 [x - \sin x] + c$$

$$= 2x - 2 \sin x + c$$

2 If $u = \cos x$, $\frac{du}{dx} = -\sin x \quad \therefore \int \frac{\sin x}{\cos^4 x} dx$

$$= \int u^{-4} \left(-\frac{du}{dx} \right) dx$$

$$= - \int u^{-4} du$$

$$= \frac{u^{-3}}{3} + c$$

$$= \frac{1}{3 \cos^3 x} + c$$

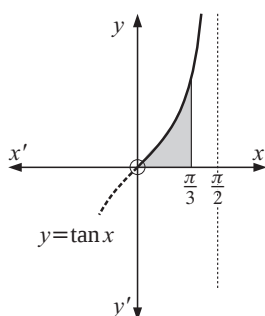
3 $\frac{d}{dx}(\sin(x^2))$

$$= 2x \cos(x^2)$$

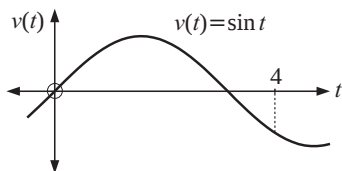
$$\therefore \int x \cos(x^2) dx$$

$$= \frac{1}{2} \int 2x \cos(x^2) dx$$

$$= \frac{1}{2} \sin(x^2) + c$$

4Now $\tan x \geq 0$ for $0 \leq x \leq \frac{\pi}{3}$

$$\begin{aligned}
 \therefore A &= \int_0^{\pi/3} \tan x \, dx & \text{Let } u = \cos x, \quad \frac{du}{dx} &= -\sin x \\
 &= \int_0^{\pi/3} \frac{\sin x}{\cos x} \, dx & \text{When } x &= \frac{\pi}{3}, \quad u = \cos \frac{\pi}{3} = \frac{1}{2} \\
 &= \int_1^{\frac{1}{2}} \frac{1}{u} \left(-\frac{du}{dx} \right) dx & \text{When } x &= 0, \quad u = \cos 0 = 1 \\
 &= \int_{\frac{1}{2}}^1 \frac{1}{u} \, du \\
 &= [\ln |u|]_{\frac{1}{2}}^1 \\
 &= \ln 1 - \ln \frac{1}{2} \\
 &= \ln 2 \text{ units}^2
 \end{aligned}$$

5

Since $v(t) = \sin t \text{ ms}^{-1}$, $v(t) = 0$ when $t = \pi$ seconds.

We may therefore see from the graph that the particle reverses its direction at time $t = \pi$.

$$\begin{aligned}
 \therefore \text{ the distance travelled is } D &= \int_0^{\pi} \sin t \, dt + \int_{\pi}^4 (-\sin t) \, dt \\
 &= [-\cos t]_0^{\pi} + [\cos t]_{\pi}^4 \\
 &= -\cos \pi + \cos 0 + \cos 4 - \cos \pi \\
 &= 1 + 1 + \cos 4 + 1 \\
 &= 3 + \cos 4 \\
 &\div 2.35 \text{ m}
 \end{aligned}$$

6

$$\begin{aligned}
 \text{a} \quad &\int_0^{\frac{\pi}{6}} \sin^2 \left(\frac{x}{2} \right) dx \\
 &= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2} \cos x \right) dx \\
 &= \left[\frac{x}{2} - \frac{1}{2} \sin x \right]_0^{\frac{\pi}{6}} \\
 &= \left(\frac{\pi}{12} - \frac{1}{4} \right) - (0 - 0) \\
 &= \frac{\pi}{12} - \frac{1}{4}
 \end{aligned}$$

b

$$\text{Let } u = \tan x, \quad \frac{du}{dx} = \frac{1}{\cos^2 x}$$

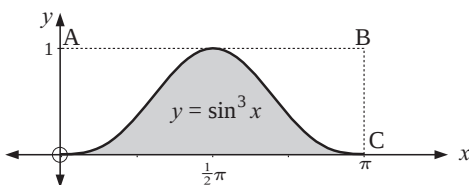
$$\text{When } x = \frac{\pi}{4}, \quad u = \tan \frac{\pi}{4} = 1$$

$$\text{When } x = \frac{\pi}{3}, \quad u = \tan \frac{\pi}{3} = \sqrt{3}$$

$$\begin{aligned}
 \therefore \quad &\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x \tan x} dx \\
 &= \int_1^{\sqrt{3}} \frac{1}{u} \frac{du}{dx} dx \\
 &= \int_1^{\sqrt{3}} \frac{1}{u} du \\
 &= [\ln |u|]_1^{\sqrt{3}} \\
 &= \ln \sqrt{3} - \ln 1 \\
 &= \frac{1}{2} \ln 3
 \end{aligned}$$

7

Consider the graph of
 $y = \sin^3 x$, $0 \leq x \leq \pi$



$$\text{Now } \int_0^{\pi} \sin^3 x \, dx = \text{shaded area}$$

But the shaded area < area of rectangle ABCO

$$\text{i.e., } \int_0^{\pi} \sin^3 x \, dx < \pi$$

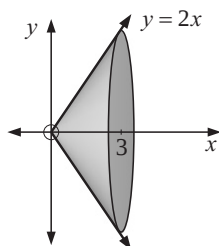
$$\therefore \int_0^{\pi} \sin^3 x \, dx < 4$$

Chapter 28

VOLUMES OF REVOLUTION

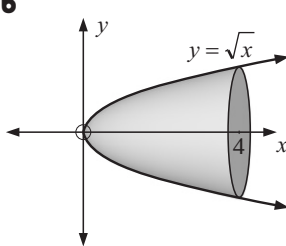
EXERCISE 28A.1

1 a



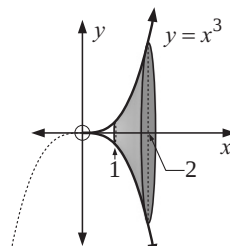
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^3 (2x)^2 dx \\
 &= 4\pi \int_0^3 x^2 dx \\
 &= 4\pi \left[\frac{x^3}{3} \right]_0^3 \\
 &= 4\pi(9 - 0) \\
 &= 36\pi \text{ units}^3
 \end{aligned}$$

b



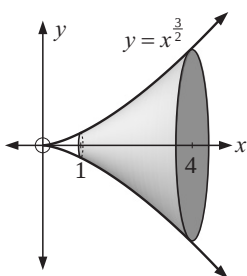
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^4 (\sqrt{x})^2 dx \\
 &= \pi \int_0^4 x dx \\
 &= \pi \left[\frac{x^2}{2} \right]_0^4 \\
 &= \pi(8 - 0) \\
 &= 8\pi \text{ units}^3
 \end{aligned}$$

c



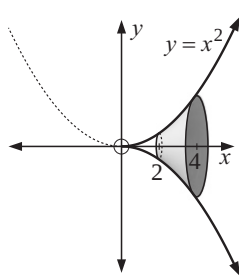
$$\begin{aligned}
 \text{Volume} &= \pi \int_1^2 (x^3)^2 dx \\
 &= \pi \int_1^2 x^6 dx \\
 &= \pi \left[\frac{x^7}{7} \right]_1^2 \\
 &= \pi \left(\frac{128}{7} - \frac{1}{7} \right) \\
 &= \frac{127\pi}{7} \text{ units}^3
 \end{aligned}$$

d



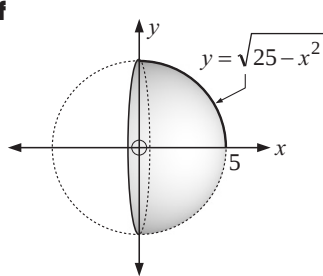
$$\begin{aligned}
 \text{Volume} &= \pi \int_1^4 \left(x^{\frac{3}{2}} \right)^2 dx \\
 &= \pi \int_1^4 x^3 dx \\
 &= \pi \left[\frac{x^4}{4} \right]_1^4 \\
 &= \pi \left(\frac{256}{4} - \frac{1}{4} \right) \\
 &= \frac{255\pi}{4} \text{ units}^3
 \end{aligned}$$

e



$$\begin{aligned}
 \text{Volume} &= \pi \int_2^4 (x^2)^2 dx \\
 &= \pi \int_2^4 x^4 dx \\
 &= \pi \left[\frac{x^5}{5} \right]_2^4 \\
 &= \pi \left[\frac{1024}{5} - \frac{32}{5} \right] \\
 &= \frac{992\pi}{5} \text{ units}^3
 \end{aligned}$$

f



$$\begin{aligned}
 \text{Volume} &= \pi \int_0^5 (25 - x^2) dx \\
 &= \pi \left[25x - \frac{x^3}{3} \right]_0^5 \\
 &= \pi \left[\left(125 - \frac{125}{3} \right) - 0 \right] \\
 &= \pi \left(\frac{2}{3} \right) 125 \\
 &= \frac{250\pi}{3} \text{ units}^3
 \end{aligned}$$

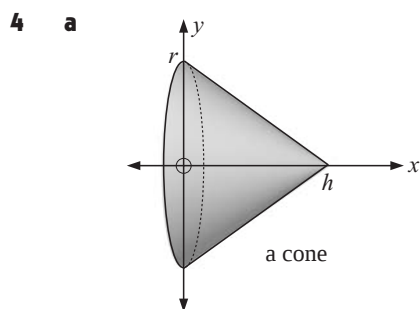
$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad V &= \pi \int_0^6 \left(\frac{x}{2} + 4\right)^2 dx \\
 &= \pi \int_0^6 \left(\frac{1}{4}x^2 + 4x + 16\right) dx \\
 &= \pi \left[\frac{x^3}{12} + \frac{4x^2}{2} + 16x \right]_0^6 \\
 &= \pi(18 + 72 + 96) - 0 \\
 &= 186\pi \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad V &= \pi \int_1^2 (x^2 + 3)^2 dx \\
 &= \pi \int_1^2 (x^4 + 6x^2 + 9) dx \\
 &= \pi \left[\frac{x^5}{5} + \frac{6x^3}{3} + 9x \right]_1^2 \\
 &= \pi \left(\left\{ \frac{32}{5} + 16 + 18 \right\} - \left\{ \frac{1}{5} + 2 + 9 \right\} \right) \\
 &= \pi \left(\frac{146}{5} \right) \\
 &= \frac{146\pi}{5} \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad V &= \pi \int_0^4 (e^x)^2 dx \\
 &= \pi \int_0^4 e^{2x} dx \\
 &= \pi \left[\frac{1}{2} e^{2x} \right]_0^4 \\
 &= \pi \left(\frac{1}{2} e^8 - \frac{1}{2} \right) \\
 &= \frac{\pi}{2} (e^8 - 1) \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad \text{Volume} &= \pi \int_5^8 y^2 dx \\
 &= \pi \int_5^8 (64 - x^2) dx \\
 &= \pi \left[64x - \frac{x^3}{3} \right]_5^8 \\
 &= \pi \left(\left\{ 512 - \frac{512}{3} \right\} - \left\{ 320 - \frac{125}{3} \right\} \right) \\
 &= 63\pi \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad &63\pi \text{ cm}^3 \\
 &\div 198 \text{ cm}^3
 \end{aligned}$$

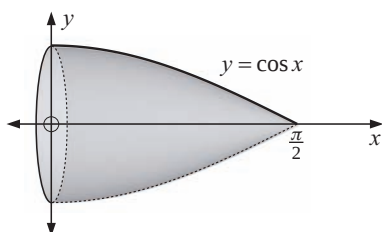


$$\begin{aligned}
 \mathbf{b} \quad \text{slope} &= \frac{r-0}{0-h} \\
 &= -\frac{r}{h} \\
 \therefore \text{equation is } y &= -\frac{r}{h}x + r
 \end{aligned}$$

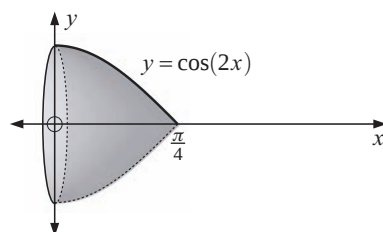
$$\begin{aligned}
 \mathbf{c} \quad V &= \pi \int_0^h \left(\frac{-r}{h}x + r \right)^2 dx \\
 &= \pi r^2 \int_0^h \left(-\frac{x}{h} + 1 \right)^2 dx \\
 &= \pi r^2 \int_0^h \left(\frac{x^2}{h^2} - \frac{2x}{h} + 1 \right) dx \\
 &= \pi r^2 \left[\frac{x^3}{3h^2} - \frac{2x^2}{2h} + x \right]_0^h \\
 &= \pi r^2 \left(\left\{ \frac{h}{3} - h + h \right\} - 0 \right) \\
 &= \frac{1}{3}\pi r^2 h \text{ units}^3
 \end{aligned}$$

5 a a sphere of radius r

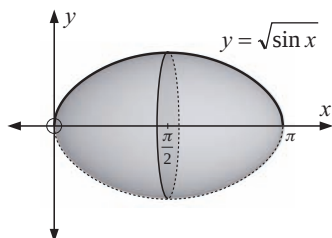
$$\begin{aligned}
 \mathbf{b} \quad V &= \pi \int_{-r}^r y^2 dx \\
 &= 2\pi \int_0^r (r^2 - x^2) dx \\
 &= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\
 &= 2\pi \left(r^3 - \frac{r^3}{3} - 0 \right) \\
 &= 2\pi \times \frac{2}{3} r^3 \\
 &= \frac{4}{3} \pi r^3 \text{ units}^3
 \end{aligned}$$

EXERCISE 28A.2**1 a**

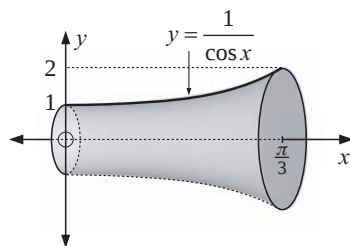
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2} \cos(2x) dx \\
 &= \pi \left[\frac{1}{2}x + \frac{1}{2} \left(\frac{1}{2} \right) \sin(2x) \right]_0^{\frac{\pi}{2}} \\
 &= \pi \left[\frac{\pi}{4} + \frac{1}{4} \sin \pi - 0 \right] \\
 &= \frac{\pi^2}{4} \text{ units}^3
 \end{aligned}$$

b

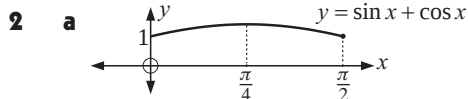
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} \cos^2(2x) dx \\
 &= \pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \pi \left[\frac{1}{2}x + \frac{1}{2} \left(\frac{1}{4} \right) \sin(4x) \right]_0^{\frac{\pi}{4}} \\
 &= \pi \left[\frac{\pi}{8} + \frac{1}{8} \sin \pi - 0 \right] \\
 &= \frac{\pi^2}{8} \text{ units}^3
 \end{aligned}$$

c

$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\pi} \sin x dx \\
 &= \pi [-\cos x]_0^{\pi} \\
 &= \pi [-\cos \pi - (-\cos 0)] \\
 &= \pi(2) \\
 &= 2\pi \text{ units}^3
 \end{aligned}$$

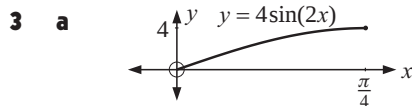
d

$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x} dx \\
 &= \pi [\tan x]_0^{\frac{\pi}{3}} \\
 &= \pi \left(\tan \frac{\pi}{3} - \tan 0 \right) \\
 &= \pi \times (\sqrt{3} - 0) \\
 &= \pi\sqrt{3} \text{ units}^3
 \end{aligned}$$



b Volume

$$\begin{aligned}
 &= \pi \int_0^{\frac{\pi}{4}} (\sin x + \cos x)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{4}} (\sin^2 x + 2 \sin x \cos x + \cos^2 x) dx \\
 &= \pi \int_0^{\frac{\pi}{4}} (1 + \sin(2x)) dx \\
 &= \pi \left[x - \frac{1}{2} \cos(2x) \right]_0^{\frac{\pi}{4}} \\
 &= \pi \left\{ \frac{\pi}{4} - \frac{1}{2} \cos\left(\frac{\pi}{2}\right) \right\} - \left\{ 0 - \frac{1}{2} \cos(0) \right\} \\
 &= \pi \left(\frac{\pi}{4} + \frac{1}{2} \right) \text{ units}^3
 \end{aligned}$$



b Volume

$$\begin{aligned}
 &= \pi \int_0^{\frac{\pi}{4}} (4 \sin(2x))^2 dx \\
 &= 16\pi \int_0^{\frac{\pi}{4}} \sin^2(2x) dx \\
 &= 16\pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2} \cos(4x) \right) dx \\
 &= 16\pi \left[\frac{x}{2} - \frac{1}{2} \left(\frac{1}{4} \right) \sin(4x) \right]_0^{\frac{\pi}{4}} \\
 &= 16\pi \left\{ \frac{\pi}{8} - \frac{1}{8} \sin \pi - \left(0 - \frac{1}{8} \sin 0 \right) \right\} \\
 &= 2\pi^2 \text{ units}^3
 \end{aligned}$$

EXERCISE 28B

1 a The graphs meet where

$$\begin{aligned}
 4 - x^2 &= 3 \\
 \therefore x^2 &= 1 \\
 \therefore x &= \pm 1 \\
 \therefore A &\text{ is } (-1, 3) \\
 \text{and } B &\text{ is } (1, 3)
 \end{aligned}$$

b

$$\begin{aligned}
 V &= 2\pi \int_0^1 ((4 - x^2)^2 - 3^2) dx \\
 &= 2\pi \int_0^1 (16 - 8x^2 + x^4 - 9) dx \\
 &= 2\pi \int_0^1 (x^4 - 8x^2 + 7) dx \\
 &= 2\pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 7x \right]_0^1 \\
 &= 2\pi \left\{ \frac{1}{5} - \frac{8}{3} + 7 - 0 \right\} \\
 &= \frac{136\pi}{15} \text{ units}^3
 \end{aligned}$$

2 a The graphs meet where

$$\begin{aligned}
 e^{\frac{x}{2}} &= e \\
 \therefore e^{\frac{x}{2}} &= e^1 \\
 \therefore \frac{x}{2} &= 1 \\
 \therefore x &= 2 \\
 \text{i.e., at } A(2, e)
 \end{aligned}$$

b

$$\begin{aligned}
 V &= \pi \int_0^2 \left(e^2 - \left(e^{\frac{x}{2}} \right)^2 \right) dx \\
 &= \pi \int_0^2 (e^2 - e^x) dx \\
 &= \pi [e^2 x - e^x]_0^2 \\
 &= \pi [2e^2 - e^2 - (0 - 1)] \\
 &= \pi [e^2 + 1] \text{ units}^3
 \end{aligned}$$

3 a The graphs meet where

$$\begin{aligned}
 x &= \frac{1}{x} \\
 \therefore x^2 &= 1 \\
 \therefore x &= \pm 1 \\
 \therefore x &= 1 \quad \{\text{as } x > 0\} \\
 \therefore A &\text{ is } (1, 1)
 \end{aligned}$$

b

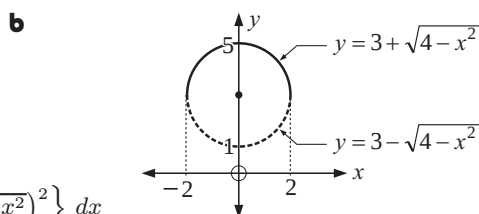
$$\begin{aligned}
 V &= \pi \int_1^2 \left(x^2 - \left(\frac{1}{x} \right)^2 \right) dx \\
 &= \pi \int_1^2 (x^2 - x^{-2}) dx \\
 &= \pi \left[\frac{x^3}{3} - \frac{x^{-1}}{-1} \right]_1^2 \\
 &= \pi \left\{ \left(\frac{8}{3} + \frac{1}{2} \right) - \left(\frac{1}{3} + 1 \right) \right\} \\
 &= \frac{11\pi}{6} \text{ units}^3
 \end{aligned}$$

- 4 a** The curves meet where

$$\begin{aligned}\sqrt{x-4} &= 1 \\ \therefore x-4 &= 1 \\ \therefore x &= 5 \\ \therefore A &\text{ is } (5, 1)\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad V &= \pi \int_5^8 \left((\sqrt{x-4})^2 - 1^2 \right) dx \\ &= \pi \int_5^8 (x-4-1) dx \\ &= \pi \int_5^8 (x-5) dx \\ &= \pi \left[\frac{x^2}{2} - 5x \right]_5^8 \\ &= \pi \left\{ (32-40) - \left(\frac{25}{2} - 25 \right) \right\} \\ &= \pi \left(\frac{9}{2} \right) = \frac{9\pi}{2} \text{ units}^3\end{aligned}$$

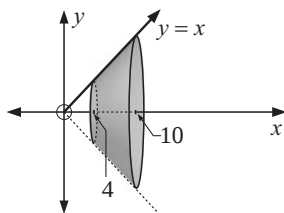
$$\begin{aligned}\mathbf{5 a} \quad x^2 + (y-3)^2 &= 4 \\ \therefore (y-3)^2 &= 4-x^2 \\ \therefore y-3 &= \pm\sqrt{4-x^2} \\ \therefore y &= 3 \pm \sqrt{4-x^2}\end{aligned}$$



$$\begin{aligned}\mathbf{c} \quad V &= \pi \int_{-2}^2 \left\{ (3 + \sqrt{4-x^2})^2 - (3 - \sqrt{4-x^2})^2 \right\} dx \\ &= 2\pi \int_0^2 \left\{ (3 + \sqrt{4-x^2})^2 - (3 - \sqrt{4-x^2})^2 \right\} dx \\ &= 2\pi \int_0^2 \left\{ (9 + 6\sqrt{4-x^2} + 4 - x^2) - (9 - 6\sqrt{4-x^2} + 4 - x^2) \right\} dx \\ &= 2\pi \int_0^2 12\sqrt{4-x^2} dx \\ &= 24\pi \int_0^2 \sqrt{4-x^2} dx \\ &= 24\pi^2 \text{ units}^3 \quad (\div 236.9 \text{ units}^3) \quad \{\text{using technology}\}\end{aligned}$$

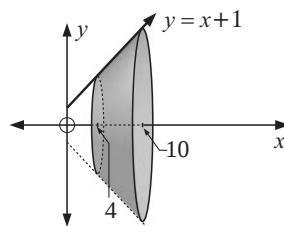
REVIEW SET 28

- 1 a**

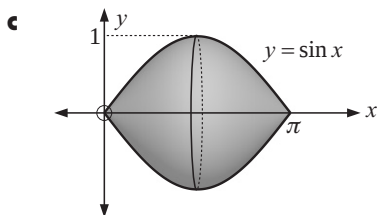


$$\begin{aligned}V &= \pi \int_4^{10} x^2 dx \\ &= \pi \left[\frac{x^3}{3} \right]_4^{10} \\ &= \pi \left(\frac{1000}{3} - \frac{64}{3} \right) \\ &= \frac{936\pi}{3} \\ &= 312\pi \text{ units}^3\end{aligned}$$

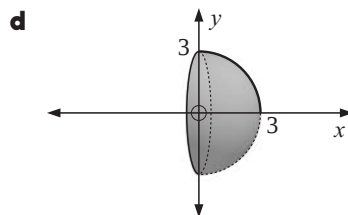
- b**



$$\begin{aligned}V &= \pi \int_4^{10} (x+1)^2 dx \\ &= \pi \left[\left(\frac{1}{3} \right) \frac{(x+1)^3}{3} \right]_4^{10} \\ &= \pi \left(\frac{11^3}{3} - \frac{5^3}{3} \right) \\ &= \frac{1206\pi}{3} = 402\pi \text{ units}^3\end{aligned}$$



$$\begin{aligned}
 V &= \pi \int_0^{\pi} \sin^2 x \, dx \\
 &= \pi \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx \\
 &= \pi \left[\frac{1}{2}x - \frac{1}{2} \left(\frac{1}{2} \right) \sin(2x) \right]_0^{\pi} \\
 &= \pi \left[\frac{1}{2}\pi - \frac{1}{4} \sin 2\pi - 0 \right] \\
 &= \frac{\pi^2}{2} \text{ units}^3
 \end{aligned}$$



$$\begin{aligned}
 V &= \pi \int_0^3 (9 - x^2) dx \\
 &= \pi \left[9x - \frac{x^3}{3} \right]_0^3 \\
 &= \pi \left\{ 27 - \frac{27}{3} - 0 \right\} \\
 &= 18\pi \text{ units}^3
 \end{aligned}$$

2 a $y = \cos(2x)$ meets the x -axis where

$$2x = \frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{4}$$

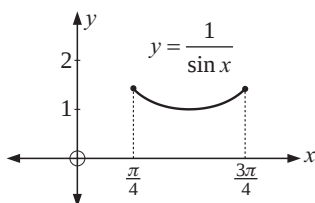
so, $V = \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2(2x) \, dx$

$$\begin{aligned}
 &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \pi \left[\frac{1}{2}x + \frac{1}{8} \sin(4x) \right]_{\frac{\pi}{16}}^{\frac{\pi}{4}} \\
 &= \pi \left\{ \left(\frac{\pi}{8} + \frac{1}{8} \sin \pi \right) - \left(\frac{\pi}{32} + \frac{1}{8} \sin \left(\frac{\pi}{4} \right) \right) \right\} \\
 &= \pi \left\{ \frac{\pi}{8} - \frac{\pi}{32} - \frac{1}{8} \left(\frac{1}{\sqrt{2}} \right) \right\} \\
 &= \pi \left\{ \frac{3\pi}{32} - \frac{1}{8\sqrt{2}} \right\} \text{ units}^3
 \end{aligned}$$

b

$$\begin{aligned}
 V &= \pi \int_0^2 (e^{-x} + 4)^2 dx \\
 &= \pi \int_0^2 (e^{-2x} + 8e^{-x} + 16) dx \\
 &= \pi \left[-\frac{1}{2}e^{-2x} + \frac{8}{-1}e^{-x} + 16x \right]_0^2 \\
 &= \pi \left\{ \left(-\frac{1}{2}e^{-4} - 8e^{-2} + 32 \right) - \left(-\frac{1}{2} - 8 \right) \right\} \\
 &= \pi \left\{ \frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2} \right\} \text{ units}^3 \quad (\doteq 123.8 \text{ units}^3)
 \end{aligned}$$

3



$$\begin{aligned}
 V &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{1}{\sin x} \right)^2 dx \\
 &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x)^{-2} dx \\
 &= \pi \times 2 \quad \{\text{using technology}\} \\
 &= 2\pi \text{ units}^3
 \end{aligned}$$

Note: This is not one of the functions we can integrate in this course, so technology is used.

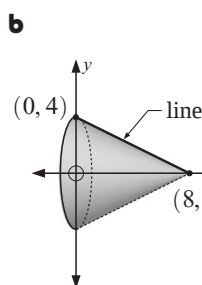
- 4** They meet where $x^2 = 4$
 $\therefore x = \pm 2$
 but $x > 0 \therefore x = 2$
 i.e., at $(2, 4)$

$$\begin{aligned} V &= \pi \int_0^2 (4^2 - (x^2)^2) dx \\ &= \pi \int_0^2 (16 - x^4) dx \\ &= \pi \left[16x - \frac{x^5}{5} \right]_0^2 \\ &= \pi \left(32 - \frac{32}{5} - 0 \right) \\ &= \frac{128\pi}{5} \text{ units}^3 \end{aligned}$$

- 5** They meet where $\sin x = \cos x$
 $\therefore \frac{\sin x}{\cos x} = 1$
 $\therefore \tan x = 1$
 $\therefore x = \frac{\pi}{4}$

$$\begin{aligned} V &= \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) dx \\ &= \pi \int_0^{\frac{\pi}{4}} \cos(2x) dx \\ &= \pi \left[\frac{1}{2} \sin(2x) \right]_0^{\frac{\pi}{4}} \\ &= \pi \left(\frac{1}{2} \sin\left(\frac{\pi}{2}\right) - \frac{1}{2} \sin 0 \right) \\ &= \pi \left(\frac{1}{2}(1) - 0 \right) \\ &= \frac{\pi}{2} \text{ units}^3 \end{aligned}$$

6 a $V = \frac{1}{3}\pi r^2 h$
 $= \frac{1}{3}\pi \times 4^2 \times 8$
 $= \frac{1}{3}\pi \times 128$
 $= \frac{128\pi}{3} \text{ units}^3$



$$\text{slope} = \frac{0-4}{8-0} = -\frac{1}{2}$$

$\therefore$ line has equation

$$y = -\frac{1}{2}x + 4$$

$$\begin{aligned} \therefore V &= \pi \int_0^8 \left(-\frac{1}{2}x + 4\right)^2 dx \\ &= \pi \int_0^8 \left(\frac{x^2}{4} - 4x + 16\right) dx \\ &= \pi \left[\frac{x^3}{12} - \frac{4x^2}{2} + 16x \right]_0^8 \\ &= \pi \left\{ \frac{128}{3} - 128 + 128 - 0 \right\} \\ &= \frac{128\pi}{3} \text{ units}^3 \end{aligned}$$

Chapter 29

STATISTICAL DISTRIBUTIONS

EXERCISE 29A

- 1
 - a The quantity of fat in a lamb chop is a continuous random variable.
 - b The mark out of 50 for the Geography test is a discrete random variable.
 - c The weight of seventeen year-old students is a continuous random variable.
 - d The volume of water in a cup of coffee is a continuous random variable.
 - e The number of trout in a lake is a discrete random variable.
 - f The number of hairs on a cat is a discrete random variable.
 - g The length of hairs on a horse is a continuous random variable.
 - h The height of a sky-scraper is a continuous random variable.
- 2
 - a
 - i The random variable is the height of water in the rain gauge.
 - ii $0 \leq x \leq 200$ mm
 - iii The variable is a continuous random variable.
 - b
 - i The random variable is the stopping distance.
 - ii $0 \leq x \leq 30$ m
 - iii The variable is a continuous random variable.
 - c
 - i The random variable could be the time taken for the switch to fail or it could be the number of times that the switch is turned on or off before it fails.
 - ii $0 \leq x \leq 10\,000$ hours if we are measuring time, or $1 \leq x \leq 2000$ if we are counting on/off.
 - iii The variable is a continuous random variable in the first case and a discrete random variable in the second.

- 3
 - a Since x is the number of weighing devices that are accurate, $x = 0, 1, 2, 3$ or 4 .

b

		YYNN		
		YNNY		
	YYYN	YNNY	NNNY	
	YYNY	NNYY	NNYN	
	YNYN	NNYY	NNYN	
YYYY	NYYY	NYYN	YNNN	NNNN
($x = 4$)	($x = 3$)	($x = 2$)	($x = 1$)	($x = 0$)

- c
 - i If two are accurate then $x = 2$
 - ii If at least two are accurate then 2, 3 or 4 are accurate i.e., $x = 2, 3$ or 4

- 4
 - a If 3 coins are tossed then the number of heads x can be 0, 1, 2 or 3.

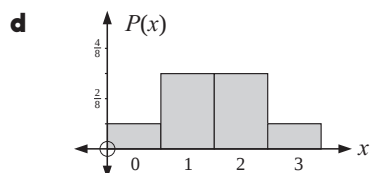
- b Suppose H represents heads, T represents tails.

	HHT	TTH		
	HTH	THT		
HHH	THH	HTT	TTT	
($x = 3$)	($x = 2$)	($x = 1$)	($x = 0$)	

- c

$$P(x = 0) = \frac{1}{8} \quad P(x = 1) = \frac{3}{8}$$

$$P(x = 2) = \frac{3}{8} \quad P(x = 3) = \frac{1}{8}$$



EXERCISE 29B

- 1 a** Since this is a probability distribution $\sum P(i) = 1$

$$\therefore a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$$

$$\therefore a + 0.4512 = 1$$

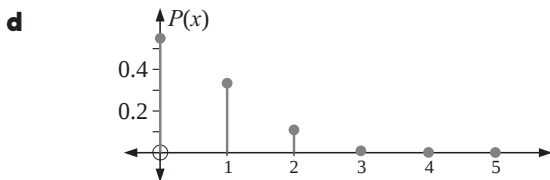
$$\therefore a = 0.5488$$

$P(0)$ is the probability that Jason does not hit a home run in a game.

- b** $P(2) = 0.1088$ (from table)

$$\begin{aligned} \text{c } P(1) + P(2) + P(3) + P(4) + P(5) &= 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.00 \\ &= 0.4512 \end{aligned}$$

This represents the probability that Jason hits *at least one* home run in a game.



- 2 a** The probabilities all lie in $0 \leq P(i) \leq 1$, $\therefore$ OK

$$\text{Sum of probabilities } \sum P(i) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$$

$$\therefore \text{sum of probabilities} \neq 1$$

$\therefore$ this is not a valid probability distribution.

- b** Notice that $P(5) = -0.2$, $\therefore$ not all of the probabilities lie in $0 \leq P(i) \leq 1$

$\therefore$ this is not a valid probability distribution.

- 3 a** The random variable represents the number of hits that Sally has in each game.

$$\text{b } 0.7 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1 \quad \{\text{since } \sum p(i) = 1\}$$

$$\therefore k + 0.77 = 1$$

$$\therefore k = 0.23$$

$$\text{c i } P(x \geq 2)$$

$$= P(x = 2 \text{ or } x = 3 \text{ or } x = 4 \text{ or } x = 5)$$

$$= P(2) + P(3) + P(4) + P(5)$$

$$= 0.23 + 0.46 + 0.08 + 0.02$$

$$= 0.79$$

$$\text{ii } P(1 \leq x \leq 3)$$

$$= P(1) + P(2) + P(3)$$

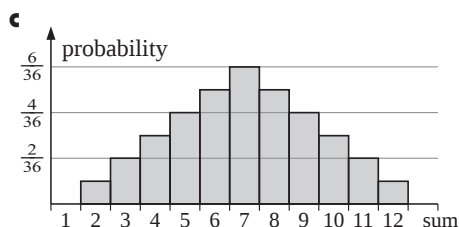
$$= 0.14 + 0.23 + 0.46$$

$$= 0.83$$

- 4 a** Rolling a die twice, sample space:

6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
roll 1 4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	1	2	3	4	5	6
	roll 2					

$$\begin{aligned} \text{b } P(0) &= 0 & P(1) &= 0 & P(2) &= \frac{1}{36} \\ P(3) &= \frac{2}{36} & P(4) &= \frac{3}{36} & P(5) &= \frac{4}{36} \\ P(6) &= \frac{5}{36} & P(7) &= \frac{6}{36} & P(8) &= \frac{5}{36} \\ P(9) &= \frac{4}{36} & P(10) &= \frac{3}{36} & P(11) &= \frac{2}{36} \\ P(12) &= \frac{1}{36} \end{aligned}$$



5 a $P(x) = k(x+2), \quad x = 1, 2, 3$

$\therefore P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$

and since this is a probability distribution, $\sum P(i) = 3k + 4k + 5k$

and $12k = 1 \quad \{\text{as } \sum P(i) = 1\}$

$\therefore k = \frac{1}{12}$

b $P(x) = \frac{k}{x+1} \quad x = 0, 1, 2, 3$

$\therefore P(0) = k, \quad P(1) = \frac{k}{2}, \quad P(2) = \frac{k}{3}, \quad P(3) = \frac{k}{4}$

Now $k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1 \quad \{\text{as } \sum P(i) = 1\}$

$\therefore \frac{12k + 6k + 4k + 3k}{12} = 1$

$\therefore \frac{25k}{12} = 1$

$\therefore k = \frac{12}{25}$

6 a $P(x) = k \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \quad x = 0, 1, 2, 3, 4$

$P(0) = k \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = \frac{16k}{81} \quad P(1) = k \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 = \frac{8k}{81} \quad P(2) = k \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \frac{4k}{81}$

$P(3) = k \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 = \frac{2k}{81} \quad P(4) = k \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = \frac{k}{81}$

b $\therefore \frac{16k}{81} + \frac{8k}{81} + \frac{4k}{81} + \frac{2k}{81} + \frac{k}{81} = 1 \quad \{\text{as } \sum P(i) = 1\}$

$\therefore \frac{31k}{81} = 1$

$\therefore k = \frac{81}{31} \div 2.6130$

$P(x \geq 2) = P(2) + P(3) + P(4)$

$= \frac{4k}{81} + \frac{2k}{81} + \frac{k}{81}$

$= \frac{7k}{81}$

$= \frac{7 \times 2.6130}{81}$

$\div 0.2258$

7 a $P(\text{no faulty component})$

$= P(x = 0)$

$= P(0)$

$= C_0^{10} (0.04)^0 (0.96)^{10-0}$

$= C_0^{10} (0.96)^{10}$

$= 0.96^{10}$

$= 0.6648$

b $P(\text{at least one faulty component})$

$= 1 - P(\text{none are faulty})$

$= 1 - 0.6648$

$= 0.3352$

EXERCISE 29C

1 $P(\text{rain}) = 0.28 \quad \therefore \text{would expect rain on } 0.28 \times 365.25 \text{ days a year}$
i.e., $\div 102$ days.

2 a $P(\text{HHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

b For 200 tosses, we expect $200 \times \frac{1}{8}$ to be '3 heads'
i.e., 25 to be '3 heads'

3 a $P(\text{back}) \div \frac{125}{325}$ (i.e., $\div 0.385$) **b** Expectation is $\frac{125}{325} \times 50 \div 19$ 'backs'

4 a i $P(\text{wins \$10}) = P(\text{rolls a 6}) = \frac{1}{6}$

ii $P(\text{wins \$4}) = P(\text{rolls 4 or 5}) = \frac{2}{6}$ (or $\frac{1}{3}$)

iii $P(\text{wins \$1}) = P(\text{rolls 1, 2 or 3}) = \frac{3}{6}$ (or $\frac{1}{2}$)

b i Expectation $= \frac{2}{6} \times \$4 \div \1.33

ii Expectation $= \frac{3}{6} \times \$1 = \0.50

iii Expectation $= \frac{1}{6} \times \$10 + \frac{2}{6} \times \$4 + \frac{3}{6} \times \$1 = \frac{1}{6}(\$21) = \$3.50$

c It costs \$4 to play and the expected return is \$3.50.

$\therefore$ you expect to lose \$0.50 per game.

d So, over 100 games you expect to lose $100 \times \$0.50 = \50 .

5 Expect to save $\frac{3}{10} \times 90 = 27$ goals

6 Expect to see snow falling on $\frac{3}{7} \times 5 \times 7$ days = 15 days

7 $P(\text{double}) = P(1, 1 \text{ or } 2, 2 \text{ or } 3, 3 \text{ or } 4, 4 \text{ or } 5, 5 \text{ or } 6, 6)$
 $= \frac{6}{36}$ {6 of the possible 36 outcomes}
 $= \frac{1}{6}$

$\therefore$ when rolling 180 times we expect $180 \times \frac{1}{6} = 30$ doubles.

8

1st 2nd $P(\text{two greens}) = P(\text{GG}) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$

$\therefore$ for 300 repetitions we expect $300 \times \frac{2}{7} \div 86$ times

9 a $165 + 87 + 48 = 300$ **i** $P(\text{votes A}) \div \frac{165}{300}$ **ii** $P(\text{votes B}) \div \frac{87}{300}$ **iii** $P(\text{votes C}) \div \frac{48}{300}$

b i Expect $7500 \times \frac{165}{300} = 4125$ to vote A **ii** Expect $7500 \times \frac{87}{300} = 2175$ to vote B

iii Expect $7500 \times \frac{48}{300} = 1200$ to vote C

10 a Expect to win $\frac{1}{6} \times \$1 + \frac{1}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$4 + \frac{1}{6} \times \$5 + \frac{1}{6} \times \$6 = \frac{1}{6} \times \$21 = \3.50

b No, as on each occasion he would expect to lose \$0.50 (on average).

11

result	win
HH	\$10
HT or TH	\$3
TT	-\$5

a Expectation $= \frac{1}{4} \times \$10 + \frac{2}{4} \times \$3 + \frac{1}{4} \times (-\$5) = \$2.75$

b Expected win per game (payout) = \$2.75
 $\therefore$ organiser would charge $\$2.75 + \$1.00 = \$3.75$ to play each game.

12

result	win
H	\$2
T	-\$1

For playing *once*,expect to win $\frac{1}{2} \times \$2 + \frac{1}{2} \times (-\$1) = \$0.50$ $\therefore$ for 3 games, expect to win \$1.50.**13**

result	win
1	\$1
2	\$2
3	\$3
4	\$5
5	\$10
6	\$25

a Expectation

$$= \frac{1}{6} \times \$1 + \frac{1}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$5 + \frac{1}{6} \times \$10 + \frac{1}{6} \times \$25$$

$$= \frac{1}{6} \times \$46$$

$$\div \$7.67$$

b For 1 game at \$10 a game, expect to lose $\$10 - \$7\frac{2}{3} = \$2\frac{2}{3}$

$$\text{So, for 100 games, expect to lose } \$2\frac{2}{3} \times 100 = \$233.33$$

$$\mathbf{14} \quad P(R) = \frac{3}{6}, \quad P(B) = \frac{2}{6}, \quad P(W) = \frac{1}{6}$$

$$\therefore \text{ expectation is } \frac{3}{6} \times \$1 + \frac{2}{6} \times \$2 + \frac{1}{6} \times \$5 = \frac{1}{6} \times \$12 = \$2$$

EXERCISE 29D**1**

x_i	0	1	2	3	4	5	> 5
$P(x_i)$	0.54	0.26	0.15	0.03	0.01	0.01	0.00

$$\mathbf{a} \quad \mu = \sum x_i p_i$$

$$= 0 \times 0.54 + 1 \times 0.26 + \dots + 5 \times 0.01$$

$$= 0.26 + 0.30 + 0.09 + 0.04 + 0.05$$

$$= 0.74 \quad \text{i.e., over a long period the mean number of deaths per dozen crayfish is 0.74.}$$

$$\mathbf{b} \quad \sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

$$= \sqrt{(0 - 0.74)^2 \times 0.54 + (1 - 0.74)^2 \times 0.26 + \dots + (5 - 0.74)^2 \times 0.01}$$

$$\div 0.9962$$

$$\mathbf{2} \quad P(x) = \frac{x^2 + x}{20} \quad \text{for } x = 1, 2, 3$$

x_i	1	2	3
$P(x_i) = p_i$	$\frac{2}{20} = 0.1$	$\frac{6}{20} = 0.3$	$\frac{12}{20} = 0.6$

$$\mu = \sum x_i p_i$$

$$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.6$$

$$= 2.5$$

$$\text{and } \sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

$$= \sqrt{(1 - 2.5)^2 \times 0.1 + (2 - 2.5)^2 \times 0.3 + (3 - 2.5)^2 \times 0.6}$$

$$\div 0.6708$$

$$\mathbf{3} \quad \mathbf{a} \quad P(x) = C_x^3 (0.4)^x (0.6)^{3-x} \quad \text{for } x = 0, 1, 2, 3$$

$$\therefore P(0) = C_0^3 (0.4)^0 (0.6)^3$$

$$= (0.6)^3$$

$$= 0.216$$

$$P(1) = C_1^3 (0.4)^1 (0.6)^2$$

$$= 3(0.4)(0.6)^2$$

$$= 0.432$$

$$P(2) = C_2^3 (0.4)^2 (0.6)^1$$

$$= 3(0.16)(0.6)$$

$$= 0.288$$

$$P(3) = C_3^3 (0.4)^3 (0.6)^0$$

$$= 1(0.4)^3$$

$$= 0.064$$

x_i	0	1	2	3
$P(x_i)$	0.216	0.432	0.288	0.064

$$\begin{aligned}\mathbf{b} \quad \mu &= \sum x_i p_i = 0(0.216) + 1(0.432) + 2(0.288) + 3(0.064) = 1.2 \\ \sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ &= \sqrt{(0 - 1.2)^2(0.216) + (1 - 1.2)^2(0.432) + (2 - 1.2)^2 \times 0.288 + (3 - 1.2)^2 \times 0.064} \\ &\div 0.8485\end{aligned}$$

$$\begin{aligned}\mathbf{4} \quad \sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ \therefore \sigma^2 &= \sum (x_i - \mu)^2 p_i \\ &= (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \dots + (x_n - \mu)^2 p_n \\ &= (x_1^2 - 2x_1\mu + \mu^2)p_1 + (x_2^2 - 2x_2\mu + \mu^2)p_2 + \dots + (x_n^2 - 2x_n\mu + \mu^2)p_n \\ &= (x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \dots + x_n^2 p_n) - 2\mu(x_1 p_1 + x_2 p_2 + \dots + x_n p_n) \\ &\quad + \mu^2(p_1 + p_2 + p_3 + \dots + p_n)\end{aligned}$$

Now $p_1 + p_2 + \dots + p_n = 1$

$$\begin{aligned}\therefore \sigma^2 &= \sum x_i^2 p_i - 2\mu(\sum x_i p_i) + \mu^2(1) \\ &= \sum x_i^2 p_i - 2\mu(\mu) + \mu^2 \quad \{\text{since } \sum x_i p_i = \mu\} \\ &= \sum x_i^2 p_i - \mu^2\end{aligned}$$

5 a

x_i	1	2	3	4	5
$P(x_i)$	0.1	0.2	0.4	0.2	0.1

$$\begin{aligned}\mathbf{b} \quad \mu &= \sum x_i p_i \\ &= 1(0.1) + 2(0.2) + \dots + 5(0.1) \\ &= 0.1 + 0.4 + 1.2 + 0.8 + 0.5 \\ &= 3 \\ \sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ &= \sqrt{\sum x_i^2 p_i - \mu^2} \\ &= \sqrt{1^2(0.1) + 2^2(0.2) + \dots + 5^2(0.1) - (3.0)^2} \\ &= \sqrt{0.1 + 0.8 + 3.6 + 3.2 + 2.5 - 9} \\ &= \sqrt{1.2} \\ &\div 1.095\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \mathbf{i} \quad &P(\mu - \sigma < x < \mu + \sigma) \\ &= P(3 - 1.095 < x < 3 + 1.095) \\ &\div P(x = 2, 3, 4) \\ &\div 0.2 + 0.4 + 0.2 \\ &\div 0.8 \\ \mathbf{ii} \quad &P(\mu - 2\sigma < x < \mu + 2\sigma) \\ &= P(3 - 2.19 < x < 3 + 2.19) \\ &= P(0.81 < x < 5.19) \\ &\div P(x = 1, 2, 3, 4 \text{ or } 5) \\ &\div 0.1 + 0.2 + 0.4 + 0.2 + 0.1 \\ &\div 1.0\end{aligned}$$

6 Let x be the payout values, then $x = \$20\,000$, $\$8000$, or $\$0$

$\therefore$ the probability distribution is

x_i	20 000	8000	0
$P(x_i) = p_i$	0.0025	0.03	0.9675

$$\begin{aligned}\text{The expectation is } \mu &= \sum x_i p_i = 20\,000(0.0025) + 8000(0.03) + 0(0.9675) \\ &= \$290\end{aligned}$$

i.e., the company expects to pay out \$290 on average in the long run

$\therefore$ the company should charge $\$290 + \$100 = \$390$

7

		die 2					
die 1		1	2	3	4	5	6
	1	1	2	3	4	5	6
	2	2	2	3	4	5	6
	3	3	3	3	4	5	6
	4	4	4	4	4	5	6
	5	5	5	5	5	5	6
	6	6	6	6	6	6	6

a

m_i	1	2	3	4	5	6
$P(m_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b

$$\begin{aligned}
 \mu &= \sum m_i p_i \\
 &= 1 \left(\frac{1}{36} \right) + 2 \left(\frac{3}{36} \right) + 3 \left(\frac{5}{36} \right) + \dots + 6 \left(\frac{11}{36} \right) \\
 &= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36} \\
 &= \frac{161}{36} \\
 &\div 4.472 \\
 \sigma &= \sqrt{\sum m_i^2 p_i - \mu^2} \\
 &= \sqrt{1^2 \left(\frac{1}{36} \right) + 2^2 \left(\frac{3}{36} \right) + \dots + 6^2 \left(\frac{11}{36} \right) - (4.4722)^2} \\
 &= \sqrt{1.97165} \\
 &\div 1.404
 \end{aligned}$$

- 8** One coin is tossed a maximum of three times, until either one tail or three heads occurs. Let x be the number of tosses needed. $\therefore x_i = 1, 2$ or 3 .

$$P(x = 1) = P(T) = \frac{1}{2} = 0.50$$

$$P(x = 2) = P(HT) = \left(\frac{1}{2} \right)^2 = \frac{1}{4} = 0.25$$

$$P(x = 3) = P(HHT \text{ or } HHH) = \left(\frac{1}{2} \right)^3 + \left(\frac{1}{2} \right)^4 = \frac{2}{8} = 0.25$$

$\therefore$ the probability distribution is

x_i	1	2	3
$P(x_i)$	0.50	0.25	0.25

$$\begin{aligned}
 \text{and } \therefore \mu &= \sum x_i p_i \\
 &= 1(0.5) + 2(0.25) + 3(0.25) \\
 &= 1.75
 \end{aligned}$$

EXERCISE 29E

- 1**
 - a** The binomial distribution applies, as tossing a coin has one of two possible outcomes (H or T) and each toss is independent of every other toss.
 - b** The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
 - c** The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
 - d** The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.
 - e** The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials.
- 2** Let X be the number of boys in the family. $\therefore X = 0, 1, 2, 3, 4, 5, 6$ and X is $\text{Bin}(6, \frac{1}{2})$.
 - a** $P(X = 6)$
 $= \text{binompdf}(6, \frac{1}{2}, 6)$
 $\div 0.0156$
 - b** $P(X = 2)$
 $= \text{binompdf}(6, \frac{1}{2}, 2)$
 $\div 0.234$
 - c** $P(\text{more than 4 girls})$
 $= P(X \leq 1)$
 $= \text{binomcdf}(6, \frac{1}{2}, 1)$
 $\div 0.109$
 - d** $P(\text{more boys than girls})$
 $= P(X \geq 4)$
 $= 1 - P(X \leq 3)$
 $= 1 - \text{binomcdf}(6, \frac{1}{2}, 3)$
 $= 0.344$

- 3** X is the random variable for the number working night-shift.

$\therefore X = 0, 1, 2, 3, 4, 5, 6, 7$ and X is Bin (7, 0.35).

a $P(X = 3)$

$$= \text{binompdf}(7, 0.35, 3)$$

$$\div 0.268$$

b $P(X < 4)$

$$= P(X \leq 3)$$

$$= \text{binomcdf}(7, 0.35, 3)$$

$$\div 0.800$$

c $P(\text{at least 4 work night-shift})$

$$= P(X \geq 4)$$

$$= 1 - P(X \leq 3)$$

$$\div 1 - 0.800$$

$$\div 0.200$$

- 4** X is the random variable for the number of apples with a blemish.

$\therefore X = 0, 1, 2, 3, 4, \dots, 25$ and X is Bin (25, 0.05).

a $P(X = 2)$

$$= \text{binompdf}(25, 0.05, 2)$$

$$\div 0.231$$

b $P(\text{at least one has a blemish})$

$$= 1 - P(\text{none of them have a blemish})$$

$$= 1 - P(X = 0)$$

$$= 1 - \text{binompdf}(25, 0.05, 0)$$

$$\div 0.723$$

- 5** X is the number of faulty items.

$\therefore X = 0, 1, 2, 3, \dots, 12$ and X is Bin (12, 0.06).

a $P(X = 0)$

$$= \text{binompdf}(12, 0.06, 0)$$

$$\div 0.476$$

b $P(\text{at most one is faulty})$

$$= P(X \leq 1)$$

$$= \text{binomcdf}(12, 0.06, 1)$$

$$\div 0.840$$

c $P(\text{at least 2 are faulty})$

$$= P(X \geq 2)$$

$$= 1 - P(X \leq 1)$$

$$\div 0.160 \quad \{\text{from b}\}$$

d $P(\text{less than 4 are faulty})$

$$= P(X < 4)$$

$$= 1 - P(X \leq 3)$$

$$= \text{binomcdf}(12, 0.06, 3)$$

$$= 0.996$$

- 6** X is the random variable for the number of correct answers.

$\therefore X = 0, 1, 2, 3, 4, \dots, 20$ and X is Bin (20, $\frac{1}{2}$)

a $P(X = 20)$

$$= \text{binompdf}(20, \frac{1}{2}, 20)$$

$$\div 9.54 \times 10^{-7}$$

b $P(\text{half correct})$

$$= P(X = 10)$$

$$= \text{binompdf}(20, \frac{1}{2}, 10)$$

$$\div 0.176$$

c $P(\text{less than half correct})$

$$= P(X \leq 9)$$

$$= \text{binomcdf}(20, \frac{1}{2}, 9)$$

$$\div 0.412$$

d $P(\text{at least 15 correct})$

$$= P(X \geq 15)$$

$$= 1 - P(X \leq 14)$$

$$= 1 - \text{binomcdf}(20, \frac{1}{2}, 14)$$

$$= 0.0207$$

- 7** X is the random variable for the number of times in a week when the bus is on time.

$\therefore X = 0, 1, 2, 3, 4, 5, 6$ or 7 and X is Bin (7, 0.6) {late 2 in 5, on time 3 in 5}

$$\begin{aligned}\mathbf{a} \quad & P(X = 7) \\ &= \text{binompdf}(7, 0.6, 7) \\ &\doteq 0.0280\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & P(X = 6) \\ &= \text{binompdf}(7, 0.6, 6) \\ &\doteq 0.131\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & P(\text{on time only on Monday}) \\ &= 0.6 \times (0.4)^6 \\ &= 0.00246\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & P(X \geq 4) \\ &= 1 - P(X \leq 3) \\ &= 1 - \text{binomcdf}(7, 0.6, 3) \\ &\doteq 0.710\end{aligned}$$

8 X is the random variable for the number with flu

$\therefore X = 0, 1, 2, 3, \dots, 25$ and X is Bin(25, 0.3)

$$\begin{aligned}\mathbf{a} \quad & P(X \geq 2) \\ &= 1 - P(X \leq 1) \\ &= 1 - \text{binomcdf}(25, 0.3, 1) \\ &\doteq 0.998\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & P(\text{test cancelled}) \\ &= P(X \geq 6) \quad \{20\% \text{ of } 25 = 5\} \\ &= 1 - P(X \leq 5) \\ &= 1 - \text{binomcdf}(25, 0.3, 5) \\ &\doteq 0.8065\end{aligned}$$

9 X is the number of goals thrown.

$\therefore X = 0, 1, 2, 3, 4, \dots, 20$ and X is Bin(20, 0.94)

$$\begin{aligned}\mathbf{a} \quad & P(X = 20) \\ &= \text{binompdf}(20, 0.94, 20) \\ &\doteq 0.290\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & P(X \geq 18) \\ &= 1 - P(X \leq 17) \\ &= 1 - \text{binomcdf}(20, 0.94, 17) \\ &\doteq 0.885\end{aligned}$$

EXERCISE 29F

1 x is distributed Bin(6, p)

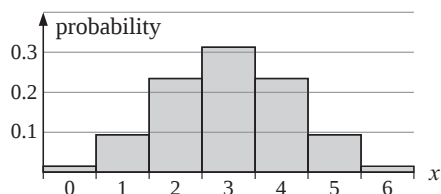
a If $p = 0.5$, x is distributed Bin(6, 0.5)

$$\begin{aligned}\mathbf{i} \quad & \mu = np \quad \text{and} \quad \sigma = \sqrt{npq} \\ &= 6 \times 0.5 \quad \quad \quad = \sqrt{6 \times 0.5 \times 0.5} \\ &= 3 \quad \quad \quad \doteq 1.2247\end{aligned}$$

$$\begin{array}{llll} \mathbf{ii} \quad & P(x = 0) & P(x = 1) & P(x = 2) & P(x = 3) \\ & = C_0^6(0.5)^0(0.5)^6 & = C_1^6(0.5)^1(0.5)^5 & = C_2^6(0.5)^2(0.5)^4 & = C_3^6(0.5)^3(0.5)^3 \\ & = 0.0156 & = 0.0938 & = 0.2344 & = 0.3125 \\ & P(x = 4) & P(x = 5) & P(x = 6) \\ & = C_4^6(0.5)^4(0.5)^2 & = C_5^6(0.5)^5(0.5)^1 & = C_6^6(0.5)^6(0.5)^0 \\ & = 0.2344 & = 0.0938 & = 0.0156 \end{array}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156

iii The distribution is bell-shaped.

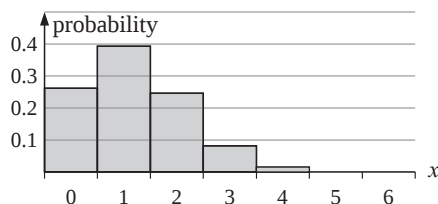


b If $p = 0.2$, x is distributed $\text{Bin}(6, 0.2)$

$$\begin{aligned} \text{i} \quad \mu &= np & \text{and} \quad \sigma &= \sqrt{npq} \\ &= 6 \times 0.2 & &= \sqrt{6 \times 0.2 \times 0.8} \\ &= 1.2 & &\div 0.980 \end{aligned}$$

$$\begin{aligned} \text{ii} \quad P(x=0) &= C_0^6(0.2)^0(0.8)^6 = 0.2621 & P(x=1) &= C_1^6(0.2)^1(0.8)^5 = 0.3932 & P(x=2) &= C_2^6(0.2)^2(0.8)^4 = 0.2458 & P(x=3) &= C_3^6(0.2)^3(0.8)^3 = 0.0819 \\ P(x=4) &= C_4^6(0.2)^4(0.8)^2 = 0.0154 & P(x=5) &= C_5^6(0.2)^5(0.8)^1 = 0.0015 & P(x=6) &= C_6^6(0.2)^6(0.8)^0 = 0.0001 \end{aligned}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001



iii The distribution is skewed to the right, i.e., positively skewed.

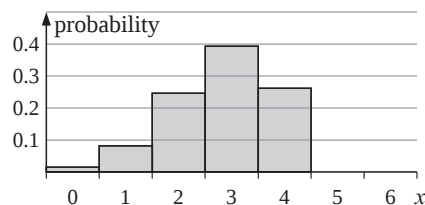
c If $p = 0.8$, x is distributed $\text{Bin}(6, 0.8)$

$$\begin{aligned} \text{i} \quad \mu &= np & \sigma &= \sqrt{npq} \\ &= 6 \times 0.8 & &= \sqrt{6 \times 0.8 \times 0.2} \\ &= 4.8 & &\div 0.980 \end{aligned}$$

$$\begin{aligned} \text{ii} \quad P(x=0) &= C_0^6(0.8)^0(0.2)^6 = 0.0001 & P(x=1) &= C_1^6(0.8)^1(0.2)^5 = 0.0015 & P(x=2) &= C_2^6(0.8)^2(0.2)^4 = 0.0154 & P(x=3) &= C_3^6(0.8)^3(0.2)^3 = 0.0819 \\ P(x=4) &= C_4^6(0.8)^4(0.2)^2 = 0.2458 & P(x=5) &= C_5^6(0.8)^5(0.2)^1 = 0.3932 & P(x=6) &= C_6^6(0.8)^6(0.2)^0 = 0.2621 \end{aligned}$$

Notice that this is the reverse of **b**.

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii This distribution is the exact reflection of **b**. It is skewed to the left or negatively skewed.

2 Number of tosses, $n = 10$

X is the number of heads obtained. $\therefore x$ is distributed $\text{Bin}(10, 0.5)$

$$\begin{aligned} \mu &= np & \sigma &= \sqrt{npq} \\ &= 10 \times 0.5 & &= \sqrt{10 \times 0.5 \times 0.5} \\ &= 5 & &\div 1.58 \end{aligned}$$

3 a x is distributed $\text{Bin}(3, p)$

$$\begin{aligned} P(x=0) &= C_0^3 p^0 q^3 & P(x=1) &= C_1^3 p^1 q^2 & P(x=2) &= C_2^3 p^2 (q)^1 \\ &= C_0^3 p^0 (1-p)^3 & &= 3p(1-p)^2 & &= 3p^2(1-p) \\ &= (1-p)^3 & & & & \end{aligned}$$

$$\begin{aligned} P(x=3) &= C_3^3 p^3 q^0 \\ &= p^3 \end{aligned}$$

x_i	0	1	2	3
$P(x_i)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

b $\mu = \sum x_i p_i$

$$\begin{aligned} &= 0(1-p)^3 + 1 \times 3p(1-p)^2 + 2 \times 3p^2(1-p) + 3p^3 \\ &= 3p(1-p)^2 + 6p^2(1-p) + 3p^3 \\ &= 3p(1-2p+p^2) + 6p^2 - 6p^3 + 3p^3 \\ &= 3p - 6p^2 + 3p^3 + 6p^2 - 6p^3 + 3p^3 \\ &= 3p \quad \text{as required} \end{aligned}$$

c $\sigma^2 = \sum x_i^2 p_i - \mu^2$

$$\begin{aligned} &= 0^2 \times (1-p)^3 + 1^2 \times 3p(1-p)^2 + 2^2 \times 3p^2(1-p) + 3^2 p^3 - (3p)^2 \\ &= 3p(1-p)^2 + 12p^2(1-p) + 9p^2(p-1) \\ &= (1-p)[3p(1-p) + 12p^2 - 9p^2] \\ &= (1-p)[3p - 3p^2 + 3p^2] \\ &= 3p(1-p) \\ &= 3pq \quad \therefore \sigma = \sqrt{3pq} \quad \text{as required} \end{aligned}$$

4 X is the number of defective bolts in the sample.

$$\begin{aligned} X \text{ is distributed } \text{Bin}(30, 0.04) \quad \mu &= np & \text{and } \sigma &= \sqrt{npq} \\ &= 30 \times 0.04 & &= \sqrt{30 \times 0.04 \times 0.96} \\ &= 1.2 & &\div 1.07 \end{aligned}$$

5 X is the number of fatalities.

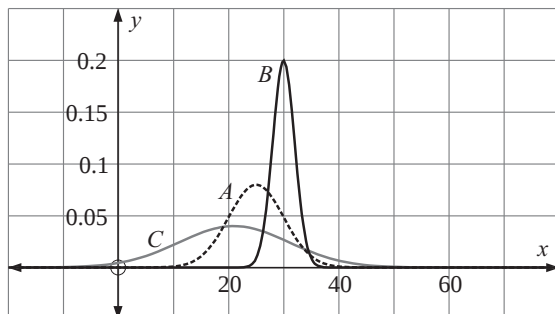
$$\begin{aligned} X \text{ is distributed } \text{Bin}(243, 0.037) \quad \mu &= np & \text{and } \sigma &= \sqrt{npq} \\ &= 243 \times 0.037 & &= \sqrt{243 \times 0.037 \times 0.963} \\ &= 8.99 & &\div 2.94 \end{aligned}$$

6 X is the number of groups that do not arrive.

$$\begin{aligned} X \text{ is distributed } \text{Bin}(5, 0.13) \quad \mu &= np & \text{and } \sigma &= \sqrt{npq} \\ &= 5 \times 0.13 & &= \sqrt{5 \times 0.13 \times 0.87} \\ &= 0.65 & &= 0.752 \end{aligned}$$

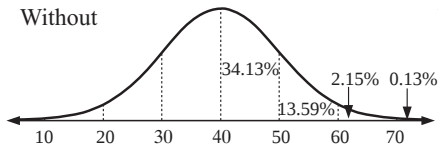
EXERCISE 29G.1

1



2 a/b/c/d

The mean volume (or life time, weight, diameter etc) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

3 Without

$$\mathbf{a} \quad P(\text{without and } x < 50)$$

$$\div 50\% + 34.13\%$$

$$\div 84.1\%$$

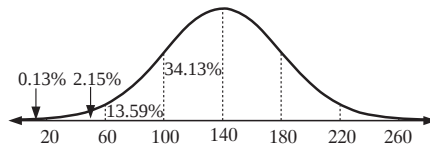
$$\mathbf{c} \quad P(\text{with and } 20 \leq x \leq 60)$$

$$\div 2.15\%$$

$$\mathbf{d} \quad P(\text{with and } x \geq 60)$$

$$\div 13.59\% + 34.13\% + 50\%$$

$$\div 97.7\%$$

With

$$\mathbf{b} \quad P(\text{with and } x < 60)$$

$$\div 0.13\% + 2.15\%$$

$$\div 2.28\%$$

$$\div 2.3\%$$

$$P(\text{without and } 20 \leq x \leq 60)$$

$$\div 2(34.13\% + 13.59\%)$$

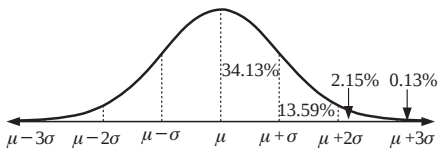
$$\div 95.44\%$$

$$\div 95.4\%$$

$$P(\text{without and } x \geq 60)$$

$$\div 2.15\% + 0.13\%$$

$$\div 2.3\%$$

4

$$\mathbf{c} \quad P(\mu < x < \mu + 2\sigma)$$

$$\div 0.3413 + 0.1359$$

$$\div 0.477$$

$$\mathbf{a} \quad P(\mu - \sigma < x < \mu + \sigma)$$

$$\div 2 \times 0.3413$$

$$\div 0.683$$

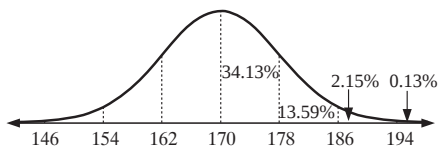
$$\mathbf{b} \quad P(\mu < x < \mu + \sigma)$$

$$\div 0.341$$

$$\mathbf{d} \quad P(\mu < x < \mu + 3\sigma)$$

$$\div 0.3413 + 0.1359 + 0.0215$$

$$\div 0.499$$

5

$$\mathbf{a} \quad P(162 < x < 170)$$

$$\div 34.1\%$$

$$\mathbf{b} \quad P(170 < x < 186)$$

$$\div 34.13\% + 13.59\%$$

$$\div 47.7\%$$

$$\mathbf{c} \quad P(178 < x < 186)$$

$$\div 13.59\%$$

$$\div 0.136$$

$$\mathbf{d} \quad P(x < 162)$$

$$\div 1 - (0.5 + 0.3413)$$

$$\div 0.159$$

$$\mathbf{e} \quad P(x < 154)$$

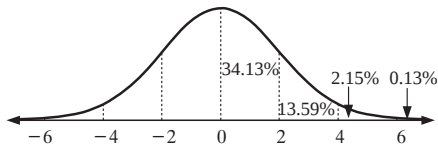
$$\div 0.0215 + 0.0013$$

$$\div 0.0228$$

$$\mathbf{f} \quad P(x > 162)$$

$$\div 1 - 0.159 \quad \{\text{using } \mathbf{d}\}$$

$$\div 0.841$$

6

$$\mathbf{b} \quad P(-6 < x < 0)$$

$$\div 0.5 - 0.0013$$

$$\div 0.4987$$

$$\therefore \text{expect } 800 \times 0.4987 \div 399$$

$$\mathbf{a} \quad P(0 < x < 4)$$

$$\div 0.3413 + 0.1359$$

$$\div 0.4772$$

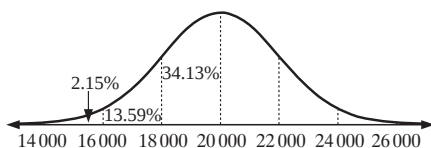
$$\therefore \text{expect } 800 \times 0.4772 \div 382$$

$$\mathbf{c} \quad P(-4 < x < 6)$$

$$\div 2 \times 0.3413 + 2 \times 0.1359 + 0.0215$$

$$\div 0.9759$$

$$\therefore \text{expect } 800 \times 0.9759 \div 781$$

7

b $P(x > 16000)$

$$\div 0.1359 + 0.3413 + 0.5$$

$$\div 0.9772$$

$$\therefore \text{expect } 260 \times 0.9772 = 254 \text{ days}$$

a $P(x < 18000)$

$$\div 1 - 0.5 - 0.3413$$

$$\div 0.1587$$

$$\therefore \text{expect } 260 \times 0.1587 = 41 \text{ days}$$

c $P(18000 \leq x \leq 24000)$

$$\div 0.3413 \times 2 + 0.1359$$

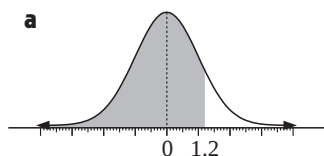
$$\div 0.8185$$

$$\therefore \text{expect } 0.8185 \times 260 = 213 \text{ days}$$

EXERCISE 29G.2 Using Technology

1 a 0.341 **b** 0.383 **c** 0.106

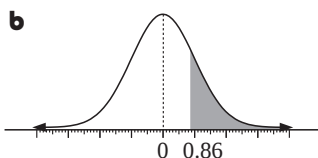
2 a 0.341 **b** 0.264 **c** 0.212 **d** 0.945 **e** 0.579 **f** 0.383

EXERCISE 29H.1**1 a**

$$P(z \leq 1.2)$$

$$= 0.8849$$

$$\div 0.885$$

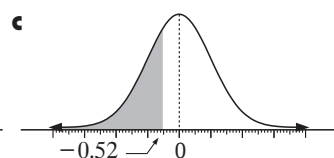
b

$$P(z \geq 0.86)$$

$$= 1 - P(z \leq 0.86)$$

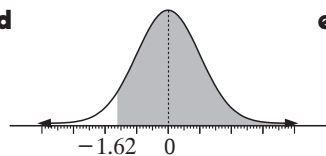
$$= 1 - 0.8051$$

$$\div 0.195$$

c

$$P(z \leq -0.52)$$

$$\div 0.3015$$

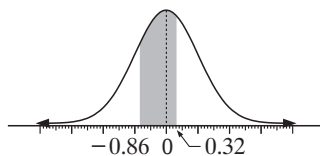
d

$$P(z \geq -1.62)$$

$$= 1 - P(z \leq -1.62)$$

$$\div 1 - 0.0526$$

$$\div 0.947$$

e

$$P(-0.86 < z < 0.32)$$

$$= P(z < 0.32) - P(z < -0.86)$$

$$= 0.6255 - 0.1949$$

$$\div 0.431$$

2

a $P(z \geq 0.837)$

$$= 1 - P(z \leq 0.837)$$

$$\div 1 - \text{normalcdf}(-E99, 0.837)$$

$$\div 0.201$$

b

$$P(z \leq 0.0614)$$

$$= \text{normalcdf}(-E99, 0.0614)$$

$$\div 0.524$$

c

$$P(z \geq -0.876)$$

$$= 1 - P(z \leq -0.876)$$

$$\div 0.809$$

d

$$P(-0.3862 \leq z \leq 0.2506)$$

$$= \text{normalcdf}(-0.3862, 0.2506)$$

$$\div 0.249$$

e

$$P(-2.367 \leq z \leq -0.6503)$$

$$= \text{normalcdf}(-2.367, -0.6503)$$

$$\div 0.249$$

3

a $P(-0.5 < z < 0.5)$

$$= \text{normalcdf}(-0.5, 0.5)$$

$$\div 0.383$$

b

$$P(-1.960 < z < 1.960)$$

$$= \text{normalcdf}(-1.96, 1.96)$$

$$\div 0.950$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & P(z \leq a) = 0.95 \\ & \therefore a \doteq 1.645 \\ & \{\text{searching in tables}\} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(z \geq a) = 0.90 \\ & \therefore 1 - P(z \leq a) = 0.90 \\ & \therefore P(z \leq a) = 0.1 \\ & \therefore a \doteq -1.28 - \frac{3}{18}(0.01) \\ & \therefore a \doteq -1.282 \end{aligned}$$

EXERCISE 29H.2

1 X is normal with mean 70, standard deviation 4.

$$\begin{aligned} \mathbf{a} \quad & P(x \geq 74) \\ & = \text{normalcdf}(74, \text{E99}, 70, 4) \\ & \doteq 0.159 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(x \leq 68) \\ & = \text{normalcdf}(-\text{E99}, 68, 70, 4) \\ & \doteq 0.309 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & P(60.6 \leq x \leq 68.4) \\ & = \text{normalcdf}(60.6, 68.4, 70, 4) \\ & \doteq 0.335 \end{aligned}$$

2 X is normal with mean 58.3 and standard deviation 8.96.

$$\begin{aligned} \mathbf{a} \quad & P(x \geq 61.8) \\ & = \text{normalcdf}(61.8, \text{E99}, 58.3, 8.96) \\ & \doteq 0.348 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(x \leq 54.2) \\ & = \text{normalcdf}(-\text{E99}, 54.2, 58.3, 8.96) \\ & \doteq 0.324 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & P(50.67 \leq x \leq 68.92) \\ & = \text{normalcdf}(50.67, 68.92, 58.3, 8.96) \\ & \doteq 0.685 \end{aligned}$$

3 L is normal with mean 50.2 mm and standard deviation 0.93 mm.

$$\begin{aligned} \mathbf{a} \quad & P(l \geq 50) \\ & = \text{normalcdf}(50, \text{E99}, 50.2, 0.93) \\ & \doteq 0.585 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(l \leq 51) \\ & = \text{normalcdf}(-\text{E99}, 51, 50.2, 0.93) \\ & = 0.805 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & P(49 \leq l \leq 50.5) \\ & = \text{normalcdf}(49, 50.5, 50.2, 0.93) \\ & \doteq 0.528 \end{aligned}$$

EXERCISE 29H.3

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & P(z \leq k) = 0.81 \\ & \therefore k \doteq 0.87 + \frac{22}{28}(0.01) \\ & \therefore k \doteq 0.878 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(z \leq k) = 0.58 \\ & \therefore k \doteq 0.20 + \frac{7}{39}(0.01) \\ & \therefore k \doteq 0.202 \end{aligned}$$

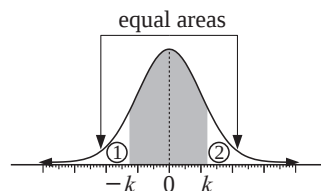
$$\begin{aligned} \mathbf{c} \quad & P(z \leq k) = 0.17 \\ & \therefore k \doteq -0.96 + \frac{15}{26}(0.01) \\ & \therefore k \doteq -0.954 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & P(z \leq k) = 0.384 \\ & \therefore k \doteq \text{invNorm}(0.384) \\ & \therefore k \doteq -0.295 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & P(z \leq k) = 0.878 \\ & \therefore k \doteq \text{invNorm}(0.878) \\ & \therefore k \doteq 1.165 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & P(z \leq k) = 0.1384 \\ & \therefore k \doteq \text{invNorm}(0.1384) \\ & \therefore k \doteq -1.088 \end{aligned}$$

$$\begin{aligned}
 3 \quad a \quad & P(-k \leq z \leq k) \\
 &= P(z \leq k) - P(z \leq -k) \\
 &= P(z \leq k) - P(z \geq k) \quad \{\text{as area ①} = \text{area ②}\} \\
 &= P(z \leq k) - [1 - P(z \leq k)] \\
 &= P(z \leq k) - 1 + P(z \leq k) \\
 &= 2P(z \leq k) - 1
 \end{aligned}$$



$$\begin{aligned}
 b \quad i \quad & P(-k \leq z \leq k) = 0.238 \\
 \therefore & 2P(z \leq k) - 1 = 0.238 \\
 \therefore & 2P(z \leq k) = 1.238 \\
 \therefore & P(z \leq k) = 0.619 \\
 \therefore & k = \text{invNorm}(0.619) \\
 \therefore & k \doteq 0.303
 \end{aligned}$$

$$\begin{aligned}
 ii \quad & P(-k \leq z \leq k) = 0.7004 \\
 \therefore & 2P(z \leq k) - 1 = 0.7004 \\
 \therefore & 2P(z \leq k) = 1.7004 \\
 \therefore & P(z \leq k) = 0.8502 \\
 \therefore & k = \text{invNorm}(0.8502) \\
 \therefore & k \doteq 1.037
 \end{aligned}$$

EXERCISE 29I

- 1 Let X be the length (in cm) of a bolt.
Then X is normally distributed with $\mu = 19.8$ and $\sigma = 0.3$.
 $P(19.7 < x < 20) = \text{normalcdf}(19.7, 20, 19.8, 0.3)$
 $\doteq 0.378$

- 2 Let X be the money collected (in \$)
Then X is normally distributed with $\mu = 40$ and $\sigma = 6$.

$$\begin{array}{ll}
 a \quad P(30.00 < x < 50.00) & b \quad P(x \geq 50) \\
 = \text{normalcdf}(30, 50, 40, 6) & = \text{normalcdf}(50, E99, 40, 6) \\
 \doteq 0.904 & \doteq 0.0478 \\
 \doteq 90.4\% & \doteq 4.78\%
 \end{array}$$

- 3 Let X be the result of the Physics test.
Then X is normally distributed with $\mu = 46$ and $\sigma = 25$.

$$\begin{aligned}
 & \text{We need to find } k \text{ such that } P(x \geq k) = 0.07 \\
 & \text{i.e., } 1 - P(x \leq k) = 0.07 \\
 & \text{i.e., } P(x \leq k) = 0.93 \\
 & \therefore k = \text{invNorm}(0.93, 46, 25) \\
 & \therefore k = 82.894\dots\dots \\
 & \therefore k \doteq 83 \quad \{k \text{ assumed to be an integer}\} \\
 & \text{i.e., lowest score would be 83 to get an A.}
 \end{aligned}$$

- 4 Let X be the length of an eel (in cm)
Then X is normally distributed with $\mu = 41$ and $\sigma = \sqrt{11}$

$$\begin{array}{ll}
 a \quad P(x \geq 50) & b \quad P(40 \leq x \leq 50) \\
 = \text{normalcdf}(50, E99, 41, \sqrt{11}) & = \text{normalcdf}(40, 50, 41, \sqrt{11}) \\
 \doteq 0.00333 & \doteq 0.615 \\
 & \doteq 61.5\% \\
 c \quad P(x \geq 45) & \\
 = \text{normalcdf}(45, E99, 41, \sqrt{11}) & \\
 \doteq 0.114 & \therefore \text{we expect } 200 \times 0.114 \doteq 23 \text{ eels}
 \end{array}$$

- 5 X is the height of a palm tree (in cm).
 X is normally distributed with $\mu = 181$, $\sigma = 4$.

$$\begin{array}{ll}
 a \quad P(x \geq 175) & b \quad P(177 \leq x \leq 180) \\
 = \text{normalcdf}(175, E99, 181, 4) & = \text{normalcdf}(177, 180, 181, 4) \\
 \doteq 0.933 & \doteq 0.243
 \end{array}$$

- 6 a** Let the mean be μ and standard deviation be σ .

$$\begin{aligned}
 \text{Then } P(x \geq 80) &= 0.1 & \text{and } P(x \leq 30) &= 0.15 \\
 \therefore P(x \leq 80) &= 0.9 & \therefore P\left(\frac{x - \mu}{\sigma} \leq \frac{30 - \mu}{\sigma}\right) &= 0.15 \\
 \therefore P\left(\frac{x - \mu}{\sigma} \leq \frac{80 - \mu}{\sigma}\right) &= 0.9 & \therefore P\left(z \leq \frac{30 - \mu}{\sigma}\right) &= 0.15 \\
 \therefore P\left(z \leq \frac{80 - \mu}{\sigma}\right) &= 0.9 & \therefore \frac{30 - \mu}{\sigma} &= \text{invNorm}(0.15) \\
 \therefore \frac{80 - \mu}{\sigma} &= \text{invNorm}(0.9) & \therefore 30 - \mu &\div -1.0364\sigma \quad \dots\dots (2) \\
 \therefore 80 - \mu &\div 1.2816\sigma \quad \dots\dots (1)
 \end{aligned}$$

From (1) and (2) $(80 - \mu) - (30 - \mu) \div 1.2816\sigma + 1.0364\sigma$

$$50 \div 2.318\sigma$$

$$\therefore \sigma \div \frac{50}{2.318} \div 21.57$$

and in (1) $80 - \mu \div 1.2816 \times 21.57 \div 27.6$

$$\therefore \mu \div 52.4$$

i.e., $\mu \div 52.4$ and $\sigma \div 21.6$

- b** X is the result of the Maths Exam.

X is normally distributed with mean μ and standard deviation σ .

Then $P(x \geq 80) = 0.1$ and $P(x \leq 30) = 0.15$

So, from **a** $\mu \div 52.4$ and $\sigma \div 21.6$

$$\begin{array}{ll}
 P(x > 50) & \text{or } P(x \geq 51) \quad \{\text{for integer marks only}\} \\
 = \text{normalcdf}(50, E99, 52.4, 21.6) & = \text{normalcdf}(51, E99, 52.4, 21.6) \\
 \div 0.544 & \div 0.526 \\
 \div 54.4\% & \div 52.6\%
 \end{array}$$

This answer assumes 'part marks' can be given.

- 7 a** Let the mean be μ and standard deviation be σ and X be the diameter (in cm).

$$\begin{aligned}
 \therefore P(x < 1.94) &= 0.02 \quad \text{and} & P(x > 2.06) &= 0.03 \\
 \therefore P\left(\frac{x - \mu}{\sigma} < \frac{1.94 - \mu}{\sigma}\right) &= 0.02 & \therefore P\left(\frac{x - \mu}{\sigma} > \frac{2.06 - \mu}{\sigma}\right) &= 0.03 \\
 \therefore P\left(z < \frac{1.94 - \mu}{\sigma}\right) &= 0.02 & \therefore P\left(z > \frac{2.06 - \mu}{\sigma}\right) &= 0.03 \\
 \therefore \frac{1.94 - \mu}{\sigma} &= \text{invNorm}(0.02) \quad \text{i.e., } P\left(z < \frac{2.06 - \mu}{\sigma}\right) &= 0.97 \\
 \therefore 1.94 - \mu &\div -2.054\sigma \quad \dots\dots (1) & \therefore \frac{2.06 - \mu}{\sigma} &= \text{invNorm}(0.97) \\
 & & \therefore 2.06 - \mu &\div 1.881\sigma \quad \dots\dots (2)
 \end{aligned}$$

From (1) and (2) $(2.06 - \mu) - (1.94 - \mu) = 1.881\sigma + 2.054\sigma$

$$\therefore 3.935\sigma = 0.12$$

$$\therefore \sigma \div 0.0305$$

and in (1) $1.94 - \mu = -2.054 \times 0.0305 \div -0.0626$

$$\therefore \mu \div 2.00$$

i.e., $\mu \div 2.00$ and $\sigma \div 0.0305$

- b** This is a binomial situation with the probability $p = 0.02 + 0.03 = 0.05$ of failure to operate and $n = 20$

$$\begin{aligned}
 \therefore P(\text{less than 2 will operate}) \\
 &= P(x \leq 1) \\
 &= \text{binomcdf}(20, 0.05, 1) \\
 &\doteq 0.736
 \end{aligned}$$

- 8** S is distributed normally with mean 60 and standard deviation 10.

$$\begin{aligned}
 P(s \geq 70) &= \text{normalcdf}(70, E99, 60, 10) \\
 &\doteq 0.158 \\
 &\doteq 16\%
 \end{aligned}$$

G is distributed normally with mean 50 and standard deviation 12.

$$\begin{aligned}
 P(g \geq 66) &= \text{normalcdf}(66, E99, 50, 12) \\
 &\doteq 0.0912 \\
 &\doteq 9\%
 \end{aligned}$$

i.e., only 9% achieved a higher grade in Geography while almost 16% achieved a higher grade in Science. The student achieved a grade higher than 91% of the class in Geography and 84% of the class in Science.

REVIEW SET 29A

1 a $P(x) = \frac{a}{x^2 + 1}$ for $a = 0, 1, 2, 3$

x_i	0	1	2	3
$P(x_i)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

$$\text{Now } a + \frac{a}{2} + \frac{a}{5} + \frac{a}{10} = 1 \quad \{\text{as } \sum P(x_i) = 1\}$$

$$\therefore 10a + 5a + 2a + a = 10 \quad \{\times \text{ each term by } 10\}$$

$$\therefore 18a = 10$$

$$\therefore a = \frac{5}{9}$$

b $P(x \geq 1) = P(x = 1, x = 2 \text{ or } x = 3) \quad \text{or} \quad P(x \geq 1) = 1 - P(x < 1)$

$$\begin{aligned}
 &= P(x = 1) + P(x = 2) + P(x = 3) &&= 1 - P(x = 0) \\
 &= \frac{5}{18} + \frac{1}{9} + \frac{5}{90} &&= 1 - \frac{5}{9} \\
 &= \frac{4}{9} &&= \frac{4}{9}
 \end{aligned}$$

2 a $P(x) = C_x^4 \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

$$P(2) = C_2^4 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 0.375$$

$$\therefore P(0) = C_0^4 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 0.0625$$

$$P(3) = C_3^4 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 = 0.25$$

$$P(1) = C_1^4 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 0.25$$

$$P(4) = C_4^4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 = 0.0625$$

i.e.,

x_i	0	1	2	3	4
$P(x_i)$	0.0625	0.25	0.375	0.25	0.0625

b $\mu = \sum x_i P(x_i)$

$$\begin{aligned}
 &= 0 \times 0.0625 + 1 \times 0.25 + 2 \times 0.375 + 3 \times 0.25 + 4 \times 0.0625 \\
 &= 2
 \end{aligned}$$

$$\sigma = \sqrt{\sum (x_i - \mu)^2 P(x_i)}$$

$$\begin{aligned}
 &= \sqrt{(-2)^2(0.0625) + (-1)^2(0.25) + 0^2(0.375) + 1^2(0.25) + 2^2(0.0625)} \\
 &= 1
 \end{aligned}$$

3 X is the number of defectives. Then X is Bin (10, 0.18). $X = 0, 1, 2, 3, \dots, 10$.

a $P(X = 1)$
 $= \text{binompdf}(10, 0.18, 1)$
 $\div 0.302$

b $P(X = 2)$
 $= \text{binompdf}(10, 0.18, 2)$
 $\div 0.298$

c $P(X \geq 2) = 1 - P(X \leq 1)$
 $= 1 - \text{binomcdf}(10, 0.18, 1)$
 $\div 0.561$

4 Let X be the number of defective toothbrushes.

X is distributed Bin(120, 0.04), $n = 120$, $p = 0.04$, $q = 0.96$,

a $\mu = np$
 $= 120 \times 0.04$
 $= 4.8$

b $\sigma = \sqrt{npq}$
 $= \sqrt{120 \times 0.4 \times 0.6}$
 $\div 2.15$

5

Result	Pays
1, 3, 5	\$2
2	\$3
4	\$6
6	\$9

a Expected return $= \frac{3}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$6 + \frac{1}{6} \times \9
 $= \frac{1}{6}(\$24)$
 $= \$4$

b For a \$5 amount to play the game the club expects a \$1 return/game
 $\therefore$ for 75 people, the return expected is \$75.

6

a

Result	Pays
1	\$2
2	\$4
3	\$6
4	\$8
5	\$10
6	\$12

Expectation
 $= \frac{1}{6} \times \$2 + \frac{1}{6} \times \$4 + \frac{1}{6} \times \$6 + \frac{1}{6} \times \$8 + \frac{1}{6} \times \$10 + \frac{1}{6} \times \12
 $= \frac{1}{6} \times \$42$
 $= \$7$

b No, as \$1 is expected to be lost per game in the long run.

7 If X is the arm length random variable (in cm) then X is normally distributed with $\mu = 64$ and $\sigma = 4$.

a i $P(60 < x < 72)$
 $= \text{normalcdf}(60, 72, 64, 4)$
 $\div 0.8186$
 $\div 81.9\%$

ii $P(X > 60)$
 $= \text{normalcdf}(60, E99, 64, 4)$
 $\div 0.841$
 $\div 84.1\%$

b $P(56 < x < 68) = \text{normalcdf}(56, 68, 64, 4)$
 $\div 0.819$

8 X is the rod length (in mm)

X is normally distributed
 with mean μ and $\sigma = 3$

Now $P(x < 25) = 0.02$
 $\therefore P\left(\frac{x - \mu}{3} < \frac{25 - \mu}{3}\right) = 0.02$
 $\therefore P\left(z < \frac{25 - \mu}{3}\right) = 0.02$
 $\therefore \frac{25 - \mu}{3} = \text{invNorm}(0.02)$
 $\therefore \frac{25 - \mu}{3} \div -2.0537$
 $\therefore 25 - \mu \div -6.161$
 $\therefore \mu \div 31.2$

REVIEW SET 29B

1 a $P(x_i) = k \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{3-x}$ for $x = 0, 1, 2, 3$

$$P(0) = k \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^3 = \frac{k}{64} \quad P(2) = k \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^1 = \frac{9k}{64}$$

$$P(1) = k \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^2 = \frac{3k}{64} \quad P(3) = k \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^0 = \frac{27k}{64}$$

x_i	0	1	2	3
$P(x_i)$	$\frac{k}{64}$	$\frac{3k}{64}$	$\frac{9k}{64}$	$\frac{27k}{64}$

Now $\frac{k}{64} + \frac{3k}{64} + \frac{9k}{64} + \frac{27k}{64} = 1$ {as $\sum p(x_i) = 1$ }

$$\therefore \frac{40k}{64} = 1$$

$$\therefore k = 1.6$$

b $P(x \geq 1) = 1 - P(x = 0)$

$$= 1 - \frac{k}{64}$$

$$= 1 - \frac{1.6}{64}$$

$$= 0.975$$

2 Let X be the payout values then $x = \$45\,000, \$30\,000, \$10\,000$ or $\$0$

x_i	0	10 000	30 000	45 000
$P(x_i)$	0.988 18	0.0088	0.0023	0.000 72

$\therefore$ expected payout over the long run

$$= \sum x_i p_i$$

$$= 0(0.988\,18) + 10\,000(0.0088) + 30\,000(0.0023) + 45\,000(0.000\,72)$$

$$= \$189.40$$

$\therefore$ to cover an expected return of \$250 they need to charge $\$189.40 + \$250 = \$439.40$

3 X is the number of hits, then $X = 0, 1, 2, 3, 4$ and X is Bin(4, 0.96)

a $P(X = 4)$
 $= \text{binompdf}(4, 0.96, 4)$
 $\div 0.849$

b $P(X = 0)$
 $= \text{binompdf}(4, 0.96, 0)$
 $\div 0.000$

c $P(X \geq 3)$
 $= 1 - P(X \leq 2)$
 $= 1 - \text{binomcdf}(4, 0.96, 2)$
 $\div 0.991$

d $P(X = 1)$
 $= \text{binompdf}(4, 0.96, 1)$
 $\div 0.000\,246$

4 X is distributed Bin(4, p)

$$\therefore P(x = 0) = C_0^4 p^0 (1-p)^4$$

$$= (1-p)^4$$

$$P(x = 1) = C_1^4 p^1 (1-p)^3$$

$$= 4p(1-p)^3$$

$$P(x = 2) = C_2^4 p^2 (1-p)^2$$

$$= 6p^2(1-p)^2$$

$$P(x = 3) = C_3^4 p^3 (1-p)^1$$

$$= 4p^3(1-p)$$

$$P(x = 4) = C_4^4 p^4 (1-p)^0$$

$$= p^4$$

x_i	0	1	2	3	4
$P(x_i)$	$(1-p)^4$	$4p(1-p)^3$	$6p^2(1-p)^2$	$4p^3(1-p)$	p^4

$$\begin{aligned}
 \mu &= \sum x_i p_i \\
 &= 0(1-p)^4 + 1 \times 4p(1-p)^3 + 2 \times 6p^2(1-p)^2 + 3 \times 4p^3(1-p) + 4 \times p^4 \\
 &= 4p(1-p)^3 + 12p^2(1-p)^2 + 12p^3(1-p) + 4p^4 \\
 &= 4p(1-3p+3p^2-p^3) + 12p^2(1-2p+p^2) + 12p^3 - 12p^4 + 4p^4 \\
 &= 4p - 12p^2 + 12p^3 - 4p^4 + 12p^2 - 24p^3 + 12p^4 + 12p^3 - 12p^4 + 4p^4 \\
 &= 4p, \text{ as required}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \sigma^2 &= \sum x_i^2 p_i - (\mu)^2 \\
 &= [0 \times (1-p)^4 + 1^2 \times 4p(1-p)^3 + 2^2 \times 6p^2(1-p)^2 + 3^2 \times 4p^3(1-p) + 4^2 p^4] \\
 &\quad - (4p)^2 \\
 &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) + 16p^4 - 16p^2 \\
 &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) - 16p^2(1-p^2) \\
 &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) - 16p^2(1+p)(1-p) \\
 &= (1-p) [4p(1-p)^2 + 24p^2(1-p) + 36p^3 - 16p^2(1+p)] \\
 &= q [4p - 8p^2 + 4p^3 + 24p^2 - 24p^3 + 36p^3 - 16p^2 - 16p^3] \\
 &= q \times 4p \\
 &= 4pq \\
 \therefore \sigma &= \sqrt{4pq}, \text{ as required.}
 \end{aligned}$$

5 X is the contents of the container (in mL).

X is normally distributed with $\mu = 377$ and $\sigma = 4.2$

- a i** $P(X < 368.6)$
 $= \text{normalcdf}(-E99, 368.6, 377, 4.2)$
 $\doteq 0.0228$
 $\doteq 2.28\%$
- ii** $P(372.8 < X < 389.6)$
 $= \text{normalcdf}(372.8, 389.6, 377, 4.2)$
 $\doteq 0.840$
 $\doteq 84.0\%$
- b** $P(364.4 < X < 381.2)$
 $= \text{normalcdf}(364.4, 381.2, 377, 4.2)$
 $\doteq 0.840$

6 X is the life of a battery (in weeks)

X is normally distributed with $\mu = 33.2$, $\sigma = 2.8$

- a** $P(X \geq 35)$
 $= \text{normalcdf}(35, E99, 33.2, 2.8)$
 $\doteq 0.260$
- b** We need to find k such that $P(X \geq k) = 0.08$
i.e., $P(X \leq k) = 0.92$
 $\therefore k = \text{invNorm}(0.92, 33.2, 2.8)$
 $\therefore k \doteq 37.134$

So, the manufacturer can expect that no more than 8% will fail for a maximum of 37 weeks and 1 day, i.e., 260 days.

REVIEW SET 29C

1

x_i	0	1	2	3	4
$P(x_i)$	0.10	0.30	0.45	0.10	k

- a** If this is a probability distribution then $\sum p(x_i) = 1$
 $\therefore 1 = 0.1 + 0.3 + 0.45 + 0.1 + k$
 $\therefore k = 1 - 0.95$
 $\therefore k = 0.05$

b $\mu = \sum x_i p_i$
 $= 0(0.1) + 1(0.3) + 2(0.45) + 3(0.1) + 4(0.05)$
 $= 0 + 0.3 + 0.9 + 0.3 + 0.2$
 $= 1.7$
 $\sigma^2 = \sum x_i^2 p_i - (\mu)^2$
 $= 0^2(0.1) + 1^2(0.3) + 2^2(0.45) + 3^2(0.1) + 4^2(0.05) - (1.7)^2$
 $= 0.3 + 1.8 + 0.9 + 0.8 - 2.89$
 $= 0.91$ $\therefore \sigma = \sqrt{0.91} \doteq 0.954$

- 2** X is the number of trees which survive the first year.

$\therefore X = 0, 1, 2, 3, 4, 5$ and X is Bin(5, 0.4)

a $P(X = 1)$ $= \text{binompdf}(5, 0.4, 1)$ $\doteq 0.259$	b $P(X \leq 1)$ $= \text{binomcdf}(5, 0.4, 1)$ $\doteq 0.337$	c $P(X \geq 1)$ $= 1 - P(X = 0)$ $= 1 - \text{binompdf}(5, 0.4, 0)$ $\doteq 0.922$
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- 3** Let X denote the number of cases of netballers needing knee surgery.

x is distributed Bin(487, 0.0132), $n = 487$, $p = 0.0132$, $q = 0.9868$

$\therefore \mu = np$ $= 487 \times 0.0132$ $\doteq 6.43$	$\sigma = \sqrt{npq}$ $= \sqrt{487 \times 0.0132 \times 0.9868}$ $\doteq 2.52$
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- 4** Let X denote the mass of a Coffin Bay Oyster. X is distributed normally with a mean of 38.6 and a standard deviation of 6.3.

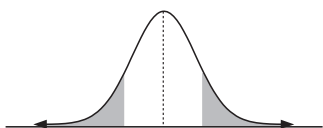
a $P(38.6 - a \leq x \leq 38.6 + a) = 0.6826$
 $P\left(\frac{38.6 - a - 38.6}{6.3} \leq \frac{x - 38.6}{6.3} \leq \frac{38.6 + a - 38.6}{6.3}\right) = 0.6826$

$$P\left(-\frac{a}{6.3} \leq z \leq \frac{a}{6.3}\right) = 0.6826$$

$$\therefore \text{by symmetry, } P\left(z \leq -\frac{a}{6.3}\right) = \frac{1 - 0.6826}{2} = 0.1587$$

$$\therefore -\frac{a}{6.3} = \text{invNorm}(0.1587)$$

$$\therefore -\frac{a}{6.3} \doteq -0.9998 \quad \text{and} \quad \therefore a \doteq 6.3 \text{ gms}$$



$$\begin{aligned}
 \mathbf{b} \quad & P(x \geq b) = 0.8413 \\
 & \therefore P(x \leq b) = 0.1587 \\
 & P\left(\frac{x - 38.6}{6.3} \leq \frac{b - 38.6}{6.3}\right) = 0.1587 \\
 & \text{i.e., } P\left(z \leq \frac{b - 38.6}{6.3}\right) = 0.1587 \\
 & \therefore \frac{b - 38.6}{6.3} \div \text{invNorm}(0.1587) \\
 & \therefore b - 38.6 \div 6.3 \times -0.9998 \\
 & \therefore b \div -6.298 + 38.6 \\
 & \therefore b \div 32.3 \text{ gms}
 \end{aligned}$$

- 5** Let X denote the life of the staplers. X is distributed normally with a mean of 3.42 and standard deviation 0.4.

$$\begin{aligned}
 P(x \leq 3) &= \text{normalcdf}(-E99, 3, 3.42, 0.4) \\
 &\div 0.147
 \end{aligned}$$

i.e., out of 2000 in the batch we would expect to have to replace $2000 \times 0.147 \div 294$

$$\therefore \text{profit} = \$15 \times (2000 - 294) = \$15 \times 1706 = \$25\,590$$

6 $f(x) = ax^2(2 - x)$ for $0 < x < 2$

- a** Since $f(x)$ is a probability distribution function the area under the curve is 1

$$\text{i.e., } \int_0^2 ax^2(2 - x) dx = 1$$

$$\therefore a \int_0^2 (2x^2 - x^3) dx = 1$$

$$\therefore a \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = 1$$

$$\therefore a \left(\frac{16}{3} - \frac{16}{4} \right) - 0 = 1$$

$$\therefore a \left(\frac{4}{3} \right) = 1$$

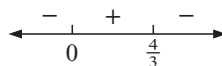
$$\therefore a = \frac{3}{4}$$

- b** The mode is the most frequently occurring score, i.e., the value of x when $f(x)$ is a maximum.

$$f(x) = \frac{3}{4}(2x^2 - x^3)$$

$$\begin{aligned}
 f'(x) &= \frac{3}{4}(4x - 3x^2) \\
 &= \frac{3}{4}x(4 - 3x)
 \end{aligned}$$

which has sign diagram:



$\therefore$ is a maximum when $x = \frac{4}{3}$

$\therefore$ the mode is $\frac{4}{3}$.

- c** If the median is m , then,

$$\int_0^m \frac{3}{4}x^2(2 - x) dx = \frac{1}{2}$$

$$\frac{3}{4} \int_0^m (2x^2 - x^3) dx = \frac{1}{2}$$

$$\therefore \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^m = \frac{2}{3}$$

$$\therefore \frac{2m^3}{3} - \frac{m^4}{4} = \frac{2}{3}$$

$$\therefore m \div 1.2285$$

{using technology}

i.e., the median is approximately 1.23.

$$\begin{aligned}
 \mathbf{d} \quad P(0.6 < x < 1.2) &= \int_{0.6}^{1.2} \frac{3}{4}x^2(2 - x) dx \\
 &= 0.3915 \\
 &\quad \text{{using technology}}
 \end{aligned}$$